THE FAILURE OF THE FIXED POINT PROPERTY FOR UNBOUNDED SETS IN c_0

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ABSTRACT. In this paper we prove that for every unbounded convex closed set C in c_0 there exists a nonexpansive mapping $T : C \rightarrow C$ which is fixed point free. This result solves in a negative sense a question that has remained open for some time in Metric Fixed Point Theory.

1. INTRODUCTION

A subset C of a Banach space is said to satisfy the Fixed Point Property for nonexpansive mappings (FPP) if every nonexpansive self-mapping T defined on C has a fixed point. In the last 40 years many papers have appeared proving the existence of fixed points for nonexpansive mappings defined on *bounded* convex closed subsets of certain classes of Banach spaces, whereas other papers have shown the failure of the FPP for some bounded convex closed subsets of other Banach spaces. For instance, it is well known [1] that every bounded convex closed subset of a Hilbert space satisfies the FPP. Many more classes of reflexive Banach spaces share this property (the monographs [4] and [6] provide detailed information on this subject). In the nonreflexive case, one of the most relevant results was proved by B. Maurey [8], namely that every convex weakly compact subset of the sequence space c_0 satisfies the FPP. Besides this positive result, the failure of the FPP for other bounded convex closed subsets of c_0 is also well-known (see, for instance, [7]). In 2004, P. Dowling et al. [3] proved the following complete characterization of the FPP for bounded convex closed subsets of c_0 : let C be a bounded convex closed set in c_0 . Then, C satisfies the FPP if, and only if, C is weakly compact. Their proof is based on the following fact, which is a consequence of Eberlein-Šmulian and Alouglu Theorems: every bounded convex closed set in c_0 , which is not weakly compact, contains a sequence which is $\sigma(\ell_\infty, \ell_1)$ -convergent to a point $u \in \ell_\infty \setminus c_0$.

However, very few results have been obtained whenever the *boundedness* assumption is removed. The problem is only completely solved in the case of Hilbert spaces, because W. Ray [10] has proved that every *unbounded* convex subset of a Hilbert space fails to satisfy the FPP. The proof is strongly based upon the structure of Hilbert spaces, nominally, on the existence of an orthogonal basis with some specific properties. A much simpler proof of the same result has been given by R. Sine

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[13], but his proof is still based upon a property that, for dimension greater than 2, is a characteristic of Hilbert spaces, namely that every convex closed subset of a Hilbert space is a nonexpansive retraction of the whole space. Due to this strong dependence on some characteristic properties of Hilbert spaces, it was conjectured ([5], Question XIV) that the failure of the Fixed Point Property (FPP) for every unbounded convex set could be a characterization of Hilbert spaces.

On the other hand, this problem is completely solved if you consider the Approximate Fixed Point Property (AFPP) instead of the FPP. Recall that C is said to satisfy the AFPP if for every nonexpansive mapping $T : C \rightarrow C$ we have that $\inf\{\|x - Tx\| : x \in C\} = 0$. S. Reich [11] proved the following characterization of the sets satisfying the AFPP in a reflexive space: let C be a convex closed subset of a reflexive Banach space X . Then, C satisfies the AFPP if, and only if, C is linearly bounded, that is, $C \cap r$ is bounded for every line r in X . (A previous result for certain classes of reflexive Banach spaces was given in [9].) I. Shafrir [12] defined the notion of directional boundedness (which is equivalent to linear boundedness in reflexive spaces) and extended the above result to arbitrary Banach spaces in the following sense: let C be a convex closed subset of a Banach space X . Then, C satisfies the AFPP if, and only if, C is directionally bounded.

In our paper we will study the failure of the FPP for unbounded convex sets of c_0 . Since a set which fails to satisfy the AFPP also fails to satisfy the FPP, Shafrir's result will be very important in our approach. We will show that every unbounded convex directionally bounded set in c_0 contains a sequence which is $\sigma(\ell_\infty, \ell_1)$ -convergent to a point $u \in \ell_\infty \setminus c_0$. As a consequence of this fact, we can adapt the proof in [3] to the unbounded case and we can prove that every unbounded convex closed set in c_0 fails to satisfy the FPP. Hence, we can respond negatively to the question raised by A. Kirk [5], proving that the failure of the FPP for every unbounded convex closed set is not a characteristic of Hilbert spaces.

2. PRELIMINARIES

In [12] the notion of a directionally bounded set is stated for any hyperbolic metric space. Since we are only interested in the case of a Banach space, we will use the following equivalent definition (see [12], Theorems 2.4 and 3.2 for the equivalence):

Definition 2.1. A convex subset C of a Banach space X is directionally bounded if for every sequence (x_n) in C such that $\|x_n\| \rightarrow \infty$ and every $f \in X^*$, $\|f\| = 1$, one has

$$\limsup_{n \rightarrow \infty} f\left(\frac{x_n}{\|x_n\|}\right) < 1.$$

The following simple consequence of the above definition will be a basic tool in this paper:

Lemma 2.2. Let C be a directionally bounded set in c_0 and (x_n) be a sequence in C such that $\|x_n\| \rightarrow \infty$. Then, for every pair of positive integers n_0, k_0 , there exist $n > n_0$ and $k > k_0$ such that $|x_n(k)| = \|x_n\|$.

Proof. Otherwise $\|x_n\| = |x_n(k)|$ for every $n > n_0$ and some $k = k(n) \leq k_0$. Thus, there exists a subsequence (x_{n_j}) and $\bar{k} \leq k_0$ such that $\pm x_{n_j}(\bar{k}) = \|x_{n_j}\|$ for every

$j \in \mathbb{N}$. Hence

$$\limsup_{n \rightarrow \infty} \frac{x_n(\bar{k})}{\|x_n\|} = \pm 1,$$

which is a contradiction according to Definition 2.1 because the mapping $x \rightarrow x(k)$ is a normalized functional in c_0^* . $\square$

3. MAIN RESULTS

Lemma 3.1. *Let C be a directionally bounded convex set in c_0 and (x_n) a sequence in C such that $\|x_n\| \rightarrow \infty$. Denote*

$$\lambda(k) = \limsup_{n \rightarrow \infty} \frac{x_n(k)}{\|x_n\|}$$

and by λ the sequence $(\lambda(k))_k \in \ell_\infty$. Assume that $\lambda \neq 0$. Then, $\|\lambda\| > |\lambda(k)|$ for every $k \in \mathbb{N}$. In particular, $\lambda \notin c_0$.

Proof. By contradiction, we assume that there exists k_1 such that $a =: \|\lambda\| = |\lambda(k_1)| > 0$. Without loss of generality we can assume that $0 \in C$. Taking a subsequence we can assume that $x_n(k_1)$ has constant sign for every $n \in \mathbb{N}$,

$$(3.1) \quad \|x_n\| \geq 2^n$$

and

$$(3.2) \quad \frac{|x_n(k_1)|}{\|x_n\|} \geq a \left(1 - \frac{1}{2^n}\right)$$

for every $n \in \mathbb{N}$. Choose $n_1 = 1$. We can choose a positive integer $k_2 > k_1$ such that $|x_{n_1}(k)| < a/2$ if $k \geq k_2$ and another integer $n_2 > n_1$ such that

$$\frac{|x_{n_2}(k)|}{\|x_{n_2}\|} < a \left(1 + \frac{1}{2^2}\right)$$

if $k_1 \leq k \leq k_2$. Following an induction argument we find two increasing sequences $(n_j), (k_j)$ of positive integers such that

$$(3.3) \quad |x_{n_j}(k)| < \frac{a}{2^j} \text{ if } k \geq k_{j+1},$$

$$(3.4) \quad \frac{|x_{n_j}(k)|}{\|x_{n_j}\|} < a \left(1 + \frac{1}{2^j}\right) \text{ if } k_1 \leq k \leq k_j.$$

Define

$$u_m = \sum_{i=1}^m \frac{x_{n_i}}{\|x_{n_i}\|}.$$

It is clear from (3.1) that u_m belongs to C . Furthermore, from (3.2)

$$|u_m(k_1)| \geq (m-1)a,$$

which, in particular, implies $(m-1)a \leq \|u_m\| \rightarrow \infty$.

Choose $j_0 > 4/a$. If $k \geq k_{j_0-1}$, there exists $j \geq j_0$ such that $k_{j-1} \leq k < k_j$. The inequality $k < k_j$ and (3.4) imply

$$\frac{|x_{n_i}(k)|}{\|x_{n_i}\|} < a \left(1 + \frac{1}{2^i}\right)$$

for every $i \geq j$. For $i < j-1$, we have

$$|x_{n_i}(k)| < \frac{a}{2^i}$$

from (3.3), because $k \geq k_{j-1} \geq k_{i+1}$. Hence, for $m \geq j$, we have

$$\begin{aligned} |u_m(k)| &\leq \sum_{i=1}^m \frac{|x_{n_i}(k)|}{\|x_{n_i}\|} \leq a \sum_{i=1}^{j-2} \frac{1}{2^i} + 1 + a \sum_{i=j}^m \left(1 + \frac{1}{2^i}\right) \\ &\leq a + 1 + (m - j + 1)a \leq a + 1 + (m - j_0 + 1)a \\ &< (m - 1)a \leq \|u_m\|. \end{aligned}$$

For $j_0 \leq m < j$ we have

$$\begin{aligned} |u_m(k)| &\leq a \sum_{i=1}^{j-2} \frac{1}{2^i} + 1 < a + 1 \leq a + 1 + (m - j_0 + 1)a \\ &< (m - 1)a \leq \|u_m\|. \end{aligned}$$

Thus, $|u_m(k)| < \|u_m\|$ for every $k \geq k_{j_0-1}$ and $m \geq j_0$, which is a contradiction according to Lemma 2.2. $\square$

Lemma 3.2. *Let C be an unbounded convex set in c_0 which is directionally bounded. Then, there exists a sequence in C which is $\sigma(\ell_\infty, \ell_1)$ -convergent to a point $u \in \ell_\infty \setminus c_0$.*

Proof. Choose a sequence (x_n) in C such that $\|x_n\| \rightarrow \infty$. Without loss of generality we can assume that $0 \in C$, and taking subsequences and using a diagonal argument we can assume that $\lambda(k) = \lim_{n \rightarrow \infty} x_n(k)/\|x_n\|$ does exist for every $k \in \mathbb{N}$. If $(\lambda(k)) \in \ell_\infty \setminus c_0$ we can finish the proof by choosing the sequence $(x_n/\|x_n\|)$. Otherwise, by Lemma 3.1, $\lambda(k) = 0$ for every $k \in \mathbb{N}$. We can assume that $\|x_n\| \geq 2^n$. By using Lemma 2.2 and the gliding hump method we can construct two increasing sequences k_j and n_j of positive integers such that:

- (i) $|x_{n_j}(k_j)| = \|x_{n_j}\|$.
- (ii) $\frac{|x_{n_j}(k)|}{\|x_{n_j}\|} < 2^{-(j+1)}$ if $k \leq k_{j-1}$.
- (iii) $|x_{n_{j-1}}(k)| < 2^{-j}$ if $k \geq k_j$.

Indeed, choose $n_1 = 1$ and $k_1 \in \mathbb{N}$ such that $\|x_{n_1}\| = |x_{n_1}(k_1)|$. Assume that k_j, n_j have been chosen satisfying (i), (ii) and (iii) for $j \leq i - 1$. We can find a positive integer k_i , large enough, such that $|x_{n_{i-1}}(k)| < 2^{-i}$ if $k \geq k_i$ and another positive integer n_i , also large enough, such that $|x_n(k)|/\|x_n\| < 2^{-(i+1)}$ if $k \leq k_{i-1}$ and $n \geq n_i$. From Lemma 2.2 we can also assume that $|x_{n_i}(k_i)| = \|x_{n_i}\|$, and so (i), (ii) and (iii) are satisfied for every $j \in \mathbb{N}$. Define the sequence

$$v_m = \sum_{i=1}^m \frac{x_{n_i}}{\|x_{n_i}\|}.$$

We claim that (v_m) is a bounded sequence. Indeed, for any $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $k_{j-1} \leq k < k_j$, where we assume $k_0 = 1$. Thus, from (ii) and (iii)

$$\frac{|x_{n_i}(k)|}{\|x_{n_i}\|} < \frac{1}{2^{i+1}}$$

if $i \neq j-1, j$. Hence

$$|v_m(k)| \leq 2 + \sum_{i=1; i \neq j, j-1}^m \frac{1}{2^{i+1}} < 3.$$

On the other hand, for $m > j$ we have

$$|v_m(k_j)| \geq \frac{|x_{n_j}(k_j)|}{\|x_{n_j}\|} - \sum_{i=1, i \neq j}^m \frac{1}{2^{i+1}} \geq \frac{1}{2}.$$

By Alaoglu's Theorem, (v_m) has a subsequence which is $\sigma(\ell_\infty, \ell_1)$ -convergent to a vector $u \in \ell_\infty \setminus c_0$. $\square$

Theorem 3.3. *Let C be a convex closed subset of c_0 . Then, C satisfies the FPP for nonexpansive mappings if and only if C is weakly compact.*

Proof. It is well-known [8] that C satisfies the FPP if it is weakly compact. To prove the converse, note that if C is a bounded convex set which is not weakly compact, then C fails to satisfy the FPP [3]. Furthermore, if C is convex and unbounded, then C fails to satisfy the AFPP, and so the FPP, whenever C is not directionally bounded. Therefore, we only need to prove that C fails to satisfy the FPP if it is an unbounded closed convex set which is directionally bounded. In this case, by Lemma 3.2, C contains a sequence which is $\sigma(\ell_\infty, \ell_1)$ -convergent to a point in $\ell_\infty \setminus c_0$. Thus, C contains a closed convex bounded subset K which is not weakly compact, and by Theorem 4 in [2], C contains a nonzero multiple of an asymptotically isometric c_0 -summing basic sequence. Under this assumption, the arguments in the proof of Theorem 1 in [3] show that there is a convex closed subset K_0 of K and a nonexpansive mapping $S : c_0 \rightarrow K_0$ which is fixed point free. Define $T : C \rightarrow K_0$ to be the restriction of S to C . Then, $T : C \rightarrow K_0 \subset C$ is a fixed point free nonexpansive mapping. $\square$

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