ALMOST PERIODIC SOLUTIONS TO SOME SECOND-ORDER NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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(Communicated by Yingfei Yi)

In memory of my friend and colleague Cora Sadosky

ABSTRACT. This paper is concerned with the existence of almost periodic mild solutions to some second-order equations. Using dichotomy tools and the Schauder fixed point theorem, the existence of almost periodic mild solutions to those second-order evolution equations is established. To illustrate our abstract results, the existence of almost periodic solutions to a damped second-order boundary value problem is also discussed.

1. INTRODUCTION

Let $\mathbb{H}$ be a separable infinite dimensional complex Hilbert space. Our main objective in this paper is to study the existence of almost periodic solutions to the classes of nonautonomous second-order differential equations

$$(1.1) \quad \frac{d^2 u}{dt^2} + a(t) \frac{du}{dt} + b(t)Au = h(t, u),$$

where $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ is a self-adjoint linear operator on $\mathbb{H}$ whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_l \rightarrow \infty$ as $l \rightarrow \infty$ with each eigenvalue having a finite multiplicity γ_j equal to the multiplicity of the corresponding eigenspace, $a, b : \mathbb{R} \mapsto \mathbb{C}$ are almost periodic and $h : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$, $(t, u) \rightarrow h(t, u)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{H}$. For that, we study the existence of almost periodic solutions to the more general second-order differential equations

$$(1.2) \quad \frac{d}{dt} \left[\frac{du}{dt} + g(t, u(t)) \right] = w(t)Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ is the same as in Eq. (1.1), the function $w : \mathbb{R} \mapsto \mathbb{C}$ given by $w(t) = \rho(t)e^{i\theta(t)}$ for all $t \in \mathbb{R}$ is almost periodic and there exist the constants $\rho_0, \rho_1 > 0$ and $\theta_0 \in (0, \frac{\pi}{4})$ such that $\rho_0 \leq \rho(t) \leq \rho_1$, and

$$(1.3) \quad \pi + 2\theta_0 < \theta(t) < 2\pi - 2\theta_0 \quad \text{for all } t \in \mathbb{R},$$

and the functions $f, g : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ are almost periodic in the first variable uniformly in the second variable.

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Our strategy consists of rewriting Eq. (1.2) as a nonautonomous first-order differential equation on $\mathbb{H} \times \mathbb{H}$ and then applying the Schauder fixed point to it with the help of dichotomy tools. Indeed, assuming that u is twice differentiable and setting $z := \begin{pmatrix} u, u' + g(t, u) \end{pmatrix}^T$, where T denotes the transpose, then Eq. (1.2) can be rewritten on $\mathbb{X} := \mathbb{H} \times \mathbb{H}$ in the following form:

$$(1.4) \quad \frac{dz}{dt} = \mathcal{A}(t)z + F(t, z(t)), \quad t \in \mathbb{R},$$

where $\mathcal{A}(t)$ is the family of 2×2 -operator matrices defined by

$$(1.5) \quad \mathcal{A}(t) = \begin{pmatrix} 0 & I_{\mathbb{H}} \\ w(t)A & 0 \end{pmatrix}, \quad t \in \mathbb{R},$$

where $I_{\mathbb{H}}$ is the identity operator of $\mathbb{H}$.

Clearly, the domain $D(\mathcal{A}(t)) = D(A) \times \mathbb{H}$ is constant in $t \in \mathbb{R}$.

Notice that the term F appearing in Eq. (1.4) is defined on $\mathbb{R} \times \mathbb{X}_{\alpha}$ for some $\alpha \in (0, 1)$ by $F(t, z(t)) = (-g(t, u), f(t, u))^T$, where $\mathbb{X}_{\alpha}$ is the real interpolation space of order (α, ∞) between $\mathbb{X}$ and $D(\mathcal{A}(t))$ given by $\mathbb{X}_{\alpha} = (\mathbb{X}, D(\mathcal{A}(t)))_{\alpha, \infty} = \mathbb{H}_{\alpha} \times \mathbb{H}$, with $\mathbb{H}_{\alpha} := (\mathbb{H}, D(A))_{\alpha, \infty}$.

Now, assuming that $a : \mathbb{R} \mapsto \mathbb{C}$ is differentiable, one can easily see that Eq. (1.1) is a special case of Eq. (1.2). Indeed, Eq. (1.1) can be rewritten as

$$(1.6) \quad \frac{d}{dt} \left[\frac{du}{dt} + \tilde{g}(t, u(t)) \right] = \tilde{w}(t)Au(t) + \tilde{f}(t, u(t)), \quad t \in \mathbb{R},$$

where $\tilde{w}(t) = -b(t)$, $\tilde{g}(t, u) = a(t)u$, and $\tilde{f}(t, u) = h(t, u) + ua'(t)$ for all $t \in \mathbb{R}$.

Let us mention that particular cases of Eq. (1.2) have been considered in the literature; see, e.g., [1, 2, 3, 11, 12, 13]. Recently, the existence of solutions to Eq. (1.2) when $w(t) = 1$ for all $t \in \mathbb{R}$ was studied and obtained in [8]. However, to the best of our knowledge, the original problem which consists of the existence of almost periodic mild solutions to both Eq. (1.2) and Eq. (1.1) remains an untreated question, which constitutes the main motivation of this paper. We essentially make extensive use of ideas of [4, 6, 7, 9] and dichotomy tools, as well as the Schauder fixed point theorem to obtain our existence results.

2. PRELIMINARIES

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{X}$, then $D(L)$, $\rho(L)$, and $\sigma(L)$ stand respectively for the domain, resolvent, and spectrum of L . Moreover, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(L)$. Furthermore, we set $Q = I - P$ for a projection P . We denote the Banach algebra of bounded linear operators on $\mathbb{X}$ by $B(\mathbb{X})$.

Let $B(\mathbb{R}, \mathbb{X})$ stand for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathbb{X}$ when equipped with the sup norm $\|\varphi\|_{\infty} := \sup_{t \in \mathbb{R}} \|\varphi(t)\|$ for $\varphi \in BC(\mathbb{R}, \mathbb{X})$. Similarly, $B(\mathbb{R}, \mathbb{X}_{\alpha})$ for $\alpha \in (0, 1)$ will stand for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathbb{X}_{\alpha}$ when equipped with the α -sup norm $\|\varphi\|_{\alpha, \infty} := \sup_{t \in \mathbb{R}} \|\varphi(t)\|_{\alpha}$ for $\varphi \in BC(\mathbb{R}, \mathbb{X}_{\alpha})$.

2.1. Almost periodic functions.

Definition 2.1. A continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t + \tau) - f(t)\| < \varepsilon \quad \text{for each } t \in \mathbb{R}.$$

The number τ above is called an ε -translation number of f , and the collection of all such functions will be denoted $AP(\mathbb{X})$.

Definition 2.2. A continuous function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $x \in K$ where $K \subset \mathbb{X}$ is any compact subset if for each $\varepsilon > 0$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|F(t + \tau, y) - F(t, y)\| < \varepsilon \quad \text{for each } t \in \mathbb{R}, y \in K.$$

The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{X})$.

The proof of our main result requires the following composition theorem.

Theorem 2.3. Let $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ be an almost periodic function. Suppose $F(t, u)$ is uniformly continuous on every bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $g \in AP(\mathbb{X})$, then $\Gamma : \mathbb{R} \rightarrow \mathbb{X}$ defined by $\Gamma(\cdot) := F(\cdot, g(\cdot))$ belongs to $AP(\mathbb{X})$.

2.2. Evolution families. The setting of this subsection and that on estimates of $U(t, s)$ are similar to that of Diagana [6].

Definition 2.4 (Acquistapace-Terreni conditions [4]). A family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfies the so-called Acquistapace-Terreni conditions. That is, there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, $K, L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$(2.1) \quad S_\theta \cup \{0\} \subset \rho(A(t) - \omega) \ni \lambda, \quad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|}$$

and

$$(2.2) \quad \|(A(t) - \omega)R(\lambda, A(t) - \omega)[R(\omega, A(t)) - R(\omega, A(s))]\| \leq L |t - s|^\mu |\lambda|^{-\nu}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

Among other things, the Acquistapace-Terreni conditions ensure that there exists a unique evolution family

$$\mathcal{U} = \{U(t, s) : t, s \in \mathbb{R} \text{ such that } t \geq s\}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t, s)\mathbb{X} \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and

- (a) $U(t, s)U(s, r) = U(t, r)$ for $t, s \in \mathbb{R}$ such that $t \geq s \geq r$;
- (b) $U(t, t) = I$ for $t \in \mathbb{R}$, where I is the identity operator of $\mathbb{X}$;
- (c) $(t, s) \mapsto U(t, s) \in B(\mathbb{X})$ is continuous for $t > s$.

In that case we say that $A(\cdot)$ generates the evolution family $U(\cdot, \cdot)$.

Definition 2.5 ([4]). One says that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

$$(d) \quad U(t, s)P(s) = P(t)U(t, s);$$

- (e) the restriction $U_Q(t, s) : Q(s)\mathbb{X} \rightarrow Q(t)\mathbb{X}$ of $U(t, s)$ is invertible (we then set $\tilde{U}_Q(s, t) := U_Q(t, s)^{-1}$); and
- (f) $\|U(t, s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|\tilde{U}_Q(s, t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

2.3. Estimates for $U(t, s)$. Let A be a sectorial operator on $\mathbb{X}$ (for that, in Eq. (2.1), replace $A(t)$ with A) and let $\alpha \in (0, 1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{ x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha(A - \omega)R(r, A - \omega)x\| < \infty \right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write $\mathbb{X}_0^A := \mathbb{X}$, $\|x\|_0^A := \|x\|$, $\mathbb{X}_1^A := D(A)$ and $\|x\|_1^A := \|(\omega - A)x\|$. Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we have the following continuous embedding:

$$(2.3) \quad D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \hookrightarrow \mathbb{X},$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection $\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A}$ for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set $\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}$ and $\hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$ for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms.

The proofs of the main results of this paper require the following assumptions:

- (H.1) The linear operators $\{A(t)\}_{t \in \mathbb{R}}$ satisfy the Acquistapace-Terreni conditions with constants as in Definition 2.4.
- (H.2) The evolution family $\mathcal{U}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$. Moreover, $0 \in \rho(A(t))$ for each $t \in \mathbb{R}$ and the following holds:

$$(2.4) \quad \sup_{t, s \in \mathbb{R}} \|A(s)A^{-1}(t)\|_{B(\mathbb{X}, \mathbb{X}_\alpha)} < c_0.$$

- (H.3) There exists $0 \leq \alpha < \beta < 1$ such that

$$\mathbb{X}_\alpha^t = \mathbb{X}_\alpha \quad \text{and} \quad \mathbb{X}_\beta^t = \mathbb{X}_\beta$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.

We have the following fundamental estimates for the evolution family $U(t, s)$.

Proposition 2.6 ([4]). *Suppose the evolution family $U = U(t, s)$ has exponential dichotomy. For $x \in \mathbb{X}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:*

- (i) *There is a constant $c(\alpha)$ such that*

$$(2.5) \quad \|U(t, s)P(s)x\|_\alpha^t \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|.$$

- (ii) *There is a constant $m(\alpha)$ such that*

$$(2.6) \quad \left\| \tilde{U}_Q(s, t)Q(t)x \right\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|.$$

3. PRELIMINARY RESULTS

Consider the nonautonomous differential equation

$$(3.1) \quad \frac{du}{dt} = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R},$$

where $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is jointly continuous.

Definition 3.1. Under assumption (H.1), a function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a mild solution to Eq. (3.1) provided that

$$(3.2) \quad u(t) = U(t, s)u(s) + \int_s^t U(t, \tau)F(\tau, u(\tau))d\tau$$

for each $\forall t \geq s, t, s \in \mathbb{R}$.

It is not difficult to see that if the function $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is bounded, then

$$(3.3) \quad u(t) = \int_{-\infty}^t U(t, s)P(s)F(s, u(s))ds - \int_t^\infty U_Q(t, s)Q(s)F(s, u(s))ds,$$

for all $t \in \mathbb{R}$, is a mild solution to Eq. (3.1).

Throughout the rest of the paper we fix α, β , real numbers satisfying $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$.

To study the existence of almost periodic solutions to Eq. (3.1), in addition to the previous assumptions, we suppose that the injection

$$\mathbb{X}_\alpha \hookrightarrow \mathbb{X}$$

is compact and that the following additional assumptions hold:

(H.4) $R(\omega, A(\cdot)) \in AP(B(\mathbb{X}, \mathbb{X}_\alpha))$.

(H.5) (a) The function $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is almost periodic in the first variable and uniformly in the second one. The function $u \mapsto F(t, u)$ is uniformly continuous on any bounded subset K of $\mathbb{X}_\alpha$ for each $t \in \mathbb{R}$. Finally,

$$\|F(t, u)\|_\infty \leq \mathcal{M}(\|u\|_{\alpha, \infty}),$$

where $\mathcal{M} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r \rightarrow \infty} \frac{\mathcal{M}(r)}{r} = 0.$$

We denote by $S = S_1 - S_2$ the nonlinear integral operators S_1 and S_2 defined by

$$(S_1 u)(t) := \int_{-\infty}^t U(t, s)P(s)F(s, u(s))ds$$

and

$$(S_2 u)(t) := \int_t^\infty U_Q(t, s)Q(s)F(s, u(s))ds.$$

The proof of the main result of the present paper requires the next lemma.

Lemma 3.2 ([6, Diagona]). *For each $x \in \mathbb{X}$, suppose that assumptions (H.1)-(H.2) hold and let α, β be real numbers such that $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$. Then there are two constants $r'(\alpha, \beta), d'(\beta) > 0$ such that*

$$(3.4) \quad \|A(t)U(t, s)P(s)x\|_\beta \leq r'(\alpha, \beta)e^{-\frac{\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|, \quad t > s,$$

and

$$(3.5) \quad \|A(t)\tilde{U}_Q(t, s)Q(s)x\|_\beta \leq d'(\beta)e^{-\delta(s-t)}\|x\|, \quad t \leq s.$$

Lemma 3.3 ([6, Diagana]). *Under assumptions (H.1)-(H.3), the mapping $S : BC(\mathbb{R}, \mathbb{X}_\alpha) \mapsto BC(\mathbb{R}, \mathbb{X}_\alpha)$ is well-defined and continuous.*

Lemma 3.4 ([6, Diagana]). *Under assumptions (H.1)-(H.5), $S = S_1 - S_2$ maps bounded sets of $BC(\mathbb{R}, \mathbb{X}_\alpha)$ into bounded sets of $BC^\gamma(\mathbb{R}, \mathbb{X}_\alpha)$ for some $0 < \gamma < 1$, where S_1, S_2 are the integral operators introduced previously.*

The proof of the next lemma follows along the same lines as that of Lemma 3.4 and is hence omitted.

Lemma 3.5. *The integral operator $S = S_1 - S_2$ maps bounded sets of $AP(\mathbb{X}_\alpha)$ into bounded sets of $BC^{1-\beta}(\mathbb{R}, \mathbb{X}_\alpha) \cap AP(\mathbb{X}_\alpha)$.*

Similarly, the next lemma is a consequence of [7, Proposition 3.3].

Lemma 3.6. *The set $BC^{1-\beta}(\mathbb{R}, \mathbb{X}_\alpha)$ is compactly contained in $BC(\mathbb{R}, \mathbb{X})$; that is, the canonical injection $id : BC^{1-\beta}(\mathbb{R}, \mathbb{X}_\alpha) \mapsto BC(\mathbb{R}, \mathbb{X})$ is compact, which yields that*

$$id : BC^{1-\beta}(\mathbb{R}, \mathbb{X}_\alpha) \cap AP(\mathbb{X}_\alpha) \mapsto AP(\mathbb{X}_\alpha)$$

is compact, too.

Theorem 3.7. *Suppose assumptions (H.1)-(H.5) hold. Then Eq. (3.1) has an almost periodic mild solution.*

Proof. The proof follows along the same lines as that of [6, Theorem 3.8]. For the sake of clarity, we reproduce it here. Let us recall that in view of Lemma 3.4, we have

$$\|Su\|_{\alpha, \infty} \leq p(\alpha, \beta, \delta)\mathcal{M}(\|u\|_{\alpha, \infty})$$

and

$$\|Su(t_2) - Su(t_1)\|_\alpha \leq s(\alpha, \beta, \delta)\mathcal{M}(\|u\|_{\alpha, \infty})|t_2 - t_1|$$

for all $u \in BC(\mathbb{R}, \mathbb{X}_\alpha)$, $t_1, t_2 \in \mathbb{R}$ with $t_1 \neq t_2$, where $p(\alpha, \beta, \delta)$ and $s(\alpha, \beta, \delta)$ are positive constants. Consequently, $u \in BC(\mathbb{R}, \mathbb{X}_\alpha)$ and $\|u\|_{\alpha, \infty} < R$ yield $Su \in BC^{1-\beta}(\mathbb{R}, \mathbb{X}_\alpha)$ and $\|Su\|_\alpha < R_1$, where $R_1 = c(\alpha, \beta, \delta)\mathcal{M}(R)$.

Therefore, there exists $r > 0$ such that for all $R \geq r$, the following holds:

$$(3.6) \quad S\left(B_{AP(\mathbb{X}_\alpha)}(0, R)\right) \subset B_{BC^{1-\beta}(\mathbb{R}, \mathbb{X}_\alpha)}(0, R) \cap B_{AP(\mathbb{X}_\alpha)}(0, R).$$

In view of the above, it follows that $S : D \mapsto D$ is continuous and compact, where D is the ball in $AP(\mathbb{X}_\alpha)$ of radius R with $R \geq r$. Using the Schauder fixed point theorem it follows that S has a fixed point, which obviously is an almost periodic mild solution to Eq. (3.1). $\square$

4. ALMOST PERIODIC SOLUTIONS TO EQ. (1.2)

Let $\mathbb{H}$ be an infinite dimensional separable complex Hilbert space equipped with the norm $\|\cdot\|$. Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j \geq 1}$. Therefore, for each

$$u \in D(A) := \{u \in \mathbb{H} : \sum_{j=1}^{\infty} \lambda_j^2 \|E_j u\|^2 < \infty\},$$

then

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u,$$

where $E_j u = \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k$.

In this section, we take $\mathbb{X} = \mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$ equipped with its natural topology. To study the existence of almost periodic solutions to Eq. (1.2), in addition to the previous assumptions, we suppose that the injection

$$\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$$

is compact and that the following additional assumption holds:

(H.6) The function $f, g : \mathbb{R} \times \mathbb{H}_\alpha \mapsto \mathbb{H}$ is almost periodic in the first variable and uniformly in the second one. The functions $u \mapsto f(t, u)$ and $u \mapsto g(t, u)$ are uniformly continuous on any bounded subset K of $\mathbb{H}_\alpha$ for each $t \in \mathbb{R}$. Finally,

$$\left(\|f(t, u)\|_\infty^2 + \|g(t, u)\|_\infty^2 \right)^{1/2} \leq \mathcal{M}(\|u\|_{\alpha, \infty}),$$

for all $u \in \mathbb{H}_\alpha$, where $\mathcal{M} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r \rightarrow \infty} \frac{\mathcal{M}(r)}{r} = 0.$$

Theorem 4.1. *Under previous assumptions and if (H.6) holds, then Eq. (1.2) has at least one almost periodic solution $u \in \mathbb{H}_\alpha$.*

Proof. For all $z := \begin{pmatrix} u \\ v \end{pmatrix} \in D = D(\mathcal{A}(t)) = D(A) \times \mathbb{H}$, we obtain the following:

$$\mathcal{A}(t)z = \sum_{n=1}^{\infty} \mathcal{A}_n(t) P_n z,$$

where $I_{\mathbb{H}}$ is the identity operator of $\mathbb{H}$,

$$P_n := \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \quad n \geq 1,$$

and

$$(4.1) \quad \mathcal{A}_n(t) := \begin{pmatrix} 0 & 1 \\ w(t)\lambda_n & 0 \end{pmatrix}, \quad n \geq 1, \quad t \in \mathbb{R}.$$

Now, the characteristic equation for $\mathcal{A}_n(t)$ is given by

$$(4.2) \quad \lambda^2 - \lambda_n w(t) = \lambda^2 - \lambda_n \rho(t) e^{i\theta(t)} = 0,$$

from which we obtain its eigenvalues given by

$$\lambda_1^n(t) = \sqrt{\lambda_n \rho(t)} e^{i \frac{\theta(t)}{2}} \quad \text{and} \quad \lambda_2^n(t) = \sqrt{\lambda_n \rho(t)} e^{-i \frac{\theta(t)}{2}},$$

and hence $\sigma(\mathcal{A}_n(t)) = \{\lambda_1^n(t), \lambda_2^n(t)\}$.

Noticing that $\mathcal{A}(t)$ is invertible for all $t \in \mathbb{R}$ and using Eq. (1.3), it follows that there exists $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that $S_\theta \cup \{0\} \subset \rho(\mathcal{A}(t))$.

On the other hand, since λ_1^n and λ_2^n are distinct, $\mathcal{A}_n(t)$ is diagonalizable. Further, it is not difficult to see that $\mathcal{A}_n(t) = K_n^{-1}(t)J_n(t)K_n(t)$, where $J_n(t)$, $K_n(t)$ and $K_n^{-1}(t)$ are respectively given by

$$J_n(t) = \begin{pmatrix} \lambda_1^n(t) & 0 \\ 0 & \lambda_2^n(t) \end{pmatrix}, \quad K_n(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1^n(t) & \lambda_2^n(t) \end{pmatrix},$$

and

$$K_n^{-1}(t) = \frac{1}{\lambda_1^n(t) - \lambda_2^n(t)} \begin{pmatrix} -\lambda_2^n(t) & 1 \\ \lambda_1^n(t) & -1 \end{pmatrix}.$$

For $\lambda \in S_\theta$ and $z \in \mathbb{X}$, one has

$$\begin{aligned} R(\lambda, \mathcal{A}(t))z &= \sum_{n=1}^{\infty} (\lambda - \mathcal{A}_n(t))^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n(t) (\lambda - J_n(t))^{-1} K_n^{-1}(t) P_n z. \end{aligned}$$

It can be shown that

$$\|R(\lambda, \mathcal{A}(t))\| \leq \frac{K}{1 + |\lambda|}$$

for all $\lambda \in S_\theta$ and $t \in \mathbb{R}$.

Now, for every $t \in \mathbb{R}$, the family of linear operators $\mathcal{A}(t)$ generate an analytic semigroup $(e^{\tau \mathcal{A}(t)})_{\tau \geq 0}$ on $\mathbb{X}$ given by

$$e^{\tau \mathcal{A}(t)} z = \sum_{n=0}^{\infty} K_n(t)^{-1} P_n e^{\tau J_n} P_n K_n(t) P_n z, \quad z \in \mathbb{X}.$$

On the other hand, we have

$$\|e^{\tau \mathcal{A}(t)} z\| = \sum_{n=0}^{\infty} \|K_n(t)^{-1} P_n\| \|e^{\tau J_n} P_n\| \|K_n(t) P_n\| \|P_n z\|,$$

with for each $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$\begin{aligned} \|e^{\tau J_n} P_n z\|^2 &= \left\| \begin{pmatrix} e^{\lambda_1^n(t)\tau} E_n & 0 \\ 0 & e^{\lambda_2^n(t)\tau} E_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 \\ &\leq \|e^{\lambda_1^n(t)\tau} E_n z_1\|^2 + \|e^{\lambda_2^n(t)\tau} E_n z_2\|^2 \\ &\leq e^{2\Re(\lambda_1^n(t))\tau} \|z\|^2. \end{aligned}$$

Clearly, using Eq. (1.3) it follows that

$$\begin{aligned}\Re(\lambda_1^n(t)) &= \sqrt{\lambda_n \rho(t)} \cos\left(\frac{\theta(t)}{2}\right) \\ &\leq \sqrt{\lambda_1 \rho_0} \cos\left(\frac{\theta(t)}{2}\right) \\ &\leq \sqrt{\lambda_1 \rho_0} \cos\left(\frac{\pi}{2} + \theta_0\right) \\ &= -\sqrt{\lambda_1 \rho_0} \sin(\theta_0).\end{aligned}$$

Setting $\delta = \sqrt{\lambda_1 \rho_0} \sin(\theta_0) > 0$, it follows that there exists $C_0 > 0$ such that

$$(4.3) \quad \|e^{\tau \mathcal{A}(t)}\| \leq C_0 e^{-\delta \tau}, \quad \tau \geq 0.$$

Therefore, the evolution family $(U(t, s))_{t \geq s}$ is exponentially stable.

Clearly, assumptions (H.2) and (H.3) hold. It remains to check assumption (H.4). Now since $t \mapsto w(t)$ and $t \mapsto w(t)^{-1}$ are almost periodic it follows that $t \mapsto \mathcal{A}(t)^{-1}$ is almost periodic with respect to operator topology. Using Theorem 3.7 it follows that Eq. (1.2) has at least one almost periodic mild solution. $\square$

5. EXISTENCE OF ALMOST PERIODIC SOLUTIONS TO EQ. (1.1)

In order to study Eq. (1.1) we still suppose that the injection $\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$ is compact and that the following additional assumptions hold:

- (h₁) The function $b : \mathbb{R} \mapsto \mathbb{C}$ is given such that $b(t) = -\tilde{\rho}(t)e^{i\tilde{\theta}(t)}$ for all $t \in \mathbb{R}$ is almost periodic and there exist $\tilde{\rho}_0, \tilde{\rho}_1 > 0$ and $\tilde{\theta}_0 \in (0, \frac{\pi}{4})$ such that $\tilde{\rho}_0 \leq \tilde{\rho}(t) \leq \tilde{\rho}_1$, and $\pi + 2\tilde{\theta}_0 < \tilde{\theta}(t) < 2\pi - 2\tilde{\theta}_0$ for all $t \in \mathbb{R}$.
- (h₂) There exist $L_0 > 0$ and $\mu \in (0, 1]$ such that $|b(t) - b(s)| \leq L_0 |t - s|^\mu$ for all $s, t \in \mathbb{R}$.
- (h₃) The function $a : \mathbb{R} \mapsto \mathbb{C}$ is uniformly continuous, almost periodic, and differentiable.
- (h₄) $h : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is almost periodic in the first variable uniformly in the second one. The function $u \mapsto h(t, u)$ is uniformly continuous on any bounded subset K' of $\mathbb{H}_\alpha$ for each $t \in \mathbb{R}$. Finally,

$$\left(2 \|h(t, u)\|_\infty^2 + 3k^2 m^2 \|u\|_{\alpha, \infty}^2\right)^{1/2} \leq \mathcal{Q}(\|u\|_{\alpha, \infty})$$

for all $u \in \mathbb{H}_\alpha$, where k is the bound of the injection $\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$, $m := \max(\|a\|_\infty, \|a'\|_\infty)$, and $\mathcal{Q} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r \rightarrow \infty} \frac{\mathcal{Q}(r)}{r} = 0.$$

Theorem 5.1. *Under previous assumptions and if (h₁)-(h₄) hold, then Eq. (1.1) has at least one almost periodic solution $u \in \mathbb{H}_\alpha$.*

Proof. It suffices to let $w(t) = -b(t)$, $g(t, u) = a(t)u$, and $f(t, u) = h(t, u) + ua'(t)$ for all $t \in \mathbb{R}$ in Eq. (1.2) and to make use of Theorem 4.1. $\square$

6. EXAMPLE

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open bounded subset with C^2 boundary $\partial\Omega$ and let $\mathbb{H} = L^2(\Omega)$ be equipped with its natural topology.

To illustrate our abstract results, we study the existence of almost periodic solutions to the following damped second-order boundary value problem:

$$(6.1) \quad \frac{\partial^2 u}{\partial t^2} + a(t, x) \frac{\partial u}{\partial t} - b(t, x) \Delta u = h(t, x, u), \quad t \in \mathbb{R}, \quad x \in \Omega,$$

$$(6.2) \quad u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial\Omega,$$

where $a : \mathbb{R} \times \Omega \mapsto \mathbb{C}$ satisfies (h_3) , $b : \mathbb{R} \times \Omega \mapsto \mathbb{C}$ satisfies (h_1) – (h_2) , and $h : \mathbb{R} \times \Omega \times L^2(\Omega) \mapsto L^2(\Omega)$ satisfies (h_4) .

Define the linear operator A as follows:

$$Au = -\Delta u \quad \text{for all } u \in D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Let $\alpha \in (0, 1)$ and $q \in (0, \infty]$. Using the fact that the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows that $H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, too. It can then be shown that the injection $\left(L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega)\right)_{\alpha, q} \hookrightarrow L^2(\Omega)$ is compact, where the space $\left(L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega)\right)_{\alpha, q}$ is the real interpolation space of order (α, q) between $L^2(\Omega)$ and $H_0^1(\Omega) \cap H^2(\Omega)$. Therefore, letting $q = \infty$, it follows that the injection $\mathbb{H}_\alpha = \left(L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega)\right)_{\alpha, \infty} \hookrightarrow L^2(\Omega)$ is compact.

Consequently, taking into account the previous assumptions, it follows that the system Eq. (6.1)–Eq. (6.2) has at least one almost periodic mild solution.

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