STEFFENSEN'S INEQUALITY AND $L^1 - L^\infty$ ESTIMATES OF WEIGHTED INTEGRALS

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ABSTRACT. Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function with $\Phi(0) = 0$. We prove that $\Phi\left(\frac{\|f\|_1}{\omega_N \|f\|_\infty}\right) \leq \frac{1}{\omega_N \|f\|_\infty} \int_{\mathbb{R}^N} |f(x)| \Phi'(|x|^N) dx$ for every $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$, where ω_N is the measure of the unit ball of $\mathbb{R}^N$. This can be used to obtain lower or upper bounds for weighted integrals $\int_{\mathbb{R}^N} |f(x)| \eta(|x|) dx$ in terms of the L^1 and L^∞ norms of f , which are often much sharper than crude estimates that may be obtained, if at all, by a visual inspection of the integrand. The basic inequality is essentially independent of Jensen's inequality, but it is closely related to Steffensen's inequality.

1. INTRODUCTION

If E is a Lebesgue measurable subset of $\mathbb{R}^N$ with finite measure $|E| > 0$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, the well-known Jensen inequality asserts that

$$(1.1) \quad |E| \Phi\left(\frac{1}{|E|} \int_E u(x) dx\right) \leq \int_E \Phi(u(x)) dx,$$

for every $u \in L^1(E)$ (the right-hand side may be ∞).

This paper is devoted to the proof of an inequality whose left-hand side bears a strong resemblance with that of the Jensen inequality, but with a very different right-hand side. We shall confine attention to continuous convex functions $\Phi : [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(0) = 0$ (continuity at 0 is not implied by convexity). For such functions, we shall prove that

$$(1.2) \quad \omega_N \|f\|_\infty \Phi\left(\frac{1}{\omega_N \|f\|_\infty} \int_E f(x) dx\right) \leq \int_E f(x) \Phi'(|x|^N) dx,$$

for every *nonnegative* $f \in L^1(E) \cap L^\infty(E)$, $f \neq 0$, where ω_N is the measure of the unit ball of $\mathbb{R}^N$ and Φ' is the derivative of Φ , well defined at all but countably many points of $(0, \infty)$. Once again, the right-hand side may be ∞ .

Thus, on the left-hand side, the only visible difference with (1.1) is that (u is replaced by f and) $|E|$ is replaced by $\omega_N \|f\|_\infty$, but the right-hand sides of (1.1) and (1.2) have a completely different structure.

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In (1.2), there is no need for E to have finite measure and, since $\Phi(0) = 0$, the inequality when $E = \mathbb{R}^N$ implies at once the general case. Also, it is simpler and equivalent to replace f by $|f|$, so that the nonnegativity assumption can be dropped. With these modifications, the inequality becomes

$$(1.3) \quad \omega_N \|f\|_\infty \Phi \left(\frac{\|f\|_1}{\omega_N \|f\|_\infty} \right) \leq \int_{\mathbb{R}^N} |f(x)| \Phi'(|x|^N) dx,$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ refer to the L^1 and L^∞ norms on $\mathbb{R}^N$, respectively.

There is no requirement that Φ or Φ' be nonnegative and the term $\Phi'(|x|^N)$ has a less special structure than it appears at first sight. Indeed, if $\psi(r)$ is any real-valued nondecreasing function on $(0, \infty)$ which is locally integrable near 0, then $\psi(|x|) = \Phi'(|x|^N)$ for some convex continuous Φ with $\Phi(0) = 0$.

In practice, (1.3) provides lower or upper bounds for integrals of the form $\int_{\mathbb{R}^N} |f(x)| \eta(|x|) dx$ in terms of the L^1 and L^∞ norms of f , which are often much sharper than the crude estimates (if any) that can be guessed from a visual inspection of the integrand.

For example, while it seems that $\int_{\mathbb{R}^N} |f(x)| |x|^\beta dx$ and $\int_{\mathbb{R}^N} |f(x)| dx = \|f\|_1$ cannot be compared since either can be arbitrarily small while the other is arbitrarily large, it turns out that $\int_{\mathbb{R}^N} |f(x)| |x|^\beta dx \geq \frac{N}{(\beta+N)\omega_N^{\beta/N}} \|f\|_1^{1+\beta/N} \|f\|_\infty^{-\beta/N}$ for every $\beta \geq 0$. Thus, the two integrals can actually be compared, provided that $\|f\|_\infty$ is also involved (and the relationship is not linear). Likewise, visual inspection does not reveal that $\int_{\mathbb{R}^N} |f(x)| \log |x| dx > 0$ if $\frac{\|f\|_1}{\|f\|_\infty} > e\omega_N$. These and several other examples are discussed in Section 3.

Although the proof (Theorem 2.5) uses only classical arguments, we have found no evidence that (1.3) or even special cases of it have previously been recorded. A main ingredient is the analogous inequality

$$(1.4) \quad \|f\|_\infty \Phi \left(\frac{1}{\|f\|_\infty} \int_0^\infty f(r) dr \right) \leq \int_0^\infty f(r) \Phi'(r) dr,$$

when $f \neq 0$ is a bounded nonnegative measurable function on $(0, \infty)$ (Corollary 2.3). Surprisingly, we have been equally unable to find any full statement of it, although a close (but awkward) variant on finite intervals was proved by Steffensen [17] as early as 1918. For more comments, see Section 2. It turns out that (1.4) is equivalent to the convexity of Φ (Theorem 2.4) and so it cannot be generalized without shrinking the set of admissible functions f .

In a limited setting, (1.4) and the Jensen inequality can be deduced from one another, as we now explain. Let $u \in L^1(\mathbb{R}^N)$ and suppose that $u > 0$ a.e. on some measurable subset E with $0 < |E| < \infty$, so that (1.1) holds. For $r > 0$, let $f(r) := |\{x \in E : u(x) > r\}|$ be the distribution function of $u\chi_E$. Then, $\|f\|_\infty = |E|$ (this uses $u > 0$ on E) and since $\Phi(0) = 0$, it follows that $\int_E \Phi(u(x)) dx = \int_{\mathbb{R}^N} \Phi(u(x)\chi_E(x)) dx = \int_0^\infty f(r) \Phi'(r) dr$ (see [20, p. 37]; the assumption that Φ is nondecreasing and nonnegative is not needed when the distribution function is bounded). Since also $\int_E u(x) dx = \int_0^\infty f(r) dr$, (1.4) follows from (1.1) when f is the distribution function of $u\chi_E$. Conversely, if (1.4) holds when $f \neq 0$ is a bounded nonnegative measurable function, then it holds when f is the distribution function of $u\chi_E$ above and so (1.1) follows by reversing the steps.¹

¹That $\Phi(0) = 0$ is not restrictive since (1.1) is unaffected by changing Φ into $\Phi - \Phi(0)$.

However, since distribution functions are nonincreasing, the above does not prove (1.4) when $f \geq 0$ is merely in $L^\infty(0, \infty)$. Likewise, if u is not positive on E , (1.4) with f the distribution function of $u\chi_E$ does not imply (1.1). Therefore, in spite of some connection, neither (1.4) nor the Jensen inequality implies the other in general. The inequality (1.3) is only related to Jensen's inequality because of its connection with (1.4), so that the relationship is even more tenuous.

It is worth pointing out that functions in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ are commonplace in the context of Sobolev spaces. Indeed, if $f \in W^{1,p}(\mathbb{R}^N)$ with $N < p < \infty$, then $f \in L^\infty(\mathbb{R}^N)$, so that $|f|^p \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. In fact, $\|f\|_\infty \leq C_{p,N} \|f\|_p^{1-\frac{N}{p}} |f|_{1,p}^{\frac{N}{p}}$, where $|f|_{1,p} := \|\nabla f\|_p$ is the $W^{1,p}$ seminorm and $C_{p,N} > 0$ is a constant (for example, $C_{p,N} = (\frac{p}{N})^{\frac{N}{p}} \frac{p}{p-N}$). Since $\| |f|^p \|_\infty = \|f\|_\infty^p$ and $|f|^p \in L^1(\mathbb{R}^N)$, it follows from (1.3) for $|f|^p$ and from the remark that $\lambda \Phi(\frac{a}{\lambda})$ is a nonincreasing function of λ for every $a > 0$ (because $\Phi(0) = 0$) that

$$(1.5) \quad M_{p,N} \|f\|_p^{p-N} |f|_{1,p}^N \Phi \left(\frac{\|f\|_p^N}{M_{p,N} |f|_{1,p}^N} \right) \leq \int_{\mathbb{R}^N} |f(x)|^p \Phi'(|x|^N) dx,$$

for every $f \in W^{1,p}(\mathbb{R}^N)$, $f \neq 0$, where $M_{p,N} := C_{p,N}^p \omega_N$ is a constant. This is the exact analog of (1.4) with f replaced by $|f|^p$, $\|f\|_1$ replaced by $\|f\|_p^p$ and the ratio $\frac{\|f\|_1}{\omega_N \|f\|_\infty}$ replaced by $\frac{\|f\|_p^N}{M_{p,N} |f|_{1,p}^N}$.

Since there are as many variants of (1.5) as there are settings in which the L^∞ norm can be controlled by some combination of other norms or seminorms, we shall not elaborate further and focus solely on (1.3).

The properties of convex functions of one variable can of course be found in a number of sources. One of the most complete expositions is given in Chapter 1 of [6], which contains almost everything needed here.

2. PROOF OF THE INTEGRAL INEQUALITIES

We shall prove the integral inequality (1.4) and its generalization (1.3) to $\mathbb{R}^N$. The following property will be used: If $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is convex and continuous, then Φ' (defined at all but countably many points of $(0, \infty)$; see [6]) is in $L^1(0, a)$ for every $a > 0$ and $\int_0^a \Phi'(r) dr = \Phi(a) - \Phi(0)$. This is of course well known but rarely emphasized (most expositions focus on the properties of convex functions on open intervals without mentioning the importance of the continuity at the endpoint when the interval is semi-open).

The argument is that Φ is bounded on $[0, a]$ and (by convexity) either monotone on $[0, a]$ or monotone on two consecutive intervals $[0, b]$ and $[b, a]$ with $0 < b < a$, so that $\Phi' \in L^1(0, a)$ by the integrability of the derivatives of bounded monotone functions. This implies $\int_0^a \Phi'(r) dr = \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^a \Phi'(r) dr = \Phi(a) - \Phi(\varepsilon)$, because Φ is absolutely continuous (even Lipschitz) on $[\varepsilon, a]$ if $0 < \varepsilon < a$. Since Φ is continuous at 0, this shows that $\int_0^a \Phi'(r) dr = \Phi(a) - \Phi(0)$, as claimed.

Also, recall that the right derivative Φ'_+ is defined and finite at every point of $(0, \infty)$ and nondecreasing. Since it coincides with Φ' whenever Φ' is defined, it follows that $\Phi'_+ \in L^1(0, a)$.

In 1918, Steffensen [17] published the following rather awkward inequality: If $f, g : [\alpha, \beta] \rightarrow \mathbb{R}$, $0 \leq f \leq 1$ and g is strictly decreasing, then

$$(2.1) \quad \int_{\alpha}^{\beta} f(r)g(r)dr \leq \int_{\alpha}^{\alpha+\gamma} g(r)dr \text{ with } \gamma := \int_{\alpha}^{\beta} f(r)dr.$$

Because it appeared in an actuarial journal, this inequality was mostly unnoticed for at least four decades and did not make its way into Hardy, Littlewood and Pólya's treatise [5], even though related discrete inequalities are discussed there. Lemma 2.1 below is a modern formulation of Steffensen's inequality, as will be explained shortly.

Lemma 2.1. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be convex and continuous with $\Phi(0) = 0$. If $a > 0$ and $f \in L^{\infty}(0, a)$, $f \geq 0$ and $\|f\|_{\infty} \leq 1$, then $f\Phi' \in L^1(0, a)$ and*

$$(2.2) \quad \Phi\left(\int_0^a f(r)dr\right) \leq \int_0^a f(r)\Phi'(r)dr.$$

Proof. That $f\Phi' \in L^1(0, a)$ follows from $f \in L^{\infty}(0, a)$ and $\Phi' \in L^1(0, a)$. To prove (2.2), assume first that $f > 0$ a.e., so that $F(r) := \int_0^r f(t)dt$ is well defined (since $L^{\infty}(0, a) \subset L^1(0, a)$) and strictly increasing. Next, $\Phi\left(\int_0^a f(r)dr\right) = \Phi(F(a)) = \int_0^{F(a)} \Phi'(s)ds = \int_0^{F(a)} \Phi'_+(s)ds$. Since F is absolutely continuous with $F' = f$ and strictly increasing and since $\Phi'_+ \in L^1(0, a)$, the change of variable $s := F(r)$ yields $\Phi\left(\int_0^a f(r)dr\right) = \int_0^a \Phi'_+(F(r))f(r)dr$. Now, $F(r) \leq r$ from the assumption $\|f\|_{\infty} \leq 1$, so that $\Phi'_+(F(r)) \leq \Phi'_+(r)$ if $r > 0$ by the monotonicity of Φ'_+ . Thus, $\Phi\left(\int_0^a f(r)dr\right) \leq \int_0^a \Phi'(r)f(r)dr$ since $f > 0$ and $\Phi'_+ = \Phi'$ a.e. This proves (2.2) when $f > 0$ a.e. on $(0, a)$.

If $f \geq 0$ (and $\|f\|_{\infty} \leq 1$), let $\varepsilon > 0$ be given and set $f_{\varepsilon}(r) := \frac{f(r) + \varepsilon a}{1 + \varepsilon a}$, so that $0 < f_{\varepsilon} \leq 1$ a.e. on $(0, a)$. From the above,

$$\Phi\left(\int_0^a f_{\varepsilon}(r)dr\right) = \Phi\left(\frac{\left(\int_0^a f(r)dr\right) + \varepsilon a^2}{1 + \varepsilon a}\right) \leq \frac{1}{1 + \varepsilon a} \left(\int_0^a \Phi'(r)f(r)dr + \varepsilon a\Phi(a)\right).$$

When $\varepsilon \rightarrow 0$, the left-hand side tends to $\Phi\left(\int_0^a f(r)dr\right)$ by the continuity of Φ and the right-hand side tends to $\int_0^a f(r)\Phi'(r)dr$. This completes the proof. $\square$

Remark 2.1. In the above proof, the approximation by f_{ε} can be avoided by using two results of Serrin and Varberg [16]: First, by [16, Corollary 6], $\int_0^{F(a)} \Phi'(s)ds = \int_0^a \Phi'(F(r))f(r)dr$ since $\Phi' \in L^1(0, F(a))$ and F is nondecreasing when $f \geq 0$ (i.e., F need not be strictly increasing). Next, by [16, Theorem 2], $\int_0^a \Phi'(F(r))f(r)dr = \int_0^a \Phi'_+(F(r))f(r)dr$ since $\Phi' = \Phi'_+$ a.e.

Formally at least, (2.2) also follows by choosing $\alpha = 0$, $\beta = a$ and $g = -\Phi'$ in Steffensen's inequality (2.1). Conversely, (2.1) follows from (2.2) with $a := \beta - \alpha$, $\Phi(r) := -\int_{\alpha}^{\alpha+r} g(s)ds$ (so that $\Phi'(r) = -g(\alpha + r)$; extend g by $g(s) := g(\beta)$ for $s > \beta$ so that Φ is defined and convex on $[0, \infty)$) and $f(r)$ changed into $f(\alpha + r)$.

Steffensen's inequality resurfaced in another paper [18] of his, 29 years after its first publication and it took another 10 years or so before it caught Bellman's attention in² [2]. Since then, a few hundred papers have been devoted to various

²Where a flawed generalization is given. Several authors published corrections; see for example [13].

aspects and generalizations of it. See for instance Mitrinović [11] or, more recently, Pečarić *et al.* [14] and the references therein.

Paradoxically, even in the more recent literature ([9] (corrected in [8]), [10], [12], [19], among many others), we were unable to find any explicit record of the arguably clearer formulation (2.2), let alone its natural extension to the infinite interval $(0, \infty)$, proved next in Theorem 2.2. However, the case $\beta = \infty$ in (2.1) is treated in Apéry³ [1], apparently unaware of Steffensen's result.

In addition to the properties mentioned at the beginning of this section, recall the simple fact that if Φ is a convex function on $[0, \infty)$, then $\Phi(\infty) := \lim_{r \rightarrow \infty} \Phi(r)$ exists, possibly $\pm\infty$. Furthermore, if $\Phi(\infty) \in \mathbb{R}$, then $\Phi(\infty) = \inf_{r \geq 0} \Phi(r)$.

Theorem 2.2. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be convex and continuous with $\Phi(0) = 0$. If $f \in L^\infty(0, \infty)$, $f \geq 0$ and $\|f\|_\infty \leq 1$, then $\int_0^\infty f(r)\Phi'(r)dr$ is well defined in $\mathbb{R} \cup \{\pm\infty\}$ and*

$$(2.3) \quad \Phi\left(\int_0^\infty f(r)dr\right) \leq \int_0^\infty f(r)\Phi'(r)dr.$$

Proof. If $\Phi' \leq 0$, then $f\Phi' \leq 0$ is measurable, so that $\int_0^\infty f(r)\Phi'(r)dr \leq 0$ is well defined, possibly $-\infty$. Otherwise, since $\Phi' = \Phi'_+$ a.e. and Φ'_+ is defined, finite and nondecreasing on $(0, \infty)$, there is $a \in \mathbb{R}$ such that $\Phi'_+(a) > 0$, whence $\Phi' > 0$ a.e. on (a, ∞) . Thus, $f\Phi' \geq 0$ a.e. on (a, ∞) , so that $\int_a^\infty f(r)\Phi'(r)dr \geq 0$ is well defined, possibly ∞ . Since $f \in L^\infty(0, a)$ and $\Phi' \in L^1(0, a)$, it follows that $\int_0^a f(r)\Phi'(r)dr$ is also well defined and finite. As a result, $\int_0^\infty f(r)\Phi'(r)dr = \int_0^a f(r)\Phi'(r)dr + \int_a^\infty f(r)\Phi'(r)dr$ is well defined, possibly ∞ .

We now prove (2.3). Let $f_n := f\chi_{(0, n)}$, so that $0 \leq f_n \leq 1$ and (f_n) is a nondecreasing sequence tending to f a.e. By the monotone convergence theorem, $\lim_{n \rightarrow \infty} \int_0^\infty f_n(r)dr = \int_0^\infty f(r)dr$, possibly ∞ . Therefore, $\lim_{n \rightarrow \infty} \Phi\left(\int_0^\infty f_n(r)dr\right) = \Phi\left(\int_0^\infty f(r)dr\right)$ (possibly $\pm\infty$) and, if we show that $\lim_{n \rightarrow \infty} \int_0^\infty f_n(r)\Phi'(r)dr = \int_0^\infty f(r)\Phi'(r)dr$, then (2.3) follows from (2.2) with f replaced by f_n by taking the limit as $n \rightarrow \infty$.

If $\Phi' \leq 0$, then $(f_n\Phi')$ is a nonincreasing sequence of nonpositive functions tending to $f\Phi'$ a.e., so that $\lim_{n \rightarrow \infty} \int_0^\infty f_n(r)\Phi'(r)dr = \int_0^\infty f(r)\Phi'(r)dr$ (possibly $-\infty$) by the monotone convergence theorem. Otherwise, with $a > 0$ as above, note that $\int_0^a f_n(r)\Phi'(r)dr = \int_0^a f(r)\Phi'(r)dr \in \mathbb{R}$ (because $f \in L^\infty(0, a)$ and $\Phi' \in L^1(0, a)$) as soon as $n \geq a$. On the other hand, since $\Phi' > 0$ on (a, ∞) , the monotone convergence theorem shows that $\lim_{n \rightarrow \infty} \int_a^\infty f_n(r)\Phi'(r)dr = \int_a^\infty f(r)\Phi'(r)dr \geq 0$, possibly ∞ . Thus, $\lim_{n \rightarrow \infty} \int_0^\infty f_n(r)\Phi'(r)dr$ exists and equals $\int_0^\infty f(r)\Phi'(r)dr$. $\square$

By simply changing f into $\|f\|_\infty^{-1}f$ in Theorem 2.2, we obtain

Corollary 2.3. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be convex and continuous with $\Phi(0) = 0$. If $f \in L^\infty(0, \infty)$, $f \geq 0$ and $f \neq 0$, then $\int_0^\infty f(r)\Phi'(r)dr$ is well defined in $\mathbb{R} \cup \{\pm\infty\}$ and*

$$(2.4) \quad \|f\|_\infty \Phi\left(\frac{1}{\|f\|_\infty} \int_0^\infty f(r)dr\right) \leq \int_0^\infty f(r)\Phi'(r)dr.$$

In Corollary 2.3, the convexity of Φ cannot be weakened, at least when $\Phi(0) = 0$. Indeed:

³An algebraic geometer and number theorist who made no other published contribution to analysis.

Theorem 2.4. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be absolutely continuous on every finite interval $[0, a]$. If $\Phi(0) = 0$ and (2.4) holds for every nonnegative step function $f \neq 0$ on $(0, \infty)$ with bounded support, then Φ is convex.*

Proof. We first show that (2.4) implies (2.3) when, in addition, $\|f\|_\infty \leq 1$. Let $a > 0$ be such that $f = 0$ on (a, ∞) and, given $n \in \mathbb{N}$ large enough that $\frac{1}{n} < a$, define $f_n(r) := 1$ on $[0, \frac{1}{n}]$ and $f_n = f$ on $(\frac{1}{n}, \infty)$. Then, f_n is a step function, $\|f_n\|_\infty = 1$ and $f_n = 0$ on (a, ∞) , so that (2.4) for f_n yields

$$\Phi\left(\frac{1}{n} + \int_{\frac{1}{n}}^a f(r)dr\right) \leq \Phi\left(\frac{1}{n}\right) + \int_{\frac{1}{n}}^a f(r)\Phi'(r)dr.$$

The right-hand side is finite since $\Phi' \in L^1(0, a)$ and $f \in L^\infty(0, a)$, so that (2.3) follows by letting $n \rightarrow \infty$.

Now, let $0 \leq a < b < c$ be arbitrary. If $f(r) := \chi_{(0,a)} + \frac{b-a}{c-a}\chi_{(a,c)}$, then $f \geq 0$, $\|f\|_\infty \leq 1$ and $f \neq 0$. By (2.3), $\Phi(b) \leq \Phi(a) + \frac{b-a}{c-a}(\Phi(c) - \Phi(a))$, i.e., $\frac{\Phi(b) - \Phi(a)}{b-a} \leq \frac{\Phi(c) - \Phi(a)}{c-a}$. Thus, Φ is convex. $\square$

It is easily checked that (2.4) also implies $\Phi(0) \leq 0$. If $\Phi(0) < 0$, then Φ need not be convex (example: $\Phi(r) := -r - \frac{1}{r+1}$), but then (2.4) is not sharp when $\|f\|_1$ is small, if at all.

The last step is to extend Corollary 2.3 to functions f on $\mathbb{R}^N$. The few existing generalizations of the Steffensen inequality to higher dimension or to arbitrary measure spaces ([3], [4], [7], [15]) have no connection with the one given in Theorem 2.5 below. Recall that ω_N is the volume of the unit ball of $\mathbb{R}^N$, so that $N\omega_N$ is the area of the unit sphere $\mathbb{S}^{N-1}$.

The proof below involves changing variables through diffeomorphisms of $\mathbb{R}^N \setminus \{0\}$ (e.g., spherical coordinates) in integrals that may be $\pm\infty$. This is always justified because they are either integrals of (measurable) functions with a constant sign, or the sum of an integrable function on a ball centered at 0 and a function with a constant sign on the exterior of that ball.

Theorem 2.5. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be convex with $\Phi(0) = 0$. If $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$, then $\int_{\mathbb{R}^N} |f(x)|\Phi'(|x|^N)dx$ is well defined in $\mathbb{R} \cup \{\infty\}$ and*

$$(2.5) \quad \omega_N \|f\|_\infty \Phi\left(\frac{\|f\|_1}{\omega_N \|f\|_\infty}\right) \leq \int_{\mathbb{R}^N} |f(x)|\Phi'(|x|^N)dx.$$

Proof. We shall actually prove (2.5) when $f \in L^\infty(\mathbb{R}^N)$, with the understanding that $\|f\|_1 = \infty$ if $f \notin L^1(\mathbb{R}^N)$. Since $f \in L^1_{loc}(\mathbb{R}^N)$, the spherical mean $f_S(r) := \frac{1}{N\omega_N} \int_{\mathbb{S}^{N-1}} |f(r\sigma)|d\sigma$ is well defined for a.e. $r > 0$ and, by Fubini's theorem, $f_S \geq 0$ is measurable. Furthermore, it is readily seen that $|f|$ is essentially bounded by $\|f\|_\infty$ on a.e. sphere centered at the origin, so that $\|f_S\|_\infty \leq \|f\|_\infty$. Also, $f_S \neq 0$, for otherwise $0 = N\omega_N \int_0^\infty f_S(r)r^{N-1}dr = \|f\|_1$, which contradicts $f \neq 0$.

By Corollary 2.3 for f_S and the use of spherical coordinates,

$$(2.6) \quad \|f_S\|_\infty \Phi\left(\frac{1}{N\omega_N \|f_S\|_\infty} \int_{\mathbb{R}^N} |x|^{1-N} |f(x)|dx\right) \leq \int_0^\infty f_S(r)\Phi'(r)dr \\ = \frac{1}{N\omega_N} \int_{\mathbb{R}^N} |x|^{1-N} |f(x)|\Phi'(|x|)dx.$$

Since $\lambda\Phi\left(\frac{a}{\lambda}\right)$ is a nonincreasing function of $\lambda > 0$ for every $a > 0$ (with $a := \frac{1}{N\omega_N} \int_{\mathbb{R}^N} |x|^{1-N} |f(x)| dx$), it follows from $\|f_S\|_\infty \leq \|f\|_\infty$ that

$$(2.7) \quad \|f\|_\infty \Phi\left(\frac{1}{N\omega_N \|f\|_\infty} \int_{\mathbb{R}^N} |x|^{1-N} |f(x)| dx\right) \leq \frac{1}{N\omega_N} \int_{\mathbb{R}^N} |x|^{1-N} |f(x)| \Phi'(|x|) dx$$

if $\int_{\mathbb{R}^N} |x|^{1-N} |f(x)| dx < \infty$. This remains true if $\int_{\mathbb{R}^N} |x|^{1-N} |f(x)| dx = \infty$, for then the left-hand side of (2.6) is $\|f_S\|_\infty \Phi(\infty)$, which is unaffected by changing $\|f_S\|_\infty$ into $\|f\|_\infty$ if $\Phi(\infty) = \pm\infty$. If $\Phi(\infty) \in \mathbb{R}$, then $\Phi(\infty) = \inf_{r \geq 0} \Phi(r) \leq \Phi(0) = 0$, so that $\|f\|_\infty \Phi(\infty) \leq \|f_S\|_\infty \Phi(\infty)$. This proves (2.7) in all cases.

In (2.7), replace $f(x)$ by $f(x|x|^{\frac{1-N}{N}})$, which does not affect $\|f\|_\infty$. Note that $f(x|x|^{\frac{1-N}{N}})$ need not be in $L^1(\mathbb{R}^N)$ when $f \in L^1(\mathbb{R}^N)$, which is the reason why the latter assumption was dropped at the beginning of the proof. We get

$$\begin{aligned} \|f\|_\infty \Phi\left(\frac{1}{N\omega_N \|f\|_\infty} \int_{\mathbb{R}^N} |x|^{1-N} \left|f\left(x|x|^{\frac{1-N}{N}}\right)\right| dx\right) \\ \leq \frac{1}{N\omega_N} \int_{\mathbb{R}^N} |x|^{1-N} \left|f\left(x|x|^{\frac{1-N}{N}}\right)\right| \Phi'(|x|) dx, \end{aligned}$$

so that the change of variable $x = y|y|^{N-1}$ yields (observe that $dx = N|y|^{N(N-1)} dy$)

$$\|f\|_\infty \Phi\left(\frac{\|f\|_1}{\omega_N \|f\|_\infty}\right) \leq \frac{1}{\omega_N} \int_{\mathbb{R}^N} |f(y)| \Phi'(|y|^N) dy.$$

This proves (2.5) when $f \in L^\infty(\mathbb{R}^N)$, $f \neq 0$. If, in addition, $f \in L^1(\mathbb{R}^N)$, as assumed in the theorem, the left-hand side is finite, so that the right-hand side is either finite or ∞ (it cannot be $-\infty$). This completes the proof. $\square$

A variety of other inequalities can be derived from (2.5) by a change of function/variable. We give only one example: The function $g(x) := |x|^{-2N} f(|x|^{-2})$ has the same L^1 norm as f and the same L^∞ norm as $|x|^{2N} f$. Thus, from (2.5) for g ,

$$(2.8) \quad \omega_N \| |x|^{2N} f \|_\infty \Phi\left(\frac{\|f\|_1}{\omega_N \| |x|^{2N} f \|_\infty}\right) \leq \int_{\mathbb{R}^N} |f(x)| \Phi'(|x|^{-N}) dx$$

if $f \in L^1(\mathbb{R}^N)$ and $|x|^{2N} f \in L^\infty(\mathbb{R}^N)$, $f \neq 0$.

Remark 2.2. The use of (2.5) and (2.8) leads to inequalities that, sometimes, may appear to be inconsistent (for example, (3.2) and (3.3) later). That this is not the case follows from the inequality $\|f\|_1 \leq 2\omega_N \sqrt{\|f\|_\infty \| |x|^{2N} f \|_\infty}$. To see this, split $\int_{\mathbb{R}^N} |f|$ over a ball B_R and its complement, estimate and minimize with respect to R . Equality holds when $f(x) = 1$ on B_1 and $f(x) = |x|^{-2N}$ outside B_1 .

The following form of (2.5) may be more convenient in practice.

Corollary 2.6. *Let $\psi : (0, \infty) \rightarrow \mathbb{R}$ be nondecreasing and locally integrable near 0. Then, $\Phi_{\psi,N}(r) := \int_0^r \psi(t^{\frac{1}{N}}) dt$ is well defined and*

$$(2.9) \quad \omega_N \|f\|_\infty \Phi_{\psi,N}\left(\frac{\|f\|_1}{\omega_N \|f\|_\infty}\right) \leq \int_{\mathbb{R}^N} |f(x)| \psi(|x|) dx,$$

for every $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$.

Proof. The assumptions imply that ψ is (nondecreasing and) locally integrable on $[0, \infty)$ so that the same thing is true of $\psi(r^{\frac{1}{N}})$ (use $\int_0^r \psi(t^{\frac{1}{N}})dt = N \int_0^{r^{\frac{1}{N}}} \psi(s)s^{N-1}ds$). Thus, $\Phi_{\psi,N}(r)$ is well defined and $\Phi'_{\psi,N}(r) = \psi(r^{\frac{1}{N}})$ for a.e. $r > 0$, so that $\Phi_{\psi,N}$ is convex, continuous on $[0, \infty)$ and $\Phi_{\psi,N}(0) = 0$. Theorem 2.5 completes the proof. $\square$

3. EXAMPLES

3.1. If $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is convex and continuous, $\Phi(0) = 0$ and $E \subset \mathbb{R}^N$ is a measurable subset of finite measure $|E|$, then $f = \chi_E$ in (2.5) yields

$$\Phi\left(\frac{|E|}{\omega_N}\right) \leq \frac{1}{\omega_N} \int_E \Phi'(|x|^N) dx.$$

Equality holds if E is a ball B_R with radius R centered at the origin since the right-hand side is $\int_0^R \frac{d}{dr}(\Phi(r^N))dr = \Phi(R^N)$ and $R^N = \frac{|B_R|}{\omega_N}$. Thus, among all the measurable subsets E with measure $m < \infty$, the minimum of $\int_E \Phi'(|x|^N)dx$ is achieved when E is the ball centered at the origin with measure m . By Corollary 2.6, this is also true of the minimum of $\int_E \psi(|x|)dx$ whenever ψ is nondecreasing on $(0, \infty)$ and integrable near 0. This is easy to verify independently when $N = 1$, but a direct proof when $N \geq 2$ seems more elusive.

3.2. If $\beta \geq 0$ and $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$, then

$$(3.1) \quad \int_{\mathbb{R}^N} |f(x)| |x|^\beta dx \geq \frac{N\omega_N^{-\frac{\beta}{N}}}{\beta + N} \|f\|_1^{1+\frac{\beta}{N}} \|f\|_\infty^{-\frac{\beta}{N}}.$$

This follows from (2.5) with $\Phi(r) := r^{1+\frac{\beta}{N}}$. By Example 3.1, the constant $\frac{N\omega_N^{-\frac{\beta}{N}}}{\beta + N}$ is optimal (equality holds when f is the characteristic function of a ball centered at the origin). Naturally, the left-hand side is finite only when f is integrable for the measure $|x|^\beta dx$. Also, by (2.8),

$$(3.2) \quad \int_{\mathbb{R}^N} \frac{|f(x)|}{|x|^\beta} dx \geq \frac{N\omega_N^{-\frac{\beta}{N}}}{N + \beta} \|f\|_1^{1+\frac{\beta}{N}} \||x|^{2N} f\|_\infty^{-\frac{\beta}{N}}.$$

On the other hand, the choice $\Phi(r) := -r^{1-\frac{\beta}{N}}$ with $0 \leq \beta < N$ yields

$$(3.3) \quad \int_{\mathbb{R}^N} \frac{|f(x)|}{|x|^\beta} dx \leq \frac{N\omega_N^{\frac{\beta}{N}}}{N - \beta} \|f\|_\infty^{\frac{\beta}{N}} \|f\|_1^{1-\frac{\beta}{N}}$$

and

$$(3.4) \quad \int_{\mathbb{R}^N} |f(x)| |x|^\beta dx \leq \frac{N\omega_N^{\frac{\beta}{N}}}{N - \beta} \||x|^{2N} f\|_\infty^{\frac{\beta}{N}} \|f\|_1^{1-\frac{\beta}{N}}.$$

Remark 2.2 and $2^{\frac{2\beta}{N}} \leq \frac{N+\beta}{N-\beta}$ if $0 \leq \beta < N$ confirm that (3.2) and (3.3) as well as (3.1) and (3.4) are indeed compatible.

3.3. Let $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and let $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$. If $x = (x_1, x_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and if $\beta > 0$, then

$$\int_{\mathbb{R}^N} |f(x)| |x_1|^\beta dx \geq \frac{N_1 \omega_{N_1}^{-\frac{\beta}{N_1}}}{N_1 + \beta} \|f\|_1^{1+\frac{\beta}{N_1}} \|f_\# \|_\infty^{-\frac{\beta}{N_1}},$$

where $f_\#(x_1) := \int_{\mathbb{R}^{N_2}} |f(x_1, x_2)| dx_2$. This follows from (3.1) with N replaced by N_1 and f replaced by $f_\#$ (because $f_\# \in L^1(\mathbb{R}^{N_1})$ and the inequality is trivial if $f_\# \notin L^\infty(\mathbb{R}^{N_1})$). The left-hand side is finite only if f is integrable for the measure $|x_1|^\beta dx$.

3.4. If $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$, then

$$(3.5) \quad \int_{\mathbb{R}^N} \frac{|f(x)| dx}{1 + |x|^N} \leq \omega_N \|f\|_\infty \text{Log} (1 + \omega_N^{-1} \|f\|_1 \|f\|_\infty^{-1}).$$

This follows from (2.5) with $\Phi(r) = -\text{Log}(1 + r)$. This is better than the trivial $\int_{\mathbb{R}^N} \frac{|f(x)| dx}{1 + |x|^N} \leq \|f\|_1$. Since $(1 + |x|^N)^{-1}$ is not integrable, there is no obvious estimate involving $\|f\|_\infty$ instead of $\|f\|_1$ when f is bounded.

If $k \in \mathbb{N}$ divides N (in particular, $k = 1$), $\int_{\mathbb{R}^N} \frac{|f(x)| dx}{1 + |x|^k}$ can be estimated by an explicit calculation of the appropriate $\Phi_{\psi, N}$ in (2.9), but the general formulas are somewhat intricate.

3.5. Let $P_{N-1}(r)$ denote the polynomial $P_{N-1}(r) := \sum_{j=0}^{N-1} \frac{(-1)^{N-j-1} r^j}{j!}$, so that $P_{N-1}(r) + P'_{N-1}(r) = \frac{r^{N-1}}{(N-1)!}$. Then, $\Phi(r) := N!(P_{N-1}(r^{\frac{1}{N}})e^{r^{\frac{1}{N}}} - (-1)^{N-1})$ satisfies $\Phi(0) = 0$ and $\Phi'(r) = e^{r^{\frac{1}{N}}}$. By (2.5),

$$\begin{aligned} & \int_{\mathbb{R}^N} |f(x)| e^{|x|} dx \\ & \geq N! \omega_N \|f\|_\infty \left(P_{N-1} \left(\omega_N^{-\frac{1}{N}} (\|f\|_1 \|f\|_\infty^{-1})^{\frac{1}{N}} \right) e^{\omega_N^{-\frac{1}{N}} (\|f\|_1 \|f\|_\infty^{-1})^{\frac{1}{N}}} - (-1)^{N-1} \right), \end{aligned}$$

for every $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$. This is significantly more revealing than the obvious $\int_{\mathbb{R}^N} |f(x)| e^{|x|} dx \geq \|f\|_1$. For example, it shows that, when finite, $\int_{\mathbb{R}^N} |f(x)| e^{|x|} dx$ tends to ∞ exponentially in $(\|f\|_1 \|f\|_\infty^{-1})^{\frac{1}{N}}$ when $\|f\|_1 \|f\|_\infty^{-1} \rightarrow \infty$ and $\|f\|_\infty$ is bounded away from 0.

By similar arguments,

$$\begin{aligned} & \int_{\mathbb{R}^N} |f(x)| e^{-|x|} dx \\ & \leq N! \omega_N \|f\|_\infty \left(1 - Q_{N-1} \left(\omega_N^{-\frac{1}{N}} (\|f\|_1 \|f\|_\infty^{-1})^{\frac{1}{N}} \right) e^{-\omega_N^{-\frac{1}{N}} (\|f\|_1 \|f\|_\infty^{-1})^{\frac{1}{N}}} \right), \end{aligned}$$

where $Q_{N-1}(r) := \sum_{j=0}^{N-1} \frac{r^j}{j!}$. Note that (2.8) yields a lower estimate for $\int_{\mathbb{R}^N} |f(x)| e^{\frac{1}{|x|}} dx$ and an upper estimate for $\int_{\mathbb{R}^N} |f(x)| e^{-\frac{1}{|x|}} dx$ involving $\||x|^{2N} f\|_\infty$ instead of $\|f\|_\infty$.

3.6. With $\Phi(r) := r \operatorname{Log} r - r$ in (2.5), we obtain

$$(3.6) \quad \int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx \geq \frac{1}{N} \|f\|_1 \left(\operatorname{Log} \left(\frac{\|f\|_1}{\omega_N \|f\|_\infty} \right) - 1 \right),$$

for every $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \neq 0$. The right-hand side may be ∞ but it may also be (finite and) negative. However, among other things, (3.6) shows that $\int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx > 0$ as soon as $\|f\|_1 \|f\|_\infty^{-1} > e \omega_N$. This is optimal (as a condition about $\|f\|_1 \|f\|_\infty^{-1}$) since $\int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx \leq 0$ when f is the characteristic function of a ball centered at the origin with radius $R \leq e^{\frac{1}{N}}$.

Since $r \operatorname{Log} r - r \geq -1$, (3.6) is always better than the easy but little informative $-\frac{\omega_N}{N} \|f\|_\infty \leq \int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx$. Because $\operatorname{Log} |x|$ is not bounded above or below, there is no obvious lower or upper estimate of $\int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx$ involving $\|f\|_1$ alone. By (2.8), an upper estimate for $\int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx$ is given by (see Remark 2.2)

$$\int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx \leq \frac{1}{N} \|f\|_1 \left(1 - \operatorname{Log} \left(\frac{\|f\|_1}{\omega_N \| |x|^{2N} f \|_\infty} \right) \right).$$

3.7. This last example is of a different nature: Let $\lambda_k \in (0, 1]$, $1 \leq k \leq n$ and let $0 = a_0 \leq a_1 \leq \dots \leq a_n$. If $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is a continuous convex function, then

$$(3.7) \quad \Phi \left(\sum_{k=1}^n \lambda_k (a_k - a_{k-1}) \right) \leq \Phi(0) + \sum_{k=1}^n \lambda_k (\Phi(a_k) - \Phi(a_{k-1})).$$

Indeed, it is not restrictive to assume $\Phi(0) = 0$, so that (3.7) follows from (2.3) with $f := \sum_{k=1}^n \lambda_k \chi_{(a_{k-1}, a_k]}$ since $\|f\|_\infty \leq 1$. In fact, (3.7) is also equivalent to (and so provides another proof of) the so-called Jensen-Steffensen inequality ([14, p. 57]):

$$(3.8) \quad \Phi \left(\sum_{k=0}^n p_k a_k \right) \leq \sum_{k=0}^n p_k \Phi(a_k),$$

whenever $p_0 \in [0, 1]$, $p_n > 0$ and $P_j := \sum_{k=0}^j p_k \geq 0$, $0 \leq j \leq n$, $P_n = 1$. First, when $\Phi(0) = 0$, (3.7) readily implies (3.8) with

$$(3.9) \quad p_0 := 1 - \lambda_1, \quad p_k := \lambda_k - \lambda_{k+1}, \quad 1 \leq k \leq n-1, \quad p_n := \lambda_n.$$

Next, if $p_0, \dots, p_n$ satisfy the conditions listed below (3.8), it is straightforward that (3.9) holds for some (unique) $\lambda_k \in [0, 1]$, $1 \leq k \leq n$. Thus, (3.7) implies (3.8) when $\Phi(0) = 0$ (and $a_0 = 0$). But since $\sum_{j=0}^n p_j = 1$, it is obvious that (3.8) remains true irrespective of $\Phi(0)$ (replace Φ by $\Phi - \Phi(0)$) and also when $a_0 \geq 0$ is arbitrary (replace $\Phi(r)$ by $\Phi(r + a_0)$ and a_k by $a_k - a_0$, $0 \leq k \leq n$). That (3.8) implies (3.7) is clear, since it implies the convexity of Φ .

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