A UNIFORM FOURIER RESTRICTION THEOREM FOR SURFACES IN $\mathbb{R}^d$

DANIEL M. OBERLIN

(Communicated by Michael T. Lacey)

ABSTRACT. We prove a Fourier restriction result, uniform over a certain collection of reference measures, for some indices in the Stein-Tomas range.

Let σ stand for Lebesgue measure on the unit sphere $S^{d-1} \subset \mathbb{R}^d$. The Stein-Tomas Fourier restriction theorem is the estimate

$$\|\hat{f}\|_{L^q(\sigma)} \leq C \|f\|_p$$

for $f \in L^p(\mathbb{R}^d)$ and $1 \leq p \leq 2(d+1)/(d+3)$, $q = (d-1)p'/(d+1)$, where p and p' are conjugate exponents and C depends on d and p . There is a well-known generalization in which σ is replaced by surface area measure on a compact subset with nonvanishing Gaussian curvature of a $(d-1)$ -dimensional submanifold Γ of $\mathbb{R}^d$. It is also of interest to investigate the possibility of similar Fourier restriction theorems when the Gaussian curvature is allowed to vanish; see [2] for a nice discussion. If one hopes to get a subset of the Stein-Tomas range of exponents in such a (degenerate) case, then it follows from results in [4] that surface area measure needs to be modified by introducing a weight which vanishes at degenerate points. When Γ is the graph

$$\{(x, \phi(x)) : x \in \Omega\}$$

of a $C^{(2)}$ function ϕ defined on an open subset $\Omega \subset \mathbb{R}^{d-1}$, the natural choice of measure λ to replace surface area measure is the so-called affine surface area measure on Γ given by

$$d\lambda = |\det H_\phi(x)|^{\frac{1}{d+1}} dx.$$

Here is a result for such λ :

Theorem. *Suppose Ω , ϕ , and λ are as above. Define $\gamma : \Omega \rightarrow \mathbb{R}^d$ by $\gamma(x) = (x, \phi(x))$. There is a positive constant C depending only on d and the generic multiplicities of the maps $\text{grad} \phi$ and*

$$\Psi : \Omega^d \rightarrow (\mathbb{R}^d)^{d-1}, \quad \Psi(x_1, \dots, x_d) = (\gamma(x_1) - \gamma(x_2), \dots, \gamma(x_1) - \gamma(x_d))$$

such that if $q = 4(d-1)/(d+1)$, then the Fourier restriction estimate

$$(1) \quad \|\hat{f}\|_{L^{q,\infty}(\lambda)} \leq C \|f\|_{4/3}$$

holds for $f \in L^{4/3}(\mathbb{R}^d)$.

Received by the editors November 10, 2010.

2010 *Mathematics Subject Classification.* Primary 42B10.

Key words and phrases. Fourier restriction.

©2011 American Mathematical Society
 Reverts to public domain 28 years from publication

(We say that a mapping Φ into $\mathbb{R}^n$ has generic multiplicity bounded by l if $\text{card}(\Phi^{-1}(x)) \leq l$ for almost all $x \in \mathbb{R}^n$.)

COMMENTS

(a) The papers [7], [1], and [3] deal with analogues of the Stein-Tomas theorem in three dimensions. If $d = 3$ the conclusion of our result is an $L^{4/3} \rightarrow L^{2,\infty}$ estimate, while the Stein-Tomas theorem gives at the endpoint the stronger $L^{4/3} \rightarrow L^2$ estimate. The range of exponents furnished by interpolating (1) with the trivial $L^1 \rightarrow L^\infty$ estimate is, for any d , a subset of the Stein-Tomas range.

(b) Examples in [2] show that the multiplicity hypotheses are necessary.

(c) An interesting feature of our result is that the bound it furnishes is uniform modulo the multiplicity hypotheses. In particular, Bézout's theorem shows that if n is fixed and if ϕ is a polynomial of degree n on $\mathbb{R}^{d-1}$, then the constant in the estimate (1) may be chosen independently of ϕ .

(d) The (very short) proof below is similar to material in [6] and [8].

Proof. The constant C may vary from line to line but will depend only on d and the multiplicities mentioned in the hypotheses. For the proof we require the estimate

$$(2) \quad \int_A \int_A \chi_E(y_1 - y_2) d\lambda(y_2) d\lambda(y_1) \leq C \lambda(A)^{(d-1)/d} |E|^{(d-1)/d}$$

for Borel sets $A \subset \gamma(\Omega)$, $E \subset \mathbb{R}^d$. Writing $\omega(x)$ for $|\det H_\phi(x)|^{1/d+1}$, we will deduce (2) from a geometric inequality which holds for Borel sets $E \subset \mathbb{R}^d$:

$$(3) \quad \int_\Omega \left[\int_{\{\omega(x_2) \geq \omega(x_1)\}} \chi_E(\gamma(x_1) - \gamma(x_2)) \omega(x_2) dx_2 \right]^d \omega(x_1) dx_1 \leq C |E|^{d-1}.$$

Estimate (3) is proved in [5]; see (2) there. To establish (2) above, identify a set $A \subset \Omega$ with its image $\gamma(A)$ in Γ and note that the bound

$$(4) \quad \int_A \int_A \chi_{\{\omega(x_2) \geq \omega(x_1)\}} \chi_E(\gamma(x_1) - \gamma(x_2)) \omega(x_2) dx_2 \omega(x_1) dx_1 \leq C \lambda(A)^{(d-1)/d} |E|^{(d-1)/d}$$

follows from (3). Replacing E by $-E$ and interchanging x_1 and x_2 gives

$$\int_A \int_A \chi_{\{\omega(x_2) \leq \omega(x_1)\}} \chi_E(\gamma(x_1) - \gamma(x_2)) \omega(x_2) dx_2 \omega(x_1) dx_1 \leq C \lambda(A)^{(d-1)/d} |E|^{(d-1)/d}.$$

With (4) this implies (2). For a measure μ on $\mathbb{R}^d$, we will write $\tilde{\mu}$ for the measure defined by $\tilde{\mu}(E) = \mu(-E)$ and will then interpret (2) as the Lorentz space estimate

$$(5) \quad \|(\chi_A d\lambda) * (\widetilde{\chi_A d\lambda})\|_{d,\infty} \leq C \lambda(A)^{(d-1)/d}.$$

Interpolating (5) with

$$\|(\chi_A d\lambda) * (\widetilde{\chi_A d\lambda})\|_1 = \lambda(A)^2$$

yields

$$\|\widehat{|\chi_A d\lambda|^2}\|_2 = \|(\chi_A d\lambda) * (\widetilde{\chi_A d\lambda})\|_2 \leq C \lambda(A)^{(3d-5)/(2d-2)},$$

and so

$$\|\widehat{\chi_A d\lambda}\|_4 \leq C \lambda(A)^{(3d-5)/(4d-4)}.$$

This is the dual of (1). □

REFERENCES

- [1] F. Abi-Khuzam and B. Shayya, Fourier restriction to convex surfaces of revolution, *Publ. Math.* **50** (2006), 71–85. MR2325011 (2008e:42019)
- [2] A. Carbery and S. Ziesler, Restriction and decay for flat hypersurfaces, *Publ. Mat.* **46** (2002), 405–434. MR1934361 (2003i:42019)
- [3] A. Carbery, C. Kenig, and S. Ziesler, Restriction for flat surfaces of revolution in $\mathbb{R}^3$, *Proc. Amer. Math. Soc.* **135** (2007), 1905–1914. MR2286103 (2008c:42021)
- [4] A. Iosevich and G. Lu, Sharpness results and Knapp’s homogeneity argument, *Canad. Math. Bull.* **43** (2000), 63–68. MR1749949 (2001g:42023)
- [5] D. Oberlin, Convolution with measures on hypersurfaces, *Math. Proc. Camb. Phil. Soc.* **129** (2000), 517–526. MR1780502 (2001j:42014)
- [6] ———, Fourier restriction for affine arclength measures in the plane, *Proc. Amer. Math. Soc.* **129** (2001), 3303–3305. MR1845006 (2002g:42013)
- [7] ———, A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^3$, *Proc. Amer. Math. Soc.* **132** (2004), 1195–1199. MR2045437 (2005h:42020)
- [8] ———, Some convolution inequalities and their applications, *Trans. Amer. Math. Soc.* **354** (2002), 2541–2556. MR1885663 (2002j:42016)

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306
E-mail address: oberlin@math.fsu.edu