A PSEUDOROTATION WITH QUADRATIC IRRATIONAL ROTATION NUMBER

MARK TURPIN

(Communicated by Bryna Kra)

ABSTRACT. We show, by explicit construction, that for any quadratic irrational number α , there exists a pseudorotation on an indecomposable cofrontier Λ with α as its rotation number. Our construction builds on a family of examples of Brechner, Guay, and Mayer. They observe that in the pseudorotations they construct, irrational numbers of constant type are not realized as rotation numbers. Circle rotations can realize any rotation number, but this is to our knowledge the first example of a pseudorotation with a quadratic irrational rotation number, and hence an irrational of constant type.

1. INTRODUCTION AND DEFINITIONS

We use the plane with the standard topology as the setting for the following mathematics.

A *cofrontier* Λ in the plane is a compact, connected set that irreducibly separates the plane into exactly two components and is the boundary of each. For example, any topological circle is a cofrontier. The cofrontiers we consider are *indecomposable*; that is, they cannot be written as the union (not necessarily disjoint) of two compact, connected subsets. The two planar components, bounded and unbounded, can be conformally mapped to the open disk, and its exterior, respectively.

Any homeomorphism f of the plane that leaves a cofrontier invariant will have on its bounded (or unbounded) component, a conjugate homeomorphism on the interior (or exterior) of the disk. These conjugacies are achieved via the above-mentioned conformal map. All of these (interior and exterior) conjugate homeomorphisms have unique extensions to homeomorphisms on their boundary circles, and in general these circle homeomorphisms may be different. A *pseudorotation* on a cofrontier Λ is a homeomorphism on Λ that extends continuously to the plane, and for which the induced circle homeomorphisms, as described above, are both conjugate to the same rotation. Although pseudorotations as defined in [BGM] include those whose cofrontiers are circles, our usage of the term pseudorotation will assume that the cofrontier is not a circle.

The *rotation number* α of a pseudorotation (or rotation) f (restricted to Λ) may be thought of as limiting average angular progress, a notion made precise through consideration of the lift of a homeomorphism to a covering space. A circle rotation

Received by the editors October 18, 2010 and, in revised form, October 30, 2010 and November 5, 2010.

2010 *Mathematics Subject Classification*. Primary 37-XX; Secondary 11-XX, 26-XX, 54-XX, 58-XX.

by angle $2\pi\alpha$ has rotation number α . A homeomorphism $f|_\Lambda$ has rotation number α if the limit $(n \rightarrow \infty) \Theta(f^n(x))/n = 2\pi\alpha$ for all $x \in \Lambda$, where here the Θ is informal, being precisely defined via the lifted homeomorphism.

In 1981 Handel [H] constructed a diffeomorphic pseudorotation whose cofrontier is the pseudocircle, a highly indecomposable continuum. Handel's example is both minimal on the cofrontier, area-preserving on the plane, and infinitely differentiable. In 1991 Brechner, Guay, and Mayer [BGM] considered a broad class of pseudorotations, and among their many results proved that none of their examples has a rotation number which is an irrational of constant type, the class of irrational numbers that are the least well approximated by rational numbers. In 1996 the author showed [T] that in Handel's original example the rotation number is very well approximated by rationals. A general question is what set of irrational numbers are realizable as rotation numbers of pseudorotations? (All real numbers are realizable as rotation numbers of circle homeomorphisms.) In particular, we consider Question 6.1 from [BGM].

Question 6.1 ([BGM]). Does there exist an indecomposable cofrontier Λ admitting an irrational pseudorotation for which the rotation number α is of constant type?

We show, by considering a construction outside the class considered in [BGM], that every quadratic irrational number (a subset of the irrationals of constant type) is realizable as the rotation number of a pseudorotation.

Theorem. *For any quadratic irrational number α there exists a pseudorotation on an indecomposable cofrontier Λ with α as its rotation number.*

2. ELEMENTS OF THE CONSTRUCTION

Our construction proceeds, up to a point, in the same manner as [BGM] from which we adopt our notation. We start with an annulus A_1 which we divide into an integer number, q_1 , of links. Having chosen a forward direction, we define on A_1 a rigid rotation g_1 which advances each link to its adjacent link. We let $g_1^{m_1}$ be a rotation which advances m_1 links, where m_1 may be any integer such that $-q_1 < m_1 < q_1$. We can extend g_1 to all of $\mathbb{R}^2$ in many ways. For our purposes any continuous extension is sufficient. See Figure 1.

Next we embed a second annulus A_2 in A_1 with the requirement that A_2 be invariant under $g_1^{m_1}$ for all m_1 . We require that A_2 be properly embedded in A_1 and that $A_1 - A_2$ be a disjoint union of two topological annuli. Further, we define g_2 on A_2 to be a "rotation" by one link within the annular structure of A_2 , again allowing $g_2^{m_2}$ as the possibility of advancing m_2 links in A_2 . We let $f_2 = g_2^{m_2} g_1^{m_1}$. See Figure 2.

Continuing inductively, given $A_1, A_2, \dots, A_{k-1}$ we embed A_k in A_{k-1} ($A_k \subset A_{k-1}$ where the inclusion is proper). Each annulus A_k is divided into q_k links on which we define a topological rotation g_k which carries links to adjacent links. The rotation $g_k^{m_k}$ advances m_k links within the annular structure of A_k . We define $f_k = g_k^{m_k} g_{k-1}^{m_{k-1}} \dots g_1^{m_1}$ to be the composition of the first k rotations and note that A_k is invariant under f_k (and f_{k-1}, f_{k-2} , etc.). We also note that f_k can be extended continuously to all of $\mathbb{R}^2$. For a proof of the existence of this extension, see [BGM].

Both the interior and exterior circle homeomorphisms induced by f_k are rigid rotations with rotation number $\alpha_k = \sum_{i=1}^k \frac{m_i}{q_i}$.

FIGURE 1. An A_1 with 6 links, so $q_1 = 6$. f_1 rotates by 2 links, so $m_1 = 2$.

FIGURE 2. A_1 has 6 links ($q_1 = 6$), A_2 has 54 links ($q_2 = 54$), f_1 rotates by 2 links ($m_1 = 2$) within A_1 , and f_2 rotates by 8 links ($m_2 = 8$) within A_2 .

We must show that the manner in which we embed these annuli guarantees that the cofrontier formed by the infinite intersection $\Lambda = \bigcap_{k=1}^{\infty} A_k$ is an indecomposable continuum.

We must also show that $\{f_k\}$ converges to a continuous function f on $\mathbb{R}^2$, invariant on Λ , with rotation number α :

$$\alpha = \lim_{k \rightarrow \infty} \alpha_k = \sum_{k=1}^{\infty} \frac{m_k}{q_k}.$$

3. THE ROTATION NUMBER α

Let α be any quadratic irrational number. The irrationality of α implies that there exist infinitely many rational numbers $\frac{b}{c}$ such that

$$\left| \alpha - \frac{b}{c} \right| \leq \frac{1}{c^2}.$$

Among these rational numbers are all the convergents of α , which are the rational numbers from each stage of the continued fraction expansion of α . We denote these convergents by $\frac{b_i}{c_i}$. It is known that an irrational number is quadratic if and only if the terms of its (simple) continued fraction expansion are (eventually) periodic:

$$\alpha = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}} = [a_1, a_2, a_3, \dots],$$

where $\exists p, K \geq 1$ such that $\forall i \geq K, a_i = a_{i+p}$.

We examine the increasing sequence of denominators, $\{c_i\}$. We show that there exists a subsequence of denominators $\{c_{i_k}\}$ with the property $c_{i_k}/c_{i_{k-1}} \in \mathbb{Z}$.

Given α , let c_{i_1} be any denominator of any convergent of α with $i_1 > K$. Let $q_1 = c_{i_1}$. Now we wish to find c_{i_2} which satisfies $c_{i_2}/c_{i_1} \in \mathbb{Z}$. We use the recurrence relation

$$(1) \quad c_{i+1} = a_{i+1}c_i + c_{i-1},$$

which determines the denominators of the convergents from the terms of the continued fraction expansion [N].

To find c_{i_2} we use a subsequence of $\{a_i\}$, namely $\{a_{i_1+jp+1}\}, j \in \mathbb{Z}^+$. Because p is the period of the sequence $\{a_i\}$, all terms in this subsequence are equal.

Consider the following triples of numbers:

$$\{(a_{i_1+jp+1}, c_{i_1+jp}(\bmod c_{i_1}), c_{i_1+jp-1}(\bmod c_{i_1}))\}.$$

The a_{i_1+jp+1} are identical, ranging over j , and the other two terms are bounded by c_{i_1} . So for some value of j we will see an ordered triple that is an exact repetition of an earlier triple. That is, $\exists j_1$ and j_2 ($j_2 > j_1$) such that

$$\begin{aligned} & (a_{i_1+j_1p+1}, c_{i_1+j_1p}(\bmod c_{i_1}), c_{i_1+j_1p-1}(\bmod c_{i_1})) \\ &= (a_{i_1+j_2p+1}, c_{i_1+j_2p}(\bmod c_{i_1}), c_{i_1+j_2p-1}(\bmod c_{i_1})). \end{aligned}$$

The equality of these terms and equation (1) implies that

$$c_{i_1+j_1p+1} \equiv c_{i_1+j_2p+1}(\bmod c_{i_1}).$$

By rearranging terms of (1), we have

$$c_{i-1} = a_{i+1}c_i - c_{i+1},$$

FIGURE 3. 9 links of A_{n+1} form a basic repetitive unit within A_n .

which holds when equality is replaced with equality mod c_i . Hence

$$\begin{aligned} c_{i_1+j_1p-2} &\equiv c_{i_1+j_2p-2} \pmod{c_{i_1}}, \\ c_{i_1+j_1p-3} &\equiv c_{i_1+j_2p-3} \pmod{c_{i_1}}, \\ &\dots \\ c_{i_1} &\equiv c_{i_1+(j_2-j_1)p} \pmod{c_{i_1}}. \end{aligned}$$

Similarly we can show

$$c_{i_1} \equiv c_{i_1+(j_2-j_1)p} \equiv c_{i_1+2(j_2-j_1)p} \equiv c_{i_1+3(j_2-j_1)p} \equiv \dots$$

This gives us an unbounded sequence of denominators of convergents of α that are all multiples of c_{i_1} , and we will choose one to be c_{i_2} . We choose c_{i_2} sufficiently large so that

$$(2) \quad n_2 = \frac{c_{i_2}}{c_{i_1}} \geq 4c_{i_1}$$

and we define $q_2 = c_{i_2}$. So $q_1 = c_{i_1}$ and $q_k = c_{i_k}$. Let $n_{k+1} = q_{k+1}/q_k \geq 4q_k$. We can say that n_{k+1} is the number of links in each basic repetitive unit of A_{k+1} . (A *basic repetitive unit* B of A_n is the smallest connected set of links of A_n for which the union over j of $g_{n-1}^j(B) = A_n$.) (See Figure 3.)

Because the fractions b_i/c_i are convergents of α , we can say

$$\alpha = \frac{b_{i_1}}{c_{i_1}} + \left(\frac{b_{i_2}}{c_{i_2}} - \frac{b_{i_1}}{c_{i_1}} \right) + \left(\frac{b_{i_3}}{c_{i_3}} - \frac{b_{i_2}}{c_{i_2}} \right) + \dots,$$

and each term in the above sequence can be written as

$$\frac{b_{i_{k+1}}}{c_{i_{k+1}}} - \frac{b_{i_k}}{c_{i_k}} = \frac{b_{i_{k+1}} - n_{k+1}b_{i_k}}{c_{i_{k+1}}}.$$

Letting $m_{k+1} = b_{i_{k+1}} - n_{k+1}b_{i_k}$ and remembering $q_{k+1} = c_{i_{k+1}}$, we can write

$$\alpha = \frac{m_1}{q_1} + \frac{m_2}{q_2} + \frac{m_3}{q_3} + \dots$$

Each finite sum is a convergent of α (but many convergents are excluded). Note that m_k could be negative.

4. PROPERTIES OF THE CONSTRUCTION

We need to show that the m_k and q_k as defined satisfy two important properties:

- (1) The cofrontier Λ , the intersection of the A_i , is indecomposable.
- (2) The limit of the functions f_k is a function.

We define r_2 to be the unique integer for which

$$2 \leq r_2 \leq \frac{n_2}{2q_1} < r_2 + 1$$

and similarly define r_{k+1} as

$$(3) \quad 2 \leq r_{k+1} \leq \frac{n_{k+1}}{2q_k} < r_{k+1} + 1.$$

We require that each segment of A_{k+1} that spans (crosses) a link of A_k consists of at least r_{k+1} links. Because each basic repetitive unit of A_{k+1} has at least twice as many links as A_k , each basic repetitive unit of A_{k+1} can span all of A_k . This condition is sufficient [CI] for the indecomposability of the cofrontier Λ .

This condition also allows us to say that the diameter of each link of A_{k+1} is less than half of the diameter of each link of A_k .

The irrationality of α gives the following for any convergent: $|\alpha - \frac{b_{i_k}}{c_{i_k}}| < \frac{1}{c_{i_k}^2}$.

FIGURE 4. Convergents of alpha

We use this to say

$$\left| \frac{m_{k+1}}{q_{k+1}} \right| = \left| \frac{b_{i_{k+1}}}{c_{i_{k+1}}} - \frac{b_{i_k}}{c_{i_k}} \right| \leq \left| \frac{b_{i_{k+1}}}{c_{i_{k+1}}} - \alpha \right| + \left| \alpha - \frac{b_{i_k}}{c_{i_k}} \right| < \frac{2}{c_{i_k}^2} = \frac{2}{q_k^2}.$$

This implies that

$$|m_{k+1}| < q_{k+1} \frac{2}{q_k^2} = n_{k+1} \frac{2}{q_k}.$$

Combining with (2) and (3) from above we have

$$\frac{|m_{k+1}|}{r_{k+1}} < n_{k+1} \frac{2}{q_k} \frac{1}{r_{k+1}} < (r_{k+1} + 1) \frac{4}{r_{k+1}} = \left(1 + \frac{1}{r_{k+1}}\right) 4 < 6.$$

So $g_{k+1}^{m_{k+1}}$ moves each link of A_{k+1} through at most 6 links of A_k . The size of each link in A_k is bounded by $(1/2)^k$, so the total distance that the composition f_k “rotates” points is summable, and therefore f_k converges to a continuous function f on Λ with a continuous extension to all of $\mathbb{R}^2$.

5. QUESTIONS

Question 1. Given any irrational number α , does there exist a pseudorotation on an indecomposable cofrontier with rotation number α ? In particular, are there irrational pseudorotations for irrationals of constant type that are not quadratic?

Question 2. Given a particular indecomposable cofrontier Λ , is there a way to characterize the set of rotation numbers of possible pseudorotations?

REFERENCES

- [BGM] B. L. Brechner, M. D. Guay, and J. C. Mayer, *The rotational dynamics of cofrontiers*, Continuum Theory and Dynamical Systems (M. Brown, ed.), Contemporary Mathematics, vol. 117, American Math. Soc., 1991. MR1112801
- [CI] H. Cook and W. T. Ingram, *A characterization of indecomposable compact continua*, Proceedings of Topology Conference (held at Arizona State University, Tempe, AZ), Arizona State Univ., 1967. MR0253298 (40:6513)
- [H] M. Handel, *A pathological area-preserving diffeomorphism of the plane*, Proc. Amer. Math. Soc., 86, 1982. MR663889 (84f:58040)
- [N] I. Niven, *Irrational Numbers*, The Carus Mathematical Monographs, no. 11, The Mathematical Association of America, distributed by John Wiley and Sons, Inc., 1956. MR0080123 (18:195c)
- [T] M. Turpin, *The restricted rotation number of an example of Handel*, Dynamical Systems and Differential Equations, Vol. II (Springfield, MO, 1996), Discrete and Continuous Dynamical Systems, 1998. MR1722474 (2000g:37046)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HARTFORD, WEST HARTFORD, CONNECTICUT 06117