SOFIC REPRESENTATIONS OF AMENABLE GROUPS

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ABSTRACT. Using probabilistic methods, Collins and Dykema proved that the free product of two sofic groups amalgamated over a monotileable amenable subgroup is sofic as well. We show that the restriction is unnecessary; the free product of two sofic groups amalgamated over an arbitrary amenable subgroup is sofic. We also prove a group-theoretical analogue of a result of Kenley Jung. A finitely generated group is amenable if and only if it has only one sofic representation up to conjugacy equivalence.

1. INTRODUCTION

First, let us recall the notion of sofic groups (see [8] for a survey). Let Γ be a finitely generated group with a symmetric generating system S . Let G be a finite graph such that each directed edge of G is labeled by an element of S . We say that G is an r -approximation of the Cayley graph of Γ , $\text{Cay}(\Gamma, S)$ if there exists a subset $W \subseteq V(G)$ such that

- $|W| > (1 - \frac{1}{r})|V(G)|$ and
- if $p \in W$, then the r -neighborhood of p is rooted isomorphic to the r -neighborhood of a vertex of the Cayley graph of Γ (as edge labeled graphs).

The group Γ is sofic if for any $r \geq 1$ there exist r -approximations of $\text{Cay}(\Gamma, S)$ by finite graphs. We call a not necessarily finitely generated group sofic if all of its finitely generated subgroups are sofic. Amenable and residually finite groups are sofic. Free products, direct products, inverse and direct limits of sofic groups are all sofic groups as well. Amenable extensions of sofic groups are also sofic. It is important to note that no nonsofic group is known at this moment. Several conjectures about groups, such as the Gottschalk Conjecture, Kaplansky's Direct Finiteness Conjecture, the Determinant Conjecture and the Connes Embedding Conjecture hold for sofic groups.

Using probabilistic arguments, Collins and Dykema [1] proved that the free products of sofic groups amalgamated over monotileable amenable groups are sofic. We show that the restriction is not necessary.

Theorem 1. *Free products of sofic groups amalgamated over arbitrary amenable groups are sofic.*

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In [6], Kenley Jung gave a characterization of amenable von Neumann algebras among “embeddable” von Neumann algebras. He proved that an embeddable von Neumann algebra M is amenable if and only if any two embeddings of M into R^ω are conjugate by a unitary element. Before stating our result, let us briefly recall an equivalent definition of soficity. Let ω be a nonprincipal ultrafilter on the natural numbers and $\lim_\omega : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$ be the associated ultralimit. Consider the product of finite symmetric groups $\prod_{n=1}^\infty S_n$. If $g \in S_n$, then let the rank of g be defined as $rk(g) := 1 - \frac{fix(g)}{n}$, where $fix(g)$ is the number of elements in the set $\{1, 2, \dots, n\}$ fixed by the permutation g . Let $H \subset \prod_{n=1}^\infty S_n$ be defined as follows:

$$H := \{(g_n)_{n=1}^\infty \mid \lim_\omega rk(g) = 0\}.$$

Then H is a normal subgroup of $\prod_{n=1}^\infty S_n$ and $\Sigma = \prod_{n=1}^\infty S_n / H$ is a simple group with a rank function

$$rk(\underline{g}) = \lim_\omega rk(g_n).$$

We call an embedding $\Gamma \rightarrow \Sigma$ faithful if $rk(\gamma) = 1$ for any $1 \neq \gamma \in \Gamma$. It is easy to see that sofic approximations lead to faithful embeddings of countable sofic groups. Conversely, any embeddable group is sofic [5].

The embeddings ϕ and ψ are *conjugate* if there exists $\underline{g} \in \Sigma$ such that $\phi(\gamma) = g\psi(\gamma)g^{-1}$ for any $\gamma \in \Gamma$. We prove the following theorem.

Theorem 2. *A finitely generated group Γ is amenable if and only if any two faithful embeddings of Γ into Σ are conjugate.*

Note added. Liviu Paunescu [7] informed us that he also has a proof of Theorem 1. using a different argument.

2. SOFIC APPROXIMATIONS OF AMENABLE GROUPS

Definition 2.1. Let $E(A)$ denote the set of edges in a colored graph A . We say that two colored graphs A and B are *r-isomorphic* for some $r > 0$ if there are subgraphs $A' \subseteq A$ and $B' \subseteq B$ such that

$$|E(A')| \geq \left(1 - \frac{1}{r}\right) |E(A)|, \quad |E(B')| \geq \left(1 - \frac{1}{r}\right) |E(B)|$$

and A' is isomorphic to B' (as colored graphs). The isomorphism between A' and B' is called an *r-isomorphism*.

Lemma 2.2. *Let A , B and C be colored graphs such that A is $2r$ -isomorphic to B and B is $2r$ -isomorphic to C . Then A is r -isomorphic to C . If A is a $2r$ -approximation of a Cayley graph and every vertex of A has degree at most d , then there is a bound $R_{\text{appr}}(r, d)$ such that every colored graph $R_{\text{appr}}(r, d)$ -isomorphic to A is an r -approximation of the Cayley graph.*

Proof. This is clear from the definition. □

Definition 2.3. Let $\mathcal{T} = T_1, T_2, \dots, T_m$ be a finite sequence of colored graphs. Their *linear combination with coefficient vector* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{N}^m$, denoted by $\alpha \cdot \mathcal{T}$, is the disjoint union of α_1 copies of T_1 , α_2 copies of T_2 , $\dots$ and α_m copies of T_m . As a special case, if $m = 1$, then we talk about the *integer multiples* of T_1 .

Definition 2.4. For vectors $0 \neq \alpha \in \mathbb{R}^m$ we define the unit vector $\sigma(\alpha) = \frac{\alpha}{\|\alpha\|}$, where $\|\alpha\|$ denotes the (usual) length of α .

Lemma 2.5. *Let $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ be a finite sequence of colored graphs, each having at least one edge. For each integer $r > 0$ there is an integer $M(r) > 0$ (also depending on $\mathcal{T}$) such that whenever $\alpha, \beta \in \mathbb{N}^m$ are nonzero lattice vectors with $\|\beta\| \geq M(r)\|\alpha\|$ and $\|\sigma(\alpha) - \sigma(\beta)\| \leq \frac{1}{M(r)}$, then the graph $\beta \cdot \mathcal{T}$ is r -isomorphic to an integer multiple of $\alpha \cdot \mathcal{T}$.*

Proof. Let E be the maximum number of edges in T_i and let t be the largest integer such that $\|t\alpha\| \leq \|\beta\|$. Then $\|\alpha\| \geq 1$. Assuming that $M(r)$ as above exists, it will clearly satisfy the condition $M(r) \leq t$. Thus

$$\|\beta - t\alpha\| \leq \|\alpha\| + t\|\sigma(\beta) - \sigma(\alpha)\| \leq \|\alpha\| + t\frac{1}{M(r)} \leq \frac{2}{M(r)}\|\alpha\|t;$$

hence each coordinate of β differs by at most $\frac{2}{M(r)}\|\alpha\|t$ from the corresponding coordinate of $t\alpha$. Therefore $\beta \cdot \mathcal{T}$ can be obtained from $(t\alpha) \cdot \mathcal{T}$ by adding or deleting at most $\frac{2}{M(r)}\|\alpha\|t$ copies of each T_i , and the total number of edges involved in this operation is at most $\frac{2mE}{M(r)}\|\alpha\|t$.

On the other hand, the sum of the coordinates of $t\alpha$ is at least $\|\alpha\|$ (triangle inequality); hence the graph $t\alpha \cdot \mathcal{T}$ has at least $\|\alpha\|t$ edges, and similarly, the graph $\beta \cdot \mathcal{T}$ has at least $\|\beta\| \geq \|\alpha\|t$ edges. If we choose $M(r) \geq 2mEr$, then we obtain that $\beta \cdot \mathcal{T}$ is r -isomorphic to $(t\alpha) \cdot \mathcal{T}$, which is indeed an integer multiple of $\alpha \cdot \mathcal{T}$. $\square$

Definition 2.6. Let Γ be a finitely generated group, $S \subseteq \Gamma$ a finite symmetric generating set. A Følner sequence in the Cayley graph $\text{Cay}(\Gamma, S)$ is a sequence $\mathcal{F} = \{F_1, F_2, \dots\}$ of spanned subgraphs such that for all $r > 0$ all but finitely many of the F_n are r -approximations. The group Γ is *amenable* if $\text{Cay}(\Gamma, S)$ has a Følner sequence. Note, that this property depends only on the group; i.e. any other Cayley graph of Γ has a Følner sequence. More generally, an arbitrary group is *amenable* if each of its finitely generated subgroups is amenable.

Proposition 2.7 (Reformulation of [2, Theorem 2]). *Let Γ be a finitely generated amenable group, $S \subseteq \Gamma$ a finite symmetric generating set and $\mathcal{F}$ a Følner sequence in $\text{Cay}(\Gamma, S)$. Then for each integer $r > 0$ there is an integer $R_0(r) > 0$ (which depends also on $\mathcal{F}$) and a finite subsequence $\mathcal{T} \subset \mathcal{F}$ such that each $R_0(r)$ -approximation of $\text{Cay}(\Gamma, S)$ is r -equivalent to some linear combination of $\mathcal{T}$.*

Proposition 2.8. *Let Γ be a finitely generated amenable group, $S \subseteq \Gamma$ a finite symmetric generating set and $\mathcal{F}$ a Følner sequence in $\text{Cay}(\Gamma, S)$. Then for each integer $r > 0$ there is an integer $R_{\text{amenable}}(r)$ (which depends also on $\mathcal{F}$) and a finite subsequence $\mathcal{T} \subset \mathcal{F}$ of say m members and a lattice vector $0 \neq \alpha \in \mathbb{N}^m$ with the following property. Each member of $\mathcal{T}$ is a $2r$ -approximation of $\text{Cay}(\Gamma, S)$, and each $R_{\text{amenable}}(r)$ -approximation of $\text{Cay}(\Gamma, S)$ is $2r$ -equivalent to an integer multiple of the (fixed) linear combination $\alpha \cdot \mathcal{T}$.*

Proof. By discarding a few elements from $\mathcal{F}$ we may assume that the remaining elements are all $2r$ -approximations of $\text{Cay}(\Gamma, S)$. We apply Proposition 2.7 with parameter $8r$, and we obtain a subsequence $\mathcal{T} \subset \mathcal{F}$ of say m members such that every $R_0(8r)$ -approximation of $\text{Cay}(\Gamma, S)$ is $8r$ -isomorphic to some linear combination of $\mathcal{T}$. Now we discard those elements of $\mathcal{F}$ that are not an $R_0(8r)$ -approximation of $\text{Cay}(\Gamma, S)$. Note that $\mathcal{T}$ is no longer a subsequence of the remaining $\mathcal{F}$.

For each member $F \in \mathcal{F}$ we choose a lattice vector $0 \neq \beta_F \in \mathbb{N}^m$ such that $\beta_F \cdot \mathcal{T}$ is $8r$ -isomorphic to F . Now $\{\sigma(\beta_F) | F \in \mathcal{F}\}$ is a sequence of unit vectors,

so it has an accumulation point $u \in \mathbb{R}^m$. We apply Lemma 2.5 to $\mathcal{T}$ with parameter $8r$ and obtain the bound $M(8r)$. Let us fix a member $H \in \mathcal{F}$ such that $\|\sigma(\beta_H) - u\| < \frac{1}{2M(8r)}$ and then replace $\mathcal{F}$ by the subsequence of those $F \in \mathcal{F}$ that satisfy $\|\sigma(\beta_F) - u\| < \frac{1}{2M(8r)}$ and $\|\beta_F\| \geq M(8r)\|\sigma(\beta_H)\|$. The remaining sequence $\mathcal{F}$ is still a Følner sequence and for each $F \in \mathcal{F}$ we have $\|\sigma(\beta_F) - \sigma(\beta_H)\| < \frac{1}{M(8r)}$; hence $\beta_F \cdot \mathcal{T}$ is $8r$ -isomorphic to an integer multiple of $\beta_H \cdot \mathcal{T}$ by Lemma 2.5.

Applying again Lemma 2.7 with parameter $8r$ we obtain a new finite subsequence $\{Q_1, Q_2, \dots, Q_n\}$ of our modified $\mathcal{F}$. Let X be any $R_0(8)$ -approximation of $\text{Cay}(\Gamma, S)$. Then X is $8r$ -isomorphic to a linear combination of the Q_i , each Q_i is $8r$ -isomorphic to the corresponding $\beta_{Q_i} \cdot \mathcal{T}$, and in turn each $\beta_{Q_i} \cdot \mathcal{T}$ is $8r$ -isomorphic to an integer multiple of $\beta_H \cdot \mathcal{T}$. Applying Lemma 2.2 twice we obtain that X is $2r$ -isomorphic to an integer multiple of $\beta_H \cdot \mathcal{T}$. The lemma holds with $R_{\text{amenable}}(r) = R_0(8r)$. $\square$

3. AMALGAMATED PRODUCTS

Definition 3.1. Let α and β be partitions of a finite set S . The *incidence graph* of α and β is a bipartite graph, whose two sets of vertices consist of the classes of α and the classes of β , and the edges are the elements of S , where each element connects its α -class with its β -class.

Proposition 3.2 (proved in [4, Step 1]). *For each triple (a, b, r) of integers there is a finite set S with two partitions α and β on it such that each α -class has a elements, each β -class has b elements, an α -class can meet a β -class in at most one element, and in the incidence graph of α and β each simple cycle is longer than $2r$.*

Corollary 3.3. *Let α, β and S be as in Proposition 3.2. Suppose that the vertices of a graph are in S and the endpoints of each edge are either in the same α -class or in the same β -class. Then every simple cycle of length less than $2r$ is either entirely in a single α -class or entirely in a single β -class.*

Proof. This follows immediately from Definition 3.1 $\square$

Definition 3.4. Let A be a colored graph and Z a subset of the colors used in A . Then $A|_Z$ denotes the subgraph obtained from A by omitting all edges whose color does not belong to Z . Suppose that C is another colored graph and ϕ is an r -isomorphism between $A|_Z$ and $C|_Z$. We build another graph called the *enhancement of C with A along ϕ* , denoted by $A \xrightarrow{\phi} C$. We start from C and add new edges to it. Namely, for each edge $a \rightarrow b$ of A whose color does not belong to Z we add a new edge $\phi(a) \rightarrow \phi(b)$ of the same color, provided that ϕ is defined at the endpoints a and b .

Lemma 3.5. *Let Γ be a finitely generated group, X be a finite symmetric generating system of Γ and $Z \subseteq X$. Let A be a $2r$ -approximation of $\text{Cay}(\Gamma, X)$ such that every vertex has degree at most d . Then there is a bound $R_{\text{enh}}(r, d)$ with the following property. Suppose that C is another colored graph with color set Y such that $X \cap Y = Z$ and ϕ is an $R_{\text{enh}}(r, d)$ -isomorphism between $A|_Z$ and $C|_Z$. Then $(A \xrightarrow{\phi} C)|_X$ is r -isomorphic to A and $(A \xrightarrow{\phi} C)|_Y$ is isomorphic to C .*

Proof. This is clear from the definition. $\square$

Lemma 3.6. *Let G and H be finitely generated sofic groups, $\Gamma \leq G$ a finitely generated amenable subgroup and $\phi : \Gamma \hookrightarrow H$ an injective homomorphism. Then the amalgamated product $G *_\phi H$ is also sofic.*

Proof. For simplicity we identify both G and H with their canonical image in $G *_\phi H$, this will also identify both Γ and $\phi(\Gamma)$ with $G \cap H$. Let $X \subseteq G$, $Y \subseteq H$ and $Z \subseteq \Gamma$ be finite symmetric generating systems such that $Z = X \cap Y$. Let d be the maximum of $|X|$ and $|Y|$. The following is clear from the definition:

Claim 3.7. For each r there is a constant $R_{\text{prod}}(r)$ with the following property. Suppose we find a colored graph C with colors from $X \cup Y$ such that $C|_X$ is an $R_{\text{prod}}(r)$ -approximation of $\text{Cay}(G, X)$, $C|_Y$ is an $R_{\text{prod}}(r)$ -approximation of $\text{Cay}(H, Y)$, and every simple cycle of length at most $2r + 1$ has either only X -colored edges or only Y -colored edges. Then C is an r -approximation of the $\text{Cay}(G *_\phi H, X \cup Y)$.

We choose any integer $r > 0$ and set

$$\begin{aligned} r_1 &= R_{\text{prod}}(r), \\ r_2 &= R_{\text{appr}}(r_1, d), \\ r_3 &= R_{\text{enh}}(r_2, d), \\ r_4 &= R_{\text{amenable}}(r_3). \end{aligned}$$

Let us start with r_4 -approximations A , resp. B , of the Cayley graphs $\text{Cay}(G, X)$, resp. $\text{Cay}(H, Y)$. Then $A|_Z$ and $B|_Z$ are both r_4 -approximations of $\text{Cay}(\Gamma, Z)$. We choose a Følner sequence in $\text{Cay}(\Gamma, Z)$. Then Proposition 2.8 provides a colored graph D (denoted there by $\alpha \cdot \mathcal{T}$) such that both $A|_Z$ and $B|_Z$ are r_3 -isomorphic to integer multiples $a \cdot D$ and $b \cdot D$ (with appropriate integers $a, b > 0$).

Applying Proposition 3.2 for the triple $(a, b, r + 1)$ we obtain a finite set S and partitions α and β of S . Let S_0 denote the graph with no edge of vertex set S . Then the graph $C_0 = S_0 \times D$ is the union of $|S|$ disjoint copies of D , namely the subgraphs $\{s\} \times D$ for $s \in S$.

Now for each α -class $\sigma \subset S$ the subgraph spanned by the subset $\sigma \times D$ is r_3 -isomorphic to $A|_Z$, so we can enhance it with A along this r_3 -isomorphism. Repeating this enhancement for all α -classes we obtain a new graph C_1 on the same vertex set $S \times D$. According to Lemma 3.5, this C_1 is r_2 -isomorphic to $|S|/a \cdot A$, and $C_1|_Z$ is still isomorphic to $|S| \cdot D$.

Next for each β -class $\rho \subset S$ we consider the subgraph $C_1^\rho \subset C_1$ spanned by the subset $\rho \times D$. Then $C_1^\rho|_Z$ is r_3 -isomorphic to $B|_Z$, so we can enhance it with B . Repeating this enhancement for all β -classes we obtain a new graph C_2 on the same vertex set $S \times D$. According to Lemma 3.5, this $C_2|_X$ is r_2 -isomorphic to $|S|/a \cdot A$ and $C_2|_Y$ is r_2 -isomorphic to $|S|/b \cdot B$.

We know from Lemma 2.2 that $C_2|_X$ is an r_1 -approximation of $\text{Cay}(G, X)$ and $C_2|_Y$ is an r_1 -approximation of $\text{Cay}(H, Y)$. Let γ be a simple cycle of length at most $2r + 1$ of C_2 . Then by Lemma 3.3, γ lies entirely in a single α -class or it lies entirely in a single β -class. According to Claim 3.7, C_2 is an r -approximation of $\text{Cay}(G *_\phi H, X \cup Y)$. Hence the lemma follows. $\square$

Proof of the main theorem. Let G, H be arbitrary sofic groups, Γ be an amenable subgroup of G and let $\phi : \Gamma \rightarrow H$ be an injective homomorphism. Then the

amalgamated product $G \star_\phi H$ is the direct limit of amalgamated products $\{G_\alpha \star_\phi H_\alpha\}$ over amenable subgroups Γ_α , where G_α , H_α and Γ_α are all finitely generated groups. Since the direct limits of sofic groups are sofic, $G \star_\phi H$ is a sofic group. $\square$

4. HYPERFINITENESS

The notion of hyperfinite graph classes was introduced in [3]. A sofic approximation $\{G_n\}_{n=1}^\infty$ is hyperfinite if for any $\epsilon > 0$ there exists $K_\epsilon > 0$ such that for any $n \geq 1$ one can erase $\epsilon|E(G_n)|$ edges of G_n to obtain a graph G'_n having components of size not greater than K_ϵ .

Proposition 4.1. *Let $\{G_n\}_{n=1}^\infty$ be a sofic approximation of the Cayley graph of a finitely generated group Γ . Then $\{G_n\}_{n=1}^\infty$ is hyperfinite if and only if Γ is amenable.*

Proof. Let Γ be amenable. Then $\{G_n\}_{n=1}^\infty$ is hyperfinite by Proposition 2.8. So let us suppose that Γ is nonamenable and let S be a symmetric generating system of Γ . The isoperimetric constant of $\text{Cay}(\Gamma, S)$ is defined as

$$i(\text{Cay}(\Gamma, S)) := \inf_{A \subset \Gamma, |A| < \infty} \frac{|L(A)|}{|A|},$$

where $L(A)$ is the set of edges having one endpoint in A and another endpoint in A^c . It is well known that $i(\text{Cay}(\Gamma, S)) = c > 0$. Now let $\{G_n\}_{n=1}^\infty$ be a sofic approximation of $\text{Cay}(\Gamma, S)$ and suppose that $\{G_n\}_{n=1}^\infty$ is hyperfinite. Let $\{G'_n\}_{n=1}^\infty$ be a graph sequence having components of size at most K such that each G'_n is obtained by erasing $\frac{c}{10}|E(G_n)|$ edges from G_n . Observe that at least half of the vertices of G_n are in such a component A of G'_n such that

$$\frac{|L_{G_n}(A)|}{|A|} \leq \frac{c}{10}.$$

Notice that the K -neighborhood of such vertices cannot be isomorphic to the K -neighborhood of the unit element in $\text{Cay}(\Gamma, S)$. This is in contradiction with the assumption that $\{G_n\}_{n=1}^\infty$ is a sofic approximation. $\square$

5. THE PROOF OF THEOREM 2

Let $\psi, \phi : \Gamma \rightarrow \Sigma$ be conjugate faithful embeddings of the group Γ . Let $[\psi], [\phi] : \Gamma \rightarrow \prod_{n=1}^\infty S_n$ be representatives of the maps ψ , resp. ϕ . This gives us for each $n \geq 1$ the S -colored graphs G_n and H_n on the set $\{1, 2, \dots, n\}$.

By the conjugacy assumption, for any $r \geq 1$ there exist subsequences $\{G_{n_k}\}_{k=1}^\infty, \{H_{n_k}\}_{k=1}^\infty$ such that

- For any $k \geq 1$, G_{n_k} and H_{n_k} are r -isomorphic to each other.
- Both G_{n_k} and H_{n_k} are r -approximations of $\text{Cay}(\Gamma, S)$.

Now let $\{G_n\}_{n=1}^\infty$ be a sofic approximation of $\text{Cay}(\Gamma, S)$ such that $|V(G_n)| = n$ and Γ is nonamenable. By Proposition 4.1, we can suppose that there exists $\epsilon > 0$ and a function $r : \mathbb{N} \rightarrow \mathbb{N}$ tending to infinity such that if we erase $\epsilon|E(G_n)|$ edges from G_n , then at least one of the remaining components has size at least $r(n)$. Now consider the sofic approximation $\{H_n\}_{n=1}^\infty$, where H_n consists of $\lfloor \frac{n}{r(n)} \rfloor$ disjoint copies of $G_{r(n)}$ and $n - \lfloor \frac{n}{r(n)} \rfloor$ single vertices. Let ψ resp. $\phi : \Gamma \rightarrow \Sigma$ be the embeddings associated to the sofic approximations $\{G_n\}_{n=1}^\infty$ resp. $\{H_n\}_{n=1}^\infty$. If ψ and ϕ are conjugate, then by the previous arguments there exists $n_k > 0$ such that G_{n_k} and

H_{n_k} are $[\frac{1}{2\epsilon}]$ -isomorphic to each other. Therefore, one can erase $\epsilon|E(G_{n_k})|$ edges from G_{n_k} to obtain a graph having components of size at most $r(n_k)$, leading to a contradiction. Thus there exist $\phi, \psi : \Gamma \rightarrow \Sigma$ nonconjugate faithful embeddings provided that Γ is nonamenable.

Now let Γ be amenable and $\psi, \phi : \Gamma \rightarrow \Sigma$ be embeddings of Γ . Let $\{G_n\}_{n=1}^\infty$ and $\{H_n\}_{n=1}^\infty$ be the associated S -graphs (for representatives $[\psi]$ and $[\phi]$). For $n \geq 1$, let $h(n)$ be the largest integer for which G_n and H_n are both $h(n)$ -approximations of $\text{Cay}(\Gamma, S)$. Using the definition of the ultraproduct it is easy to see that for any $t \geq 1$,

$$A_t = \{n \mid h(n) \geq t\} \in \omega,$$

where ω is a nonprincipal ultrafilter. By Proposition 2.8, there exists a sofic approximation $\{J_n\}_{n=1}^\infty$ such that:

- $|V(J_n)| = n$.
- G_n and J_n , resp. H_n and J_n , are $2r$ -isomorphic whenever $h(n)$ is large enough.

Therefore G_n and H_n are r -isomorphic if $h(n)$ is large enough. Now we construct elements $g \in \prod_\omega S_n$ such that $g\psi g^{-1} = \phi$. For each $n \geq 1$, let $l(n)$ be the largest integer such that G_n and H_n are $l(n)$ -isomorphic. Let $g_n \in S_n$ be a bijection such that $g_n(G'_n) = H'_n$, where G'_n resp H'_n are isomorphic subgraphs obtained by erasing $\frac{1}{l_n}|E(G_n)|$ resp. $\frac{1}{l_n}|E(H_n)|$ edges from G_n resp. H_n . Then obviously $g\psi g^{-1} = \phi$.

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