ON THE BOUNDARY OF KÄHLER CONES

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ABSTRACT. We study some geometric properties of a compact Kähler manifold (M^n, g) under a certain condition on the bisectional curvature. As an application, we give a new proof for an earlier result which asserts that any boundary class of the Kähler cone of M^n can be represented by a C^∞ closed $(1,1)$ form that is parallel and everywhere nonnegative.

Let (M^n, g) be a compact Kähler manifold. Denote by

$$H(M) = H_{\mathbb{C}}^{1,1}(M) = H^{1,1}(M) \cap H^2(M, \mathbb{C})$$

the vector space of real $(1,1)$ classes. Write $\mathcal{K}(M)$ for the Kähler cone in $H(M)$, namely, the convex cone formed by all the cohomology classes that can be represented by smooth closed $(1,1)$ forms that are everywhere positives. We are interested in the boundary set $\mathcal{B} = \overline{\mathcal{K}} \setminus \mathcal{K}$ of the Kähler cone. We will call a (nontrivial) cohomology class α in $\mathcal{B}$ a boundary Kähler class of M . It would be an interesting question to ask when α can be represented by a closed, smooth $(1,1)$ form that is everywhere nonnegative. The existence of such forms often has direct geometric applications in the study of Monge-Ampère foliations or Kähler submersions.

Recently, this problem has been studied by D. Wu, S.T. Yau and F. Zheng [5]. They interpreted the geometric problem by a degenerate complex Monge-Ampère equation and proved the existence of a smooth solution to the equation through a delicate way. The main result in their paper is as follows:

Theorem (Wu – Yau – Zheng). *Let (M^n, g) be a compact manifold satisfying the following curvature condition: for any orthogonal tangent frame $e_1, \dots, e_n$ at any $x \in M$, and for any real numbers $a_1, \dots, a_n$:*

$$(*) \quad \sum_{i,j=1}^n R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0.$$

Then any boundary class of the Kähler cone of M^n can be represented by a C^∞ closed $(1,1)$ form that is everywhere nonnegative.

As pointed out in [5], when $n = 2$, the curvature condition $(*)$ simply implies $R_{u\bar{u}v\bar{v}} \geq 0$ for any pair of orthogonal tangent vectors u, v of type $(1,0)$, which means that M^2 has nonnegative orthogonal bisectional curvature. For $n > 2$, it's easy to give an example and see that the condition $(*)$ becomes much less restrictive than nonnegative orthogonal bisectional curvature. So, it should be interesting to

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study some geometric properties of the compact manifolds under this curvature condition.

In 1981, A. Howard, B. Smyth and H. Wu [6] proved that

Lemma (Howard – Smyth – Wu). *If a compact Kähler manifold M satisfies curvature condition $(*)$, then all harmonic forms of type $(1,1)$ are parallel. In particular, the Hodge number $h^{1,1}$ is equal to 1 if M is locally irreducible.*

Inspired by this nice result, we can prove the following geometric property for the manifolds with curvature condition $(*)$.

Theorem 1. *Let (M^n, g) be a compact Kähler manifold satisfying the curvature condition $(*)$. Then, for any closed $(1,1)$ form Φ on (M^n, g) , we can find $\tilde{\Phi} \in [\Phi]$, such that $\tilde{\Phi}$ is parallel. In particular, for any closed $(1,1)$ form α , we have*

$$[\alpha] = [\beta + \lambda_s \omega_0],$$

where β is a nonnegative C^∞ closed $(1,1)$ form on the boundary of the Kähler cone, λ_s is a constant depending on β , and ω_0 is the Kähler form on (M^n, g) .

Before giving the proof for our main theorem, we state a fundamental lemma which can be found in [1].

Lemma ([1] (2.33)). *Let ϕ be a closed $(1,1)$ form on a compact Kähler manifold. Then ϕ is harmonic if and only if its trace is constant.*

Proof of Theorem 1. Consider the equation

$$(0.1) \quad \sigma_1(\omega_0 + \Phi + \partial\bar{\partial}v) = C,$$

where C is some constant to be determined. The above equation is equivalent to

$$\Delta v = C - (n + \text{Tr}_g \Phi).$$

By the standard theory of partial differential equations, we know that $\Delta u = f$ is solvable if and only if $\int_M f = 0$. So, if we choose

$$C = n + \frac{1}{\text{Vol}(M)} \int_M \omega_0^{n-1} \wedge \Phi,$$

there is a smooth solution of the equation (0.1).

Let $\tilde{\Phi} = \Phi + \partial\bar{\partial}v$. Then the equation is

$$(0.2) \quad \sigma_1(\omega_0 + \tilde{\Phi}) = C.$$

By the previous two lemmas, we can assert that $\tilde{\Phi}$ is a smooth parallel $(1,1)$ form and $\tilde{\Phi} \in [\Phi]$.

Let λ_s be the smallest eigenvalue of the $(1,1)$ form $\tilde{\Phi}$ (under the fixed orthonormal frame) and define

$$\phi = \tilde{\Phi} - \lambda_s \omega_0.$$

Then it is not hard to see that ϕ is nonnegative everywhere on M^n and on the boundary of the Kähler cone.

Thus, for any closed $(1,1)$ form α , we can find a nonnegative closed $(1,1)$ form β on the boundary of the Kähler cone such that

$$[\alpha] = [\beta + \lambda_s \omega_0],$$

where ω_0 is the Kähler form. □

As an application of the above theorem, we give a new proof for the main theorem in [5].

Corollary 1. *Let (M^n, g) be a compact manifold satisfying the curvature condition $(*)$. Then any boundary class of the Kähler cone of M^n can be represented by a C^∞ closed $(1, 1)$ form that is parallel and everywhere nonnegative.*

Proof of Corollary 1. Suppose α is a closed $(1, 1)$ form on the boundary of the Kähler cone. By the definition of boundary Kähler class, we know that there exists a sequence of smooth closed $(1, 1)$ forms $\{\omega_m\}$ which are everywhere positive and $\omega_m \rightarrow \alpha$ as $m \rightarrow \infty$.

Now, consider the integration to be a continuous functional on the form space $h^{1,1}(M)$, and by the convergence, we get

$$(0.3) \quad \int_M \omega_m^k \wedge \omega_0^{n-k} \rightarrow \int_M \alpha^k \wedge \omega_0^{n-k}, \quad k = 0, 1, \dots, n.$$

So, $\int_M \alpha^k \wedge \omega_0^{n-k} \geq 0$ for any $k = 0, 1, \dots, n$.

According to the proof of Theorem 1, there is a parallel closed $(1, 1)$ form $\tilde{\alpha} \in [\alpha]$. Thus, the eigenvalues of $\tilde{\alpha}$ are all constant on M^n and

$$(0.4) \quad \begin{aligned} 0 &\leq \int_M \alpha^k \wedge \omega_0^{n-k} = \int_M \tilde{\alpha}^k \wedge \omega_0^{n-k} = \sigma_k(\tilde{\alpha}) \int_M \omega_0^n \\ \implies \quad \sigma_k(\tilde{\alpha}) &\geq 0, \quad k = 0, 1, \dots, n, \end{aligned}$$

which means that $\tilde{\alpha}$ is in the Γ_n convex cone. So, $\tilde{\alpha} \in [\alpha]$ is nonnegative everywhere. $\square$

Remark 1. Let (M, g) be a compact Kähler manifold and let ω be the Kähler form of g . For any real $(1, 1)$ form α , we can define the k -th symmetric function for α with respect to ω as

$$\alpha^k \wedge \omega^{n-k} = \sigma_k(\alpha) \omega^n.$$

Denote the space of real $(1, 1)$ forms by $\Lambda_{\mathbb{R}}^{1,1}$. We define the k -convex cone by

$$\Gamma_k = \{\alpha \in \Lambda_{\mathbb{R}}^{1,1} \mid \sigma_j(\alpha) \geq 0, j = 1, \dots, k\}.$$

To study the Betti number for a compact connected Kähler manifold under some curvature conditions is also a very interesting topic in geometry. In 1965, Bishop and Goldberg [2] showed that any compact Kähler manifold M^n with positive bisectional curvature must have its second Betti number equal to 1. Later, Goldberg and Kobayashi [4] introduced the conception of holomorphic bisectional curvature and proved that the second Betti number of a compact connected Kähler manifold M with positive holomorphic bisectional curvature is one. Now, if we restrict the curvature condition to be the following so-called *quasi- $(*)$* , we can also get the similar geometric property.

Definition 1. We say that a compact manifold (M_n, g) satisfies the *quasi- $(*)$* curvature condition if it satisfies: for any orthogonal tangent frame $e_1, \dots, e_n$ at any $x \in M$, and for any real numbers $a_1, \dots, a_n$, $\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0$ holds everywhere and is strictly positive at least at one point.

Theorem 2. *Let (M_n, g) be a compact Kähler manifold satisfying the quasi- $(*)$ curvature condition. Then $\dim h^{1,1}(M, \mathbb{R}) = 1$.*

Proof. The proof follows the well-known technique due to Bochner and Lichnerowicz.

Let ξ be a closed (1,1) form on a compact Kähler manifold M^n . For any fixed point on M^n , we can choose local coordinates $\{z_1, \dots, z_n\}$ such that

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}, \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial z_i} = \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{z}_{\bar{i}}} = 0 \quad \text{and} \quad \xi = \xi_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}.$$

We consider the function $\phi = \sigma_2(g^{i\bar{l}}\xi_{l\bar{j}})$ and denote $W = (w_{i\bar{j}}) = (g^{i\bar{l}}\xi_{l\bar{j}})$. Then we have

$$(0.5) \quad \begin{aligned} \phi_\alpha &= \sigma_1(W|i)(g^{i\bar{l}}\xi_{l\bar{i}})_\alpha, \\ \phi_{\alpha\bar{\beta}} &= \sigma_1(W|i)(g^{i\bar{l}}\xi_{l\bar{i}})_{\alpha\bar{\beta}} + (g^{i\bar{l}}\xi_{l\bar{i}})_\alpha (g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\beta}} - (g^{i\bar{l}}\xi_{l\bar{i}})_\alpha (g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\beta}}. \end{aligned}$$

By the proof of Theorem 1, we know there is a closed (1,1) form (which we still denote by ξ) in $[\xi]$ such that

$$(0.6) \quad F(\xi) = \sigma_1(g^{i\bar{l}}\xi_{l\bar{j}}) = C,$$

where C is some constant.

By (0.6), we have

$$(0.7) \quad \begin{aligned} F^{\alpha\bar{\beta}} &= g^{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}, \\ \sigma_1(g^{\alpha\bar{\eta}}\xi_{\eta\bar{\beta}}) = C &\implies \sum_i^n \xi_{i\bar{i},\alpha} = 0, \\ (0.8) \quad \sigma_1(g^{\alpha\bar{\eta}}\xi_{\eta\bar{\beta}}) = C &\implies g^{\alpha\bar{\eta}}_{,i\bar{i}}\xi_{\eta\bar{\beta}} + g^{\alpha\bar{\eta}}\xi_{\eta\bar{\beta},i\bar{i}} = 0 \\ &\implies \xi_{\alpha\bar{\alpha},i\bar{i}} = -g^{\alpha\bar{\eta}}_{,i\bar{i}}\xi_{\eta\bar{\alpha}} = -g^{\alpha\bar{\alpha}}_{,i\bar{i}}\xi_{\alpha\bar{\alpha}}. \end{aligned}$$

Thus, by direct computation, we can get that

$$(0.9) \quad \begin{aligned} F^{\alpha\bar{\beta}}\phi_{\alpha\bar{\beta}} &= \sigma_1(W|i)(g^{i\bar{l}}\xi_{l\bar{i}})_{\alpha\bar{\alpha}} + (g^{i\bar{l}}\xi_{l\bar{i}})_\alpha (g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\alpha}} - (g^{i\bar{l}}\xi_{l\bar{i}})_\alpha (g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\alpha}} \\ &= \sigma_1(W|i)(g^{i\bar{i}}_{,\alpha\bar{\alpha}}\xi_{i\bar{i}} + \xi_{i\bar{i},\alpha\bar{\alpha}}) + \xi_{i\bar{i},\alpha}\xi_{j\bar{j},\bar{\alpha}} - \xi_{i\bar{j},\alpha}\xi_{i\bar{j},\bar{\alpha}} \\ &= \sigma_1(W|i)(g^{i\bar{i}}_{,\alpha\bar{\alpha}}\xi_{i\bar{i}} + \xi_{\alpha\bar{\alpha},i\bar{i}}) + \xi_{i\bar{i},\alpha}\xi_{j\bar{j},\bar{\alpha}} - \xi_{i\bar{j},\alpha}\xi_{i\bar{j},\bar{\alpha}} \\ &= \sigma_1(W|i)(g^{i\bar{i}}_{,\alpha\bar{\alpha}}\xi_{i\bar{i}} - g^{\alpha\bar{\alpha}}_{,i\bar{i}}\xi_{\alpha\bar{\alpha}}) - \sum_{i,j} |\nabla \xi_{i\bar{j}}|^2 + \left(\sum_{i=1}^n \xi_{i\bar{i},\alpha}\right)^2 \\ &= \sigma_1(W|i)(-R_{i\bar{i}\alpha\bar{\alpha}}\xi_{i\bar{i}} + R_{i\bar{i}\alpha\bar{\alpha}}\xi_{\alpha\bar{\alpha}}) - \sum_{i,j} |\nabla \xi_{i\bar{j}}|^2 \\ &= -\frac{1}{2} \sum_{i\alpha} R_{i\bar{i}\alpha\bar{\alpha}}(\sigma_1(W|i) - \sigma_1(W|\alpha))(\xi_{i\bar{i}} - \xi_{\alpha\bar{\alpha}}) - \sum_{i,j} |\nabla \xi_{i\bar{j}}|^2 \\ &= -\frac{1}{2} \sum_{i\alpha} R_{i\bar{i}\alpha\bar{\alpha}}(\xi_{i\bar{i}} - \xi_{\alpha\bar{\alpha}})^2 - \sum_{i,j} |\nabla \xi_{i\bar{j}}|^2. \end{aligned}$$

From the second to third equality, we used the fact that ξ is a closed (1,1) form, which gives us

$$(0.10) \quad \xi_{\alpha\bar{\alpha},i} = \xi_{i\bar{\alpha},\alpha}, \quad \xi_{\alpha\bar{\alpha},\bar{i}} = \xi_{\alpha\bar{i},\bar{\alpha}}.$$

From equality (0.9), we can see that if the compact Kähler manifold M^n satisfies the curvature condition (*), then

$$(0.11) \quad F^{\alpha\bar{\beta}}\phi_{\alpha\bar{\beta}} = -\frac{1}{2} \sum_{i,\alpha} R_{i\bar{i}\alpha\bar{\alpha}}(\xi_{i\bar{i}} - \xi_{\alpha\bar{\alpha}})^2 - \sum_{i,j} |\nabla \xi_{i\bar{j}}|^2 \leq 0.$$

Finally, by the strong maximal principle, we assert that $\phi \equiv \text{constant}$. Thus,

$$\sigma_1(g^{i\bar{i}}\xi_{i\bar{j}}) = \text{constant and } \sigma_2(g^{i\bar{i}}\xi_{i\bar{j}}) = \text{constant},$$

from which we can get that $\xi_{i\bar{i}}$ are constants on M^n . Furthermore, if the manifold satisfies the quasi-(*) curvature condition, by (0.11), we can see that $\xi_{i\bar{i}} = \xi_{\alpha\bar{\alpha}}$ for $i, \alpha = 1, \dots, n$. Thus, $\xi = \lambda\omega_0$, where λ is a constant. So, $\dim h^{1,1}(M) = 1$. $\square$

Remark 2. For the results in [2], [4], [6], the restriction of the bisectional curvature makes the Ricci tensor of M to be positive. So there are no nontrivial holomorphic 2-forms on M (cf. Bochner [8]), i.e., $H^{2,0}(M) = H^{0,2}(M) = 0$, i.e., $b_2(M) = \dim H^2(M) = \dim h^{1,1}(M)$. But, in our case, we have no information about the positivity of the Ricci tensor from the quasi-(*) curvature condition. Thus, we cannot get $b_2(M) = 1$.

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