DIFFEOMORPHISMS WITH GLOBAL DOMINATED SPLITTINGS CANNOT BE MINIMAL

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ABSTRACT. Let M be a closed manifold and f be a diffeomorphism on M . We show that if f has a nontrivial dominated splitting $TM = E \oplus F$, then f cannot be minimal. The proof mainly uses Mañé's argument and Liao's selecting lemma.

1. INTRODUCTION

In [4] Herman constructed a (family of) C^∞ diffeomorphism(s) on a compact manifold that is minimal and has positive topological entropy simultaneously. Therefore, positive entropy is insufficient to guarantee the nonminimality. This draws forth the problem of finding some natural structure of the system that is incompatible with the minimality. In [7] Mañé gave an argument to locate some nonrecurrent point if the map admits some invariant expanding foliation (also see [1]). In particular, this argument shows that a partially hyperbolic diffeomorphism always has some nonrecurrent point and hence cannot be minimal. In this paper we show that a global dominated splitting is sufficient to exclude the minimality of the system.

Let M be a closed Riemannian manifold and $f : M \rightarrow M$ be a C^1 diffeomorphism on M . The map f is said to have a global dominated splitting on M if there exist an invariant splitting $TM = E \oplus F$ and two numbers $\lambda \in (0, 1)$ and $C \geq 1$ such that

$$(1.1) \quad \|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n x)}\| < C\lambda^n \text{ for all } n \geq 1, x \in M.$$

A Riemannian metric on M is said to be *adapted* to the dominated splitting if we can take $C = 1$ in (1.1) with respect to this metric. Adapted metrics always exist; see [3] for details.

Although the restriction of the existence of a dominated splitting is much weaker than that of a (partially) hyperbolic splitting, we show that the restriction is strong enough to exclude the possibility of minimality. Recall that the map $f : M \rightarrow M$ is said to be minimal if for each $x \in M$ the orbit $\mathcal{O}_f(x) = \{f^n x : n \in \mathbb{Z}\}$ is a dense subset in M . The following is our main result.

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Main Theorem. *Let M be a closed Riemannian manifold and $f : M \rightarrow M$ be a diffeomorphism on M . If f has a global dominated splitting, then f cannot be minimal.*

There are vast results in the case that the dimension of M is 2. Pujals and Sambarino gave several good characteristics in [8, 9] for the topology of invariant sets having some dominated splitting for C^2 diffeomorphisms on surfaces. On the other hand, Xia proved in [11] that for every compact surface M with nonzero Euler characteristic, any homeomorphism on M admits some periodic points. In [12] X. Zhang considered a diffeomorphism f on a closed surface and an f -invariant set Λ with a dominated splitting. Under some assumptions of Λ he showed that there exists a periodic orbit near Λ by Liao's selecting lemma and also showed some results about Crovisier's central models.

Our proof combines Mañé's argument and Liao's sifting lemma (or Liao's selecting lemma). Firstly we can find a nonrecurrent point if we are in the case to apply Mañé's argument. Secondly we can find a hyperbolic periodic point if we are in the case to apply Liao's lemmas. Besides these two cases we show that there always exists a proper f -invariant subset of M . Hence, f is not minimal and our Main Theorem follows.

2. PARTIAL HYPERBOLICITY AND QUASI-HYPERBOLIC STRINGS

In this section we review several useful results for later discussions. Let M be a closed Riemannian manifold and $f : M \rightarrow M$ be a C^1 diffeomorphism on M . Let $TM = E \oplus F$ be a dominated splitting on M . We always assume the Riemannian metric on M is chosen to be *adapted*. That is, there exists $\lambda \in (0, 1)$ such that

$$\|Df|_{E(x)}\| \cdot \|Df^{-1}|_{F(fx)}\| < \lambda, \text{ for all } x \in M.$$

In this case we say $TM = E \oplus F$ is a λ -dominated splitting. Observe that $\|Df^n|_{E(x)}\| \leq \prod_{k=0}^{n-1} \|Df|_{E(f^k x)}\|$ for all $n \geq 1$. We have similar observations for the subbundle F .

The first result we recall is an argument due to Mañé (see [7, Lemma 5.2]) which can locate a *nonrecurrent* point. Also see [1, Corollary 1].

Proposition 2.1. *Let f be a diffeomorphism on M and $\mathcal{W}$ be an f -invariant foliation tangent to a distribution $E \subset TM$ such that Df is uniformly expanding (or uniformly contracting) on E . Then there exists a nonrecurrent point of f . Moreover, the set $\{z \in M : z \notin \omega(z)\}$ of points that are nonrecurrent in the future is dense in every leaf of $\mathcal{W}$.*

Let's sketch the proof here. Without loss of generality we assume E is ν -expanding for some $\nu > 1$. First we show that each leaf $W(x)$ contains at most one periodic point. Suppose that there are two periodic points $p, q \in W(x)$ for some $x \in M$. Pick some $n \geq 1$ with $f^n p = p$ with $f^n q = q$. Let γ be a smooth curve in $W(x)$ connecting p and q with length $|\gamma| \leq \nu^{1/2} \cdot d_W(p, q)$. Then $W(x)$ is f^n -invariant and $f^{-n}\gamma$ is also a path connecting p and q in $W(x)$ with length $d_W(p, q) \leq |f^{-n}\gamma| \leq \nu^{-n} \cdot |\gamma| \leq \nu^{-1/2} \cdot d_W(p, q)$. This is impossible unless $p = q$. This shows that each leaf $W(x)$ contains at most one periodic point. Let $N \geq 1$ such that $\nu^N \geq 5$. Then for a nonperiodic point $y \in W(x)$ we pick $\epsilon > 0$ small enough such that $f^k B(y, \epsilon)$, $0 \leq k \leq N$, are pairwise disjoint. Then choose $\delta > 0$ to be much smaller than ϵ . We define inductively a sequence of closed disks $D_n \subset f^n W(y, \delta)$ for

$n \geq 1$ with $D_{n+1} \subset fD_n$ and $D_n \cap B(y, 2\delta) = \emptyset$. Then the intersection $\bigcap_{n \geq 1} f^{-n}D_n$ is nonempty, and each point $z \in \bigcap_{n \geq 1} f^{-n}D_n$ satisfies $f^n z \notin B(y, 2\delta)$ for all $n \geq 1$. Such z is nonrecurrent, and we finish the proof. See [1, Lemma 5.2] for more details about the construction of D_n .

The second one is Liao's *Sifting Lemma* (see [5, 6]), which helps us to locate some *periodic* point. Also see [10, Lemma 2.2].

Proposition 2.2. *Let $1 \leq I \leq d-1$ and Λ be a compact f -invariant subset of M with a λ -dominated splitting $T_\Lambda M = E \oplus F$ of index I . Assume the following:*

- (1) *There is a point $b \in \Lambda$ satisfying $\prod_{k=0}^{n-1} \|Df|_{E(f^k b)}\| \geq 1$ for all $n \geq 1$.*
- (2) *(The tilde condition.) There are λ_1 and λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$ such that if a point $x \in \Lambda$ satisfies $\prod_{k=0}^{n-1} \|Df|_{E(f^k x)}\| \geq \lambda_2^n$ for all $n \geq 1$, then the omega set $\omega(x)$ contains a point c satisfying $\prod_{k=0}^{n-1} \|Df|_{E(f^k c)}\| \leq \lambda_1^n$ for all $n \geq 1$.*

Then for each $\lambda_3 \in (\lambda_2, 1)$ and each $l \in \mathbb{N}$, there are l positive integers $n_1 < n_2 < \dots < n_l$ with the following property: for every $j = 1, \dots, l-1$ and every $k = n_j + 1, \dots, n_{j+1}$,

$$(2.1) \quad \prod_{i=n_j}^{k-1} \|Df|_{E(f^i b)}\| \leq \lambda_3^{k-n_j} \quad \text{and} \quad \prod_{i=k-1}^{n_{j+1}-1} \|Df|_{E(f^i b)}\| \geq \lambda_2^{n_{j+1}-k+1}.$$

Note that the point b in the first condition of Proposition 2.2 satisfies the assumption of the *tilde condition*. So the *tilde condition* is a nontrivial restriction of the proposition.

In the following we sketch how to find a hyperbolic periodic point near Λ . Proposition 2.2 shows that there are many 'double' uniform strings when $f^n b$ approaches some 'good' point $c \in \omega(b)$: for each $\lambda_3 \in (\lambda_2, 1)$ and each $l \in \mathbb{N}$, there are l positive integers $n_1 < n_2 < \dots < n_l$ with the following property: for every $j = 1, \dots, l-1$ and every $k = n_j + 1, \dots, n_{j+1}$,

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i b)}\| \leq \lambda_3^{k-n_j} \quad \text{and} \quad \prod_{i=k-1}^{n_{j+1}-1} \|Df|_{E(f^i b)}\| \geq \lambda_2^{n_{j+1}-k+1}.$$

Then by λ -domination assumption we have that

$$\prod_{i=k-1}^{n_{j+1}-1} \|Df^{-1}|_{F(f^{i+1} b)}\| \leq \prod_{i=k-1}^{n_{j+1}-1} \frac{\lambda}{\|Df|_{E(f^i b)}\|} \leq \left(\frac{\lambda}{\lambda_2}\right)^{n_{j+1}-k+1}$$

for every $k = n_j + 1, \dots, n_{j+1}$ and every $j = 1, \dots, l-1$. Let $\tilde{\lambda} = \max\{\sqrt{\lambda}, \lambda_3, \lambda/\lambda_2\} < 1$. Then the segment $(f^{n_j} b, f^{n_{j+1}} b) = \{f^k b : n_j \leq k \leq n_{j+1}\}$ forms a ' $\tilde{\lambda}$ -quasi-hyperbolic string' (see [2]) for each $j = 1, \dots, l-1$.

Let $L \geq 1$ and d_0 be given by [2, Theorem 1.1] with respect to $\tilde{\lambda}$. For $\epsilon \in (0, d_0]$ let's pick an integer $l = l(\epsilon) \geq 1$ large enough such that given arbitrary l points $x_1, \dots, x_l$ in M , there exist i and j with $1 \leq i < j \leq l$ such that $d(x_i, x_j) < \epsilon$. For this l we let $n_1 < \dots < n_l$ be given by Proposition 2.2 such that (2.1) holds. Then $d(f^{n_i} b, f^{n_j} b) < \epsilon$ for some $1 \leq i < j \leq l$. Therefore, $\{(f^{n_i} b, f^{n_{i+1}} b), (f^{n_{i+1}} b, f^{n_{i+2}} b), \dots, (f^{n_{j-1}} b, f^{n_j} b)\}$ forms a 'periodic $\tilde{\lambda}$ -quasi-hyperbolic ϵ -pseudo-orbit', and we can apply Liao-Gan's shadowing lemma (see [2, Theorem 1.1]) to find a hyperbolic periodic point x whose orbit $L\epsilon$ -shadows the

periodic pseudo-orbit $\{(f^{n_i}b, f^{n_{i+1}}b), (f^{n_{i+1}}b, f^{n_{i+2}}b), \dots, (f^{n_{j-1}}b, f^{n_j}b)\}$. Clearly this periodic orbit lies in the $L\epsilon$ -neighborhood of Λ . For more details see [2, 10]. Also see Liao's Selecting Lemma (for example [10, Lemma 2.3]) for more information.

3. DOMINATED SPLITTING AND MINIMALITY

With the preparations in Section 2 we prove the main theorem that if the map f has a global dominated splitting, then it cannot be minimal.

Proof of the Main Theorem. Let $TM = E \oplus F$ be a λ -dominated splitting for some $\lambda \in (0, 1)$ with respect to some adapted Riemannian metric. If F is uniformly expanding, then F is uniquely integrable and tangent to the strongly unstable foliation $\mathcal{W}^{su}$. By Proposition 2.1 there exists some nonrecurrent point of f , and hence f cannot be minimal on M . Similarly, if E is uniformly contracting, then f cannot be minimal either. Then we are left with the case that neither E is uniformly contracting nor F is uniformly expanding. In this case we have that:

- (1) since E is not uniformly contracting, there exists $p \in M$ such that $\|Df^n|_{E(p)}\| \geq 1$ for all $n \geq 1$,
- (2) since F is not uniformly expanding, there exists $q \in M$ such that $\|Df^{-n}|_{F(f^n q)}\| \geq 1$ for all $n \geq 1$.

We first observe that, by the λ -domination assumption, for all $n \geq 1$,

$$(3.1) \quad \prod_{k=0}^{n-1} \|Df|_{E(f^k q)}\| \leq \prod_{k=0}^{n-1} \frac{\lambda}{\|Df^{-1}|_{F(f^{k+1} q)}\|} \leq \frac{\lambda^n}{\|Df^{-n}|_{F(f^n q)}\|} \leq \lambda^n.$$

Also note that the first condition in Proposition 2.2 is already satisfied if we take $\Lambda = M$ and $b = p$ since $\prod_{k=0}^{n-1} \|Df|_{E(f^k p)}\| \geq \|Df^n|_{E(p)}\| \geq 1$ for each $n \geq 1$. Then we divide the discussion into two subcases:

Subcase I. The *tilde condition* in Proposition 2.2 holds on M for some λ_1, λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$. Then by Proposition 2.2 and Liao–Gan's shadowing lemma (Theorem 1.1 in [2]), there does exist a hyperbolic periodic point of f : the map f cannot be minimal.

Subcase II. The *tilde condition* fails. Hence, for each pair λ_1, λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$, there exists some point $\tilde{x} \in M$ such that

- $\prod_{k=0}^{n-1} \|Df|_{E(f^k \tilde{x})}\| \geq \lambda_2^n$ for all $n \geq 1$,
- for each $y \in \omega(\tilde{x})$, there exists some $n(y) \geq 1$ with $\prod_{k=0}^{n(y)-1} \|Df|_{E(f^k y)}\| \geq \lambda_1^{n(y)}$.

According to (3.1), we see that $q \notin \omega(\tilde{x})$ since $\lambda < \lambda_1$. Therefore, $\omega(\tilde{x}) \subsetneq M$ for some point $\tilde{x} \in M$, and the map f is not minimal either.

This finishes the verification for both subcases and ends the proof of the theorem. $\square$

Remark 3.1. The result is not true if we consider invariant subsets instead of the whole manifold, since there are various kinds of minimal subsets on which the map f has a dominated splitting with respect to the ambient system. For example, let

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $f_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the induced diffeomorphism. Let $R : \mathbb{T} \rightarrow \mathbb{T}$ be an irrational rotation. Then f_A is Anosov with a fixed point $o \in \mathbb{T}^2$ and R is minimal. Moreover, the product system $(R, f_A) : \mathbb{T} \times \mathbb{T}^2 \rightarrow \mathbb{T} \times \mathbb{T}^2$ is partially hyperbolic with an invariant minimal subset $\Lambda = \mathbb{T} \times \{o\}$.

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