QUANTITATIVE UNIQUENESS ESTIMATE FOR THE MAXWELL SYSTEM WITH LIPSCHITZ ANISOTROPIC MEDIA

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ABSTRACT. We study quantitative uniqueness estimates for the time harmonic Maxwell system with Lipschitz anisotropic media. Our main results are a three-balls inequality and a minimal vanishing rate at a point of any nontrivial solution. The proof relies on a Carleman estimate with a divergence term.

1. INTRODUCTION

In this paper we study local properties of solutions of the time-harmonic Maxwell system with anisotropic media

$$(1.1) \quad \begin{cases} \operatorname{curl} H &= i\omega \varepsilon E & \text{in } \Omega, \\ \operatorname{curl} E &= -i\omega \mu H & \text{in } \Omega. \end{cases}$$

Here Ω is an open subset of $\mathbb{R}^3$, $\omega \in \mathbb{C} \setminus \{0\}$ and $\varepsilon(x), \mu(x)$ are two real symmetric matrix-valued functions in Ω satisfying

$$(1.2) \quad \lambda |\xi|^2 \leq \varepsilon(x) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \lambda |\xi|^2 \leq \mu(x) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^3,$$

and

$$(1.3) \quad \|\varepsilon\|_{W^{1,\infty}(\Omega)} + \|\mu\|_{W^{1,\infty}(\Omega)} \leq M,$$

where $0 < \lambda < 1$ and M are positive constants.

We prove a quantitative uniqueness estimate in the form of a three balls inequality for (E, H) satisfying (1.1). As a by-product, we derive a minimal vanishing rate at an arbitrary point of Ω for any nontrivial (E, H) , which implies a unique continuation property for (1.1). When $\Omega = \mathbb{R}^3$, we also obtain a minimal decay rate of any nontrivial bounded (E, H) at infinity. We now state our main results. Let us denote $B_R(x_0) = \{|x - x_0| < R\}$.

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Theorem 1.1. *Assume that (1.2) and (1.3) hold. Then there exist $\rho, s > 0$, depending on λ, M , such that for any $(E, H) \in (L^2_{loc}(\Omega))^6$ solving (1.1) and $r_0 < r_1 < r_2/2 < \rho$ with $B_{r_2}(x_0) \subset \Omega$, we have*

$$(1.4) \quad \int_{B_{r_1}(x_0)} (|E|^2 + |H|^2) dx \leq C \left(\int_{B_{r_0}(x_0)} (|E|^2 + |H|^2) dx \right)^\tau \left(\int_{B_{r_2}(x_0)} (|E|^2 + |H|^2) dx \right)^{1-\tau},$$

where C depends on λ, M, r_1, r_2, s and

$$\tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}}.$$

In fact, the constant C is explicitly given by

$$C = \max\{C_{\lambda, M}, e^{2s[(r_1)^{-s} - (r_2/2)^{-s}]} \}$$

with $C_{\lambda, M}$ depending on λ and M .

Corollary 1.2. *Assume that the hypotheses in Theorem 1.1 hold. Let r_0, r_1, r_2 be described as above. Then for any nontrivial $(E, H) \in (L^2_{loc}(\Omega))^6$ satisfying (1.1),*

$$(1.5) \quad \int_{B_{r_0}(x_0)} (|E|^2 + |H|^2) dx \geq \exp(-Cr_0^{-s}) \int_{B_{r_2}(x_0)} (|E|^2 + |H|^2) dx,$$

where C depends on λ, M, r_1, r_2, s and

$$\frac{\int_{B_{r_2}(x_0)} (|E|^2 + |H|^2) dx}{\int_{B_{r_1}(x_0)} (|E|^2 + |H|^2) dx}.$$

From Corollary 1.2, it immediately follows that if Ω is connected and for some $x_0 \in \Omega$, (E, H) satisfies

$$\int_{B_r(x_0)} (|E|^2 + |H|^2) dx \leq C_N \exp(-Nr^{-s}), \quad \forall r \ll 1, \quad \forall N \in \mathbb{N},$$

then $E = H \equiv 0$ in Ω . This is the unique continuation property with exponential vanishing rate for (1.1). Note that we do not impose any structural assumptions on the matrices ε and μ at x_0 .

Similar to the result in [9], using Corollary 1.2, one can also study the minimal decay rate of any nontrivial bounded solution (E, H) to (1.1) with $\Omega = \mathbb{R}^3$. Denote

$$M_r(t) = \inf_{|x|=t} \int_{|y-x|<r} (|E|^2 + |H|^2) dy.$$

We have the following:

Corollary 1.3. *Assume that the hypotheses of Theorem 1.1 hold. Let $(E, H) \in (L^2_{loc}(\mathbb{R}^3))^6$ be a nontrivial solution to*

$$\begin{cases} \operatorname{curl} H &= i\omega \varepsilon E & \text{in } \mathbb{R}^3, \\ \operatorname{curl} E &= -i\omega \mu H & \text{in } \mathbb{R}^3. \end{cases}$$

Suppose that there exists $K > 0$ such that

$$\|E\|_{L^\infty(\mathbb{R}^3)} + \|H\|_{L^\infty(\mathbb{R}^3)} \leq K.$$

Then there exist ρ_0 depending on λ, M, K , and $c > 1$ depending on λ, M , and L depending on $\int_{B_r(0)} (|E|^2 + |H|^2) dx$, λ, M, r , such that for all $0 < r \leq \rho_0$,

$$(1.6) \quad M_r(t) \geq L^{t/r}.$$

We now mention some related results. When ε and μ are C^2 smooth, the unique continuation property was proved by Leis [6]. If ε and μ are C^1 smooth, the uniqueness of the Cauchy problem for (1.1) was established by Eller and Yamamoto [4]. Their result implies that if (E, H) vanishes in an open subset of Ω , then it vanishes identically in Ω . Our result is obviously an improvement of those in [6] and [4].

When both ε and μ are Lipschitz, the strong unique continuation property for (1.1) was proved by Ōkaji [10] and by Vogelsang [12]. A recent result by Colombini and Koch [3] also implies the strong unique continuation property for (1.1) when ε and μ are in the Gevrey class. However, it is important to point out that additional structural assumptions on ε and μ are required in [3], [10], and [12]. In view of the counterexamples by Alinhac [1], it seems that the strong unique continuation property for (1.1) may not hold without such assumptions.

As with previous unique continuation results for (1.1), our results are proved using a Carleman estimate, which we derive in Section 2. To handle the Lipschitz smoothness of coefficients, our Carleman estimate contains a divergence term. This type of Carleman estimate was first introduced for the Stokes equations in [5] and was also useful in treating the Lamé system with less regular coefficients [7], [8]. In Section 3, we first reduce the Maxwell system (1.1) to a weakly coupled second order elliptic system, following [6] and [2]. We then apply our Carleman estimate to this reduced system to obtain Theorem 1.1. The proof of Corollary 1.3 is given in Section 4.

2. CARLEMAN ESTIMATE

Denote $\varphi(x) = |x|^{-s} = r^{-s}$. The proof of our main result relies on the following Carleman estimate.

Proposition 2.1. *Let $P = \sum_{ij} a_{ij}(x) \partial_i \partial_j$ be a second order elliptic operator with Lipschitz coefficients $\{a_{ij}\}$ satisfying*

$$\lambda |\xi|^2 \leq \sum_{ij} a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2 \quad \forall x \in \mathbb{R}^3, \xi \in \mathbb{R}^3.$$

Assume that there exists $M > 0$ such that

$$\|a\|_{L^\infty(\mathbb{R}^3)} + \|\nabla a\|_{L^\infty(\mathbb{R}^3)} \leq M.$$

Then there exist constants $s_0 = s_0(\lambda, M)$ such that if $\beta \geq s \geq s_0$, $\epsilon > 0$ we can find $R = R(s, \lambda, M, \epsilon) \in (0, 1)$ so that for any $u \in H_0^1(B_R \setminus \{0\})$, $f \in L_0^2(B_R \setminus \{0\})^3$ with $Pu + \operatorname{div} f \in L^2(B_R \setminus \{0\})$, the following inequality holds:

$$(2.1) \quad \begin{aligned} & \beta^3 s^4 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx + \beta s^2 \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx \\ & \leq C_{\lambda, M} \int r^7 e^{2\beta\varphi} |Pu + \operatorname{div} f|^2 dx + C_{\lambda, M} \beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta\varphi} |f|^2 dx. \end{aligned}$$

Remark. It is clear that this estimate also holds when P is of divergence form, i.e., $P = \sum_{ij} \partial_i (a_{ij} \partial_j)$.

Proof. Using standard approximation arguments, it is easy to see that we can assume that $u \in C_0^\infty(B_R \setminus \{0\})$ and $\operatorname{div} f \in L_0^2(B_R \setminus \{0\})$. Let $m = s - 5$ and

$$v = r^{-m/2} e^{\beta\varphi} u, \quad g = r^{-m/2} e^{\beta\varphi} f.$$

Then we have

$$\begin{aligned} & r^{\frac{7}{2}} e^{\beta\varphi} (Pu + \operatorname{div} f) \\ &= A + B + D + r^{\frac{m+7}{2}} \operatorname{div} g + r^{\frac{m+7}{2}} \left(\frac{m}{2} r^{-2} + \beta s r^{-s-2} \right) x \cdot g, \end{aligned}$$

where

$$\begin{aligned} A &= r^{\frac{m+7}{2}} a_{ij} \partial_{ij} v, \\ B &= 2r^{\frac{m+7}{2}} \left(\frac{m}{2} r^{-2} + \beta s r^{-s-2} \right) a_{ij} \partial_i v x_j, \\ D &= r^{\frac{m+7}{2}} v a_{ij} q_{ij}, \\ q_{ij} &= x_i x_j \left(\beta^2 s^2 r^{-2s-4} - 7\beta s r^{-s-4} + \frac{m(m-4)}{4} r^{-s} \right) \\ &\quad + \delta_{ij} \left(\frac{m}{2} r^{-2} + \beta s r^{-s-2} \right). \end{aligned}$$

Let $w \in H_0^1(B_R \setminus \{0\})$ be a solution of

$$\partial_i (a_{ij} \partial_j w) = \operatorname{div} g$$

and let

$$\begin{aligned} A' &= r^{\frac{m+7}{2}} a_{ij} \partial_{ij} (v + w), \\ B' &= 2r^{\frac{m+7}{2}} \left(\frac{m}{2} r^{-2} + \beta s r^{-s-2} \right) a_{ij} x_j \partial_i (v + w). \end{aligned}$$

Then we have

$$\begin{aligned} & r^{\frac{7}{2}} e^{\beta\varphi} (Pu + \operatorname{div} f) - (A' + B' + D) \\ &= r^{\frac{m+7}{2}} \partial_i a_{ij} \partial_j w - 2r^{\frac{m+7}{2}} \left(\frac{m}{2} r^{-2} + \beta s r^{-s-2} \right) a_{ij} x_j \partial_i w \\ &\quad + r^{\frac{7}{2}} e^{\beta\varphi} \left(\frac{m}{2} r^{-2} + \beta s r^{-s-2} \right) x \cdot f. \end{aligned}$$

Using Lemma 2.2 below, it follows that

$$\begin{aligned} (2.2) \quad & \int \left| r^{\frac{7}{2}} e^{\beta\varphi} (Pu + \operatorname{div} f) - (A' + B' + D) \right|^2 dx \\ & \leq \beta^2 s^2 \int r^{-2s+5} e^{2\beta\varphi} |f|^2 dx + C_M \beta^2 s^2 \int r^{-s} |\nabla w|^2 dx \\ & \leq \beta^2 s^2 \int r^{-2s+5} e^{2\beta\varphi} |f|^2 dx + C_{\lambda, M} \beta^2 \int r^{-2s-\epsilon} |g|^2 dx \\ & \leq C_{\lambda, M} \beta^2 \int r^{-3s+5-\epsilon} e^{2\beta\varphi} |f|^2 dx. \end{aligned}$$

We have $\int |A' + B' + D|^2 dx \geq 2\operatorname{Re} \int (A'\overline{B'} + B'\overline{D}) dx$. First consider

$$\begin{aligned} I &:= 2\operatorname{Re} \int A'\overline{B'} dx \\ &= 4\operatorname{Re} \int a_{ij}a_{kl}x_j \left(\frac{m}{2}r^s + \beta s \right) \partial_i(v+w) \partial_{kl}(\overline{v+w}) dx \\ &= -4\operatorname{Re} \int \partial_l \left[\left(\frac{m}{2}r^s + \beta s \right) a_{kl}a_{ij}x_j \right] \partial_k(v+w) \partial_i(\overline{v+w}) dx \\ &\quad + 2\operatorname{Re} \int \partial_i \left[\left(\frac{m}{2}r^s + \beta s \right) a_{kl}a_{ij}x_j \right] \partial_k(v+w) \partial_l(\overline{v+w}) dx. \end{aligned}$$

Here in the last identity we have used integration by parts three times. By using elliptic regularity, we have

$$\begin{aligned} |I| &\leq C_M \beta s \int |\nabla(v+w)|^2 dx \leq C_M \beta s \int (|\nabla v|^2 + |\nabla w|^2) dx \\ &\leq C_M \beta s \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx + C_M \beta^3 s^3 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx \\ (2.3) \quad &+ C_M \beta s \int r^{-s+5} e^{2\beta\varphi} |f|^2 dx. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} II &:= 2\operatorname{Re} \int B'\overline{D} dx \\ &= 2\operatorname{Re} \int a_{ij}a_{kl} \left(\frac{m}{2}r^s + \beta s \right) q_{kl}x_j \partial_i(|v|^2) dx \\ &\quad + 4\operatorname{Re} \int \left(\frac{m}{2}r^s + \beta s \right) a_{ij}x_j \partial_i w a_{kl} q_{kl} \overline{v} dx \\ &= -2 \int |v|^2 \partial_i \left[a_{ij}a_{kl} \left(\frac{m}{2}r^s + \beta s \right) q_{kl}x_j \right] dx \\ &\quad + 4\operatorname{Re} \int \left(\frac{m}{2}r^s + \beta s \right) a_{ij}x_j \partial_i w a_{kl} q_{kl} \overline{v} dx \\ &=: II' + II''. \end{aligned}$$

When $\beta \geq s$ is large enough, depending on λ and M , the dominating term in II' is

$$\begin{aligned} &-2 \int |v|^2 \partial_i \left[a_{ij}a_{kl} \left(\frac{m}{2}r^s + \beta s \right) \beta^2 s^2 r^{-2s-4} x_k x_l x_j \right] dx \\ &\geq \beta^3 s^4 \int |v|^2 r^{-2s-6} a_{ij}a_{kl} x_i x_j x_k x_l \geq \lambda^2 \beta^3 s^4 \int r^{-2s-2} |v|^2 dx \\ (2.4) \quad &= \lambda^2 \beta^3 s^4 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx. \end{aligned}$$

Next, using Cauchy-Schwarz and Lemma 2.2 below, II'' can be bounded by

$$\begin{aligned} (2.5) \quad &\frac{\lambda^2 \beta^3 s^4}{4} \int r^{-2s-2} |v|^2 dx + C_{\lambda, M} \beta^3 s^2 \int r^{-2s} |\nabla w|^2 \\ &\leq \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx + C_{\lambda, M} \beta^3 s^2 \int r^{-2s-\epsilon} |g|^2 \\ &= \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx + C_{\lambda, M} \beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta\varphi} |f|^2. \end{aligned}$$

From (2.2)-(2.5), we get

$$(2.6) \quad \int r^7 e^{2\beta\varphi} |Pu + \operatorname{div} f|^2 dx \geq \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx \\ - C_{\lambda,M} \beta s \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx - C_{\lambda,M} \beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta\varphi} |f|^2 dx.$$

To get the $|\nabla u|^2$ term, we use integration by parts to derive

$$\int r^{-s+5} e^{2\beta\varphi} (Pu + \operatorname{div} f) \bar{u} dx = - \int r^{-s+5} e^{2\beta\varphi} a_{ij} \partial_i u \partial_j \bar{u} dx \\ - \int \partial_j (a_{ij} r^{-s+5} e^{2\beta\varphi}) \partial_i u \bar{u} dx - \int f \cdot \nabla (r^{-s+5} e^{2\beta\varphi} \bar{u}) dx.$$

Multiplying this with βs^2 and using Cauchy-Schwarz, we get

$$\lambda \beta s^2 \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx \leq \beta s^2 \int r^{-s+5} e^{2\beta\varphi} a_{ij} \partial_i u \partial_j \bar{u} dx \\ \leq \int r^7 e^{2\beta\varphi} |Pu + \operatorname{div} f|^2 dx + \frac{\lambda \beta s^2}{2} \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx \\ + C_{\lambda,M} \beta^3 s^4 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx + C_{\lambda,M} \beta s^2 \int r^{-s+5} e^{2\beta\varphi} |f|^2 dx,$$

which implies that

$$(2.7) \quad \int r^7 e^{2\beta\varphi} |Pu + \operatorname{div} f|^2 dx \geq \frac{\lambda \beta s^2}{2} \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx \\ - C_{\lambda,M} \beta^3 s^4 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx - C_{\lambda,M} \beta s^2 \int r^{-s+5} e^{2\beta\varphi} |f|^2 dx.$$

Multiplying (2.6) with $8\lambda^{-2}C_{\lambda,M}$, then adding (2.7), we get (2.1), provided $s > 4\lambda^{-1}C_{\lambda,M}$. $\square$

Now we prove a technical lemma used in the proof above.

Lemma 2.2. *Assume that the hypotheses in Proposition 2.1 hold. There exists $s_0 > 3$ such that for any $s > s_0$ and $\epsilon > 0$, there exists $R = R(s, \lambda, M, \epsilon) > 0$ such that if $w \in H_0^1(B_R \setminus \{0\})$, $g \in L_0^2(B_R \setminus \{0\})^3$ satisfying*

$$\partial_i (a_{ij} \partial_j w) = \operatorname{div} g,$$

then

$$(2.8) \quad \int r^{-2s} |\nabla w|^2 dx \leq \int r^{-2s-\epsilon} |g|^2 dx.$$

Proof. By an orthonormal change of coordinates, we can assume, without loss of generality, that $a_{ij}(0) = \alpha_i \delta_{ij}$, where $\lambda \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \lambda^{-1}$. Let

$$(2.9) \quad h_i(x) = -g_i(x) + [a_{ij}(x) - a_{ij}(0)] \partial_j w(x).$$

Then we get

$$(2.10) \quad \partial_i (a_{ij}(0) \partial_j w) + \operatorname{div} h = 0.$$

Let $x = \Lambda y$, where $\Lambda = \operatorname{diag}(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3})$, $\hat{w}(y) = w(\Lambda y)$, and $\hat{h}_i(y) = h_i(\Lambda y)/\sqrt{\alpha_i}$. Then (2.10) is equivalent to

$$\Delta_y \hat{w} + \operatorname{div}_y \hat{h} = 0,$$

which implies that

$$|y|^2 \Delta_y \widehat{w} + |y| \operatorname{div}_y(|y| \widehat{h}) = y \cdot \widehat{h}.$$

Using the Carleman inequality with a divergence term in [8, Lemma 2.1] (taking $n = 3$ there), we obtain

$$\begin{aligned} & s \int |y|^{-2s+1} |\log |y||^{-4s+2} |\nabla_y \widehat{w}|^2 dy \\ & \leq C \int |y|^{-2s+1} \left| |y|^2 \Delta_y \widehat{w} + |y| \operatorname{div}_y(|y| \widehat{h}) \right|^2 dy + Cs^2 \int |y|^{-2s-1} \left| |y| \widehat{h} \right|^2 dy \\ & \leq Cs^2 \int |y|^{-2s+1} |\widehat{h}|^2 dy. \end{aligned}$$

Here C is an absolute constant. Undoing the change of variables and using (2.9) we get

$$\begin{aligned} (2.11) \quad & \int |x|^{-2s+1} \left| \log \frac{|x|}{\sqrt{\lambda}} \right|^{-4s+2} |\nabla_x w|^2 \\ & \leq \frac{Cs}{\lambda^{2s+3}} \int |x|^{-2s+1} [|g|^2 + M^2 |x|^2 |\nabla_x w|^2]. \end{aligned}$$

Note that (2.11) holds true as long as s is sufficiently large and the supports of w and h are contained in $B_{\sqrt{\lambda}} \setminus \{0\}$. Choose $R = R(s, \lambda, \epsilon) < \sqrt{\lambda}$ small enough so that

$$\left| \log \frac{|x|}{\sqrt{\lambda}} \right|^{-4s+2} \geq \frac{2Cs}{\lambda^{2s+3}} |x|^\epsilon \quad \text{if } |x| \leq R.$$

Then

$$2 \int |x|^{-2s+1+\epsilon} |\nabla_x w|^2 \leq \int |x|^{-2s+1} [|g|^2 + M^2 |x|^2 |\nabla_x w|^2].$$

If furthermore, $M^2 R^{2-\epsilon} < 1$, then the second terms of the right-hand side will be absorbed by the left-hand side, and we obtain an equivalent form of (2.8). $\square$

3. THREE-BALL INEQUALITY

In this section, we prove the main result: the three-ball inequality in Theorem 1.1. We first reduce the Maxwell system (1.1) to a weakly coupled second order elliptic system, following [2, Lemma 1] and [6, page 168].

Denote

$$\gamma_{jl}^k = \begin{cases} 1 & \text{if } (k, j, l) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (k, j, l) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise,} \end{cases}$$

so that by (1.1),

$$\partial_k H = \nabla H_k + i\omega \gamma^k \varepsilon E \quad \text{and} \quad \partial_k E = \nabla E_k - i\omega \gamma^k \mu H.$$

Taking the divergence of (1.1), we get

$$\operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0;$$

hence for $k = 1, 2, 3$,

$$\begin{aligned} (3.1) \quad 0 &= \partial_k \operatorname{div}(\mu H) = \operatorname{div}(\mu \partial_k H) + \operatorname{div}(\partial_k \mu H) \\ &= \operatorname{div}(\mu \nabla H_k) + \operatorname{div}(\partial_k \mu H + i\omega \mu \gamma^k \varepsilon E). \end{aligned}$$

Similarly, we have

$$(3.2) \quad 0 = \operatorname{div}(\varepsilon \nabla E_k) + \operatorname{div}(\partial_k \varepsilon E - i\omega \varepsilon \gamma^k \mu H).$$

Together, (3.1) and (3.2) constitute our weakly coupled system.

Without loss of generality, we can assume $x_0 = 0$. Let $0 \leq \chi \leq 1$ be a cutoff function satisfying

- $\chi(x) = 1$ if $2r_0/3 < |x| < r_2/2$,
- $\chi(x) = 0$ if $|x| \leq r_0/2$ or $|x| \geq 2r_2/3$, and
- $|\partial^\alpha \chi(x)| \leq 10|x|^{-|\alpha|} \forall x$, for $|\alpha| = 1, 2$.

Let $U = (E, H) \in (L^2_{loc}(\Omega))^6$, $\tilde{E} = \chi E$, $\tilde{H} = \chi H$, $\tilde{U} = \chi U$. Then from (3.1) and (3.2) we obtain

$$\begin{aligned} |\operatorname{div}(\varepsilon \nabla \tilde{E}_k) + \operatorname{div}(\partial_k \varepsilon \tilde{E})| &\leq C_M |x|^{-2} (|U| + |\nabla U|), \\ |\operatorname{div}(\mu \nabla \tilde{H}_k) + \operatorname{div}(\partial_k \mu \tilde{H})| &\leq C_M |x|^{-2} (|U| + |\nabla U|). \end{aligned}$$

Let $\epsilon = 1$ and $R(s, \lambda, M, 1)$ be the constant given in Proposition 2.1. Note that when ε and μ are Lipschitz and $(E, H) \in (L^2_{loc}(\Omega))^6$, by the regularity theorem of [13], we have $(E, H) \in (H^1_{loc}(\Omega))^6$. Hence, if $\rho \leq R(s, \lambda, M, 1)$, we can apply the Carleman estimate (2.1) to $\tilde{E}$ and $\tilde{H}$ to obtain

$$\begin{aligned} &\beta^3 s^4 \int |x|^{-3s+3} e^{2\beta\varphi} |\tilde{U}|^2 dx + \beta s^2 \int |x|^{-s+5} e^{2\beta\varphi} |\nabla \tilde{U}|^2 dx \\ &\leq C_{\lambda, M} \int_{2r_0/3 < |x| < r_2/2} |x|^3 e^{2\beta\varphi} (|U|^2 + |\nabla U|^2) dx \\ &\quad + C_{\lambda, M} \int_{\{r_0/2 < |x| < 2r_0/3\} \cup \{r_2/2 < |x| < 2r_2/3\}} |x|^3 e^{2\beta\varphi} (|U|^2 + |\nabla U|^2) dx \\ (3.3) \quad &+ C_{\lambda, M} \beta^3 s^2 \int |x|^{-3s+4} e^{2\beta\varphi} |\tilde{U}|^2 dx. \end{aligned}$$

Choose $s = \max\{C_{\lambda, M}, s_0\}$. Then the first term of the right-hand side of (3.3) is absorbed by its left-hand side. For sufficiently small ρ , the third term of the right-hand side of (3.3) is also absorbed by the left-hand side. Consequently, we obtain

$$\begin{aligned} (3.4) \quad &\int_{2r_0/3 < |x| < r_1} e^{2\beta\varphi} |U|^2 dx \leq \int_{2r_0/3 < |x| < r_2/2} |x|^{-3s+3} e^{2\beta\varphi} |U|^2 dx \\ &\leq \int_{\{r_0/2 < |x| < 2r_0/3\} \cup \{r_2/2 < |x| < 2r_2/3\}} |x|^3 e^{2\beta\varphi} (|U|^2 + |\nabla U|^2) dx. \end{aligned}$$

It follows that

$$\begin{aligned} (3.5) \quad &e^{2\beta r_1^{-s}} \int_{2r_0/3 < |x| < r_1} |U|^2 dx \\ &\leq (r_0/2)^3 e^{2\beta(r_0/2)^{-s}} \int_{r_0/2 < |x| < 2r_0/3} (|U|^2 + |\nabla U|^2) dx \\ &\quad + (r_2/2)^3 e^{2\beta(r_2/2)^{-s}} \int_{r_2/2 < |x| < 2r_2/3} (|U|^2 + |\nabla U|^2) dx. \end{aligned}$$

Adding $e^{2\beta r_1^{-s}} \int_{|x| < 2r_0/3} |U|^2 dx$ to both sides of (3.5) and using a Caccioppoli-type inequality, we obtain

$$(3.6) \quad \begin{aligned} \int_{|x| < r_1} |U|^2 dx &\leq C_{\lambda, M} e^{2\beta[(r_0/2)^{-s} - r_1^{-s}]} \int_{|x| < r_0} |U|^2 dx \\ &+ C_{\lambda, M} e^{2\beta[(r_2/2)^{-s} - r_1^{-s}]} \int_{|x| < r_2} |U|^2 dx, \end{aligned}$$

for all $\beta \geq s$.

By standard arguments, we can then deduce that

$$\int_{|x| < r_1} |U|^2 dx \leq C \left(\int_{|x| < r_0} |U|^2 dx \right)^\tau \left(\int_{|x| < r_2} |U|^2 dx \right)^{1-\tau},$$

where

$$C = \max\{C_{\lambda, M}, e^{2s[(r_1)^{-s} - (r_2/2)^{-s}]} \} \quad \text{and} \quad \tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}}.$$

Corollary 1.2 is an easy consequence of the three-ball inequality (1.4). The arguments can be found in [7].

4. MINIMAL DECAY RATE AT INFINITY

In this section, we prove Corollary 1.3. Choose $\rho_0 < \rho/5$ small depending on K so that for any x_0 ,

$$\int_{B_{5\rho_0}(x_0)} (|E|^2 + |H|^2) dx \leq 1.$$

Then from the three-ball inequality (1.4) with $r_0 = r$, $r_1 = 2r$, $r_2 = 5r$ where $r \leq \rho_0$ we get

$$(4.1) \quad \int_{B_{2r}(x_0)} (|E|^2 + |H|^2) dx \leq C \left(\int_{B_r(x_0)} (|E|^2 + |H|^2) dx \right)^\tau,$$

where C and τ are defined as in Theorem 1.1.

If $|x_0 - x_1| \leq r$ so that $B_r(x_1) \subset B_{2r}(x_0)$, then we obtain from (4.1),

$$(4.2) \quad C^{-1/\tau} \left(\int_{B_r(x_1)} (|E|^2 + |H|^2) dx \right)^{1/\tau} \leq \int_{B_r(x_0)} (|E|^2 + |H|^2) dx.$$

Now if $|x_0| = t$, we can find a sequence of points $x_1, \dots, x_N = 0$ such that $|x_{j+1} - x_j| \leq r$ for $j = 0, \dots, N-1$, where $N \leq 1 + t/r$. Applying (4.2) repeatedly with

x_j and x_{j+1} in place of x_0 and x_1 , we get

$$\begin{aligned}
\int_{B_r(x_0)} (|E|^2 + |H|^2) dx &\geq C^{-1/\tau} \left(\int_{B_r(x_1)} (|E|^2 + |H|^2) dx \right)^{1/\tau} \\
&\geq C^{-1/\tau - 1/\tau^2} \left(\int_{B_r(x_2)} (|E|^2 + |H|^2) dx \right)^{1/\tau^2} \\
&\geq \dots \geq C^{-\frac{1-\tau^N}{\tau^N(1-\tau)}} \left(\int_{B_r(0)} (|E|^2 + |H|^2) dx \right)^{1/\tau^N} \\
&\geq \left(C^{-\frac{1}{1-\tau}} \int_{B_r(0)} (|E|^2 + |H|^2) dx \right)^{1/\tau^N}.
\end{aligned}$$

Noting that

$$\tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}} = \frac{4^{-s} - 5^{-s}}{1 - 5^{-s}}$$

is independent of r and $N \leq 1 + t/r$, we obtain (1.6) with $c = 1/\tau$ and

$$L = \left(C^{-\frac{1}{1-\tau}} \int_{B_r(0)} (|E|^2 + |H|^2) dx \right)^{1/\tau}.$$

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