ON COMMUTATIVITY OF THE COMMUTANT OF STRONGLY IRREDUCIBLE OPERATOR

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ABSTRACT. In 2006, C. L. Jiang and Z. Y. Wang posed an open problem: If T is a strongly irreducible operator, is $\mathcal{A}'(T)/\text{rad } \mathcal{A}'(T)$ commutative? They conjectured that the answer is positive. In this paper, to negatively answer their problem, a counterexample is given.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators acting on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $\mathcal{A}'(T)$ and $\sigma(T)$ denote the commutant and the spectrum of T , respectively. Letting $\mathcal{A}$ be a unital Banach algebra, we write $\text{rad } \mathcal{A}$ for the Jacobson radical of $\mathcal{A}$, i.e.

$$\text{rad } \mathcal{A} = \{A \in \mathcal{A} : \sigma(AB) = \sigma(BA) = \{0\}, \forall B \in \mathcal{A}\}.$$

An operator T in $\mathcal{B}(\mathcal{H})$ is called *strongly irreducible*, denoted by $T \in SI$, if $\mathcal{A}'(T)$ does not include any nontrivial idempotent operators. Obviously, strong irreducibility of an operator is invariant under similarity sense. Moreover, it is shown in [4] and [5] that the class of such operators is a suitable replacement of the Jordan blocks in $\mathcal{B}(\mathcal{H})$.

When $\mathcal{H}$ is a finite dimensional space, from the Jordan Canonical Form Theorem, T in $\mathcal{B}(\mathcal{H})$ is strongly irreducible if and only if there exists a basis of $\mathcal{H}$ such that the representing matrix of T with respect to this basis is a Jordan block:

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}.$$

It is easily known that $\mathcal{A}'(J(\lambda))$ consists of all upper-triangular matrices as follows:

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ & \ddots & \ddots & \vdots \\ & & a_1 & a_2 \\ 0 & & & a_1 \end{pmatrix}.$$

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As a result, $\mathcal{A}'(J(\lambda))$ is exactly commutative. However, if $\mathcal{H}$ is an infinite dimensional space, an example was given in [3] to show that there exists a strongly irreducible operator such that its commutant is not commutative. Further, C. L. Jiang and Z. Y. Wang posed the following problem (cf. [5], p. 147):

Open problem JW. *If $T \in SI$, is $\mathcal{A}'(T)/\text{rad } \mathcal{A}'(T)$ commutative?*

Moreover, they [5] showed that the operator class

$$\{T \in SI : \mathcal{A}'(T)/\text{rad } \mathcal{A}'(T) \text{ is commutative}\}$$

is dense in the set of strongly irreducible operators in the norm topology. Also, Jiang [2] proved that if T is a Cowen-Douglas operator and $T \in SI$, then $\mathcal{A}'(T)/\text{rad } \mathcal{A}'(T)$ is commutative. Based on these conclusions, they conjectured that the answer to the above problem is positive ([5], p. 85).

In this paper, a counterexample is given to negatively answer this problem.

2. PRELIMINARIES

A generalization of a nilpotent Jordan block in an infinite dimensional space is a weighted shift operator. However, the commutant of a weighted shift is also commutative [8]. In view of this fact, we consider the following shifts with operator weights introduced by A. L. Lambert in [6].

Let $\mathbb{C}^n$ be an n -dimensional complex Hilbert space and let $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathbb{C}^n$. Suppose that $\{W_k\}_{k=1}^{\infty}$ is a sequence of operators in $\mathcal{B}(\mathbb{C}^n)$ such that each W_k is invertible and $\sup_k \{\|W_k\|\} < \infty$. Call

$$S = \begin{pmatrix} 0 & W_1 & & & \\ & 0 & W_2 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 0 & W_{k+1} \\ & & & \ddots & \ddots \end{pmatrix} \begin{matrix} \mathbb{C}^n \\ \mathbb{C}^n \\ \vdots \\ \mathbb{C}^n \\ \vdots \end{matrix}$$

an (a unilateral backward) operator weighted shift with weight sequence $\{W_k\}_{k=1}^{\infty}$, denoted by $S \sim \{W_k\}$. Moreover, n is called the multiplicity of S .

First, by a routine computation, we have the following.

Proposition 2.1. *Let $S \sim \{W_k\}$ be an operator weighted shift. Then $T \in \mathcal{A}'(S)$ if and only if T is upper-triangular and*

$$T_{k+k+l} = (W_1 \cdots W_k)^{-1} T_{0l} W_{l+1} \cdots W_{l+k} \quad (k \geq 1, l \geq 0).$$

Obviously, $\ker S = \{(x_k) \in \mathcal{H} : x_k = 0, \forall k \geq 1\}$. Set $\mathcal{A}_0(S) = \mathcal{A}'(S)|_{\ker S}$. In [3], C. L. Jiang and J. X. Li proved the following.

Theorem 2.1. *Let $S \sim \{W_k\}$ be an operator weighted shift. Then $S \in SI$ if and only if $\sigma(T_0)$ contains only one point for every T_0 in $\mathcal{A}_0(S)$.*

3. THE CONSTRUCTION OF THE COUNTEREXAMPLE

We consider the operator weighted shift $S \sim \{W_k\}$ and let $W_k = \lambda_k + J_2$, where $0 \neq \lambda_k \in \mathbb{C}$ and J_2 is a 2×2 nilpotent Jordan block. A simple computation shows that

$$W_1 W_2 \cdots W_k = \begin{pmatrix} u_k & u'_k \\ 0 & u_k \end{pmatrix},$$

where

$$u_k = \prod_{i=1}^k \lambda_i,$$

$$u'_k = \sum_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq k} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_{k-1}}.$$

Thus, $u'_k u_k^{-1} = \sum_{i=1}^k \frac{1}{\lambda_i}$. Also, we have

$$(W_1 W_2 \dots W_k)^{-1} = \begin{pmatrix} v_k & v'_k \\ 0 & v_k \end{pmatrix},$$

where $v_k = u_k^{-1}$ and $v'_k = -u'_k u_k^{-2}$.

Theorem 3.1. *Let $S \sim \{W_k\}$ be given as above. Then $S \in SI$ if and only if*

$$(3.1) \quad \sup_k \left\{ \left| \sum_{i=1}^k \lambda_i^{-1} \right| \right\} = \infty.$$

Proof. Assume that (3.1) is false. Set $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$P_k := (W_1 W_2 \dots W_k)^{-1} P_0 W_1 W_2 \dots W_k = \begin{pmatrix} 1 & v_k u'_k \\ 0 & 0 \end{pmatrix}.$$

Since $|v_k u'_k| = |u'_k u_k^{-1}| \leq \sup_k \left\{ \left| \sum_{i=1}^k \frac{1}{\lambda_i} \right| \right\} < \infty$ we have $\sup_k \{\|P_k\|\} < \infty$. By Proposition 2.1, it follows that the nontrivial idempotent operator

$$\text{diag}(P_0, \dots, P_k, \dots)$$

is in $\mathcal{A}'(S)$. Thus, $S \notin SI$.

On the other hand, suppose that (3.1) is true. For every $T_0 = (t_{ij}) \in \mathcal{A}_0(S)$, there exists an operator $T \in \mathcal{A}'(S)$ such that $T_0 = T|_{\ker S}$. Let the k th diagonal element of T be $T_k = (t_{ij}^{(k)})$. By Proposition 2.1,

$$T_k = (W_1 W_2 \dots W_k)^{-1} T_0 W_1 W_2 \dots W_k.$$

Hence, we have

$$|t_{11}^{(k)}| = |v_k t_{11} u_k + v'_k t_{21} u_k| \geq |u'_k u_k^{-1}| |t_{21}| - |t_{11}|.$$

Note that $|t_{11}^{(k)}| \leq \|T_k\| \leq \|T\|$; it follows that $t_{21} = 0$. Thus, we have

$$|t_{12}^{(k)}| = |v_k t_{11} u'_k + v_k t_{12} u_k + v'_k t_{22} u_k| \geq |u'_k u_k^{-1}| \cdot |t_{11} - t_{22}| - |t_{12}|.$$

Still, since $|t_{12}^{(k)}| \leq \|T\|$, we see that $t_{11} = t_{22}$. Thus $\sigma(T_0)$ contains only one point. By Theorem 2.1, it follows that $S \in SI$. $\square$

Now, we can give our example as follows.

Example 3.1. Let $\{\lambda_k\}_{k=1}^\infty$ be given by the sequence

$$\{1, -1, 2^{-1}, -2^{-1}, \dots, \overbrace{1, -1, \dots, 1, -1}^{2j}, (j+1)^{-1}, -(j+1)^{-1}, \dots\}.$$

Then (3.1) holds. It follows from Theorem 3.1 that $S \in SI$. Also, by Corollary 2.1 of [7] and Theorem 4 of [8], we obtain that $\sigma(S) = \mathbb{B} := \{z \in \mathbb{C} : |z| \leq 1\}$. Set $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$A_k := (W_1 W_2 \dots W_k)^{-1} A_0 W_1 W_2 \dots W_k = A_0.$$

Thus,

$$A := \text{diag}(A_0, \dots, A_k, \dots) \in \mathcal{A}'(S).$$

Next, set $B_{01} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. It follows that

$$\begin{aligned} B_{k \ k+1} &:= (W_1 W_2 \cdots W_k)^{-1} B_{01} W_2 W_3 \cdots W_{k+1} \\ &= \begin{pmatrix} v_k & v'_k \\ 0 & v_k \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{u}_k & \bar{u}'_k \\ 0 & \bar{u}_k \end{pmatrix} = \begin{pmatrix} v'_k \bar{u}_k & v'_k \bar{u}'_k \\ v_k \bar{u}_k & v_k \bar{u}'_k \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \bar{u}_k &= \prod_{i=2}^{k+1} \lambda_i, \\ \bar{u}'_k &= \sum_{2 \leq i_1 < i_2 < \cdots < i_{k-1} \leq k+1} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{k-1}}. \end{aligned}$$

By calculating, we have that

$$\begin{aligned} |v_k \bar{u}_k| &= \left| \frac{\bar{u}_k}{u_k} \right| = |\lambda_{k+1}| \leq 1, \\ |v'_k \bar{u}_k| &= \left| \frac{\bar{u}_k u'_k}{u_k} \right| = |\lambda_{k+1}| \sum_{i=1}^k \frac{1}{\lambda_i} \leq 1, \\ |v_k \bar{u}'_k| &= \left| \frac{\bar{u}'_k}{u_k} \right| = |\lambda_{k+1}| \sum_{i=2}^{k+1} \frac{1}{\lambda_i} \leq 1, \\ |v'_k \bar{u}'_k| &= \left| \frac{u'_k \bar{u}'_k}{u_k} \right| = |\lambda_{k+1}| \sum_{i=1}^k \frac{1}{\lambda_i} \sum_{i=2}^{k+1} \frac{1}{\lambda_i} \leq 1. \end{aligned}$$

Thus, $\sup_k \{\|B_{k \ k+1}\|\} < \infty$. Proposition 2.1 implies that

$$B := \begin{pmatrix} 0 & B_{01} & & & \\ & 0 & B_{12} & & 0 \\ & & \ddots & \ddots & \\ & & & 0 & B_{k \ k+1} \\ 0 & & & & \ddots & \ddots \end{pmatrix} \in \mathcal{A}'(S).$$

Let $T = AB - BA$. Then

$$T_{01} = A_0 B_{01} - B_{01} A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

But

$$T^2 \equiv X = \begin{pmatrix} 0 & 0 & X_{02} & & & \\ & 0 & 0 & X_{13} & & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & X_{k \ k+2} \\ 0 & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$X_{02} = T_{01} T_{12} = T_{01} W_1^{-1} T_{01} W_2 = (T_{01} W_1^{-1})^2 W_1 W_2 = W_1 W_2.$$

As a result, we obtain $T^2 = S^2$. This implies that $\sigma(T^2) = (\sigma(S))^2 = \mathbb{B}$. Thus $\sigma(T) \neq \{0\}$. So, $T \notin \text{rad } \mathcal{A}'(S)$. This indicates that $\mathcal{A}'(S)/\text{rad } \mathcal{A}'(S)$ is not commutative.

Remark. J. X. Li, Y. Q. Ji and S. L. Sun [7] proved that $S \sim \{W_k\}$ is a Cowen-Douglas operator if and only if $\sup_k \{\|W_k^{-1}\|\} < \infty$. Clearly, $S \sim \{W_k\}$ in Example 3.1 is not a Cowen-Douglas operator.

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