STANLEY DEPTH AND SIZE OF A MONOMIAL IDEAL

JÜRGEN HERZOG, DORIN POPESCU, AND MARIUS VLADOIU

(Communicated by Irena Peeva)

ABSTRACT. Lyubeznik introduced the concept of size of a monomial ideal and showed that the size of a monomial ideal increased by 1 is a lower bound for its depth. We show that the size increased by 1 is also a lower bound for its Stanley depth. Applying Alexander duality we obtain upper bounds for the regularity and Stanley regularity of squarefree monomial ideals.

INTRODUCTION

Let $I \subset S$ be a monomial ideal, where $S = K[x_1, \dots, x_n]$ is the polynomial ring in the indeterminates $x_1, \dots, x_n$ over the field K . In [5] Lyubeznik showed that $\text{depth } I \geq 1 + \text{size } I$. In the case that I is a squarefree monomial ideal with minimal prime ideals $P_1, \dots, P_s$, the size of I is the number $v + (n - h) - 1$, where h is the height of $\sum_{j=1}^s P_j$ and v is the minimal number t for which there exist integers $i_1 < i_2 < \dots < i_t$ such that $\sum_{k=1}^t P_{i_k} = \sum_{j=1}^s P_j$. Replacing in the previous definition “there exist integers $i_1 < i_2 < \dots < i_t$ ” by “for all integers $i_1 < i_2 < \dots < i_t$ ”, one obtains the definition of bigsize, first considered by Popescu in [7]. See Section 1, where the general definition of size and bigsize is given.

Let $u \in I$ be a monomial and $uK[Z]$, $Z \subset \{x_1, \dots, x_n\}$ the linear K -subspace of I of all elements uf , $f \in K[Z]$. A presentation of I as a finite direct sum of such spaces $\mathcal{D} : I = \bigoplus_{i=1}^r u_i K[Z_i]$ is called a Stanley decomposition of I . Set $\text{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \dots, r\}$ and

$$\text{sdepth } I := \max\{\text{sdepth } (\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } I\}.$$

It is conjectured by Stanley that $\text{sdepth } I \geq \text{depth } I$. Assuming Stanley’s conjecture and combining it with Lyubeznik’s inequality one should expect that $\text{sdepth } I \geq 1 + \text{size } I$, where $\text{sdepth } I$ denotes the Stanley depth of I . In Section 3 we show that this is indeed the case; see Theorem 3.1. We also expect that $\text{sdepth } S/I \geq \text{size } I$. Assuming the conjectured inequality $\text{sdepth } I \geq 1 + \text{sdepth } S/I$, our inequality $\text{sdepth } I \geq 1 + \text{size } I$ would also follow from the inequality $\text{sdepth } S/I \geq \text{size } I$.

Received by the editors November 30, 2010.

2010 *Mathematics Subject Classification.* Primary 13C15; Secondary 13P10, 13F55, 13F20.

This paper was partially written during the visit of the first author at the Institute of Mathematics “Simion Stoilow” of the Romanian Academy supported by a BitDefender Invited Professor Scholarship, 2010.

The second and third authors were partially supported by the CNCSIS grant PN II-542/2009, respectively CNCSIS grant TE_46 no. 83/2010, of the Romanian Ministry of Education, Research and Innovation. They also want to express their gratitude to ASSMS of GC University Lahore for creating a very appropriate atmosphere for research work.

In Section 1 we introduce a cocomplex $\mathbb{G}$ which is attached to a set $\mathcal{P} = \{J_1, \dots, J_s\}$ of monomial ideals in S/I , where $I \subset S$ is a monomial ideal. This cocomplex is acyclic (Theorem 1.1), and in the case that $I = 0$ and the ideals J_k are irreducible monomial ideals, $\mathbb{G}$ may be viewed as the Alexander dual of the Taylor complex. The complex allows us in some cases to compute or to estimate the depth of $\bigcap_{k=1}^s J_k$. In terms of this complex, we give in Theorem 1.2 a criterion for a monomial ideal I to have minimal depth, that is, to satisfy the equation $\text{depth } I = 1 + \text{size } I$, which is in particular the case when $\text{bigsize } I = \text{size } I$. We conclude Section 1 by giving upper bounds for the regularity of a squarefree monomial ideal I in terms of the cosize of I . This upper bound is obtained by applying Alexander duality; see Corollary 1.4.

In Section 2 we describe the method of splitting the variables in order to obtain lower bounds for the Stanley depth of monomial ideals. This technique was first introduced by A. Popescu [6] in a special case, and then generalized by D. Popescu [7] for all squarefree monomial ideals. Here we further extend it to arbitrary monomial ideals, and use it in Section 3 to prove the inequality $\text{sdepth } I \geq 1 + \text{size } I$. We conclude the paper by giving upper bounds for the Stanley regularity in terms of the cosize of I .

1. SIZE, BIGSIZE AND DEPTH OF A SQUAREFREE MONOMIAL IDEAL

Let K be a field and $S = K[x_1, \dots, x_n]$ be the polynomial ring over K in the indeterminates $x_1, \dots, x_n$. Let $I \subset S$ be a squarefree monomial ideal, and $I = \bigcap_{i=1}^s P_i$ its presentation as an irredundant intersection of irreducible monomial ideals. This unique presentation establishes a bijection between monomial ideals and finite sets of irreducible monomial ideals, that is, monomial ideals of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \dots, x_{i_k}^{a_k})$, with no inclusions among them. By means of this intersection we would like to give some depth estimates for I .

More generally, let $\mathcal{P} = \{I_1, \dots, I_s\}$ be an arbitrary set of monomial ideals, and $I \subset S$ a monomial ideal. We set $T = S/I$. Given these data, we are going to introduce a (co)complex $\mathbb{G}$ of T -modules which allows us in some cases to compute the depth of $J = (\bigcap_{j=1}^s I_j + I)/I$. For $k = 1, \dots, s$, let $J_k = (I_k + I)/I \subset T$. Then the i th component G^i of $\mathbb{G}$ is given by

$$G^i = \bigoplus_{U \subset [s], |U|=i} T_U,$$

where $T_U = T/(\sum_{k \in U} J_k)$. The chain map from $G^i \rightarrow G^{i+1}$ is defined on the component $T_U \rightarrow T_V$ to be the canonical residue class map multiplied with $(-1)^\ell$, where ℓ is the number of elements $i \in U$ with $i < k$ if $V = U \cup \{k\}$, and to be the zero map if $U \not\subset V$.

For $s = 2$ and $I = 0$ the complex $\mathbb{G}$ is the standard complex

$$\begin{aligned} 0 &\longrightarrow S \longrightarrow S/I_1 \oplus S/I_2 \longrightarrow S/(I_1 + I_2) \longrightarrow 0 \\ a &\mapsto (a + I_1, a + I_2) \\ (a + I_1, b + I_2) &\mapsto (a - b) + I_1 + I_2. \end{aligned}$$

In general, $\mathbb{G}$ is of the form

$$0 \rightarrow T \rightarrow \bigoplus_{i=1}^s T_i \rightarrow \bigoplus_{i < j} T_{ij} \rightarrow \cdots \rightarrow T_{12 \cdots s} \rightarrow 0.$$

It is easy to see that $\mathbb{G}$ is indeed a complex. We call $\mathbb{G}$ the complex attached to the set $\mathcal{P} = \{J_1, \dots, J_s\}$ of monomial ideals in $T = S/I$. Now we have

Theorem 1.1. *Let $\mathbb{G}$ be the complex attached to $\mathcal{P} = \{J_1, \dots, J_s\}$. Then*

$$H_i(\mathbb{G}) = \begin{cases} J, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $J = \bigcap_{k=1}^s J_k$.

Proof. We prove the theorem by induction on s . The assertion is trivial for $s = 1$. Now let $s > 1$, and let $\mathbb{G}'$ be the complex attached to the set $\mathcal{Q} = \{J_1, \dots, J_{s-1}\}$. Furthermore, let $\mathbb{G}''$ be the complex attached to the set of monomial ideals

$$\mathcal{R} = \{(J_1 + J_s)/J_s, \dots, (J_{s-1} + J_s)/J_s\} \quad \text{in } T/J_s.$$

Then $\mathbb{G}''$ may be viewed as a subcomplex of $\mathbb{G}$, and we get a short exact sequence of complexes

$$0 \rightarrow \mathbb{G}''[-1] \rightarrow \mathbb{G} \rightarrow \mathbb{G}' \rightarrow 0,$$

where $\mathbb{G}''[-1]$ is the complex $\mathbb{G}''$ homologically shifted by -1 .

This short exact sequence gives rise to the following long exact sequence:

$$\cdots \rightarrow H_{i-1}(\mathbb{G}'') \rightarrow H_i(\mathbb{G}) \rightarrow H_i(\mathbb{G}') \rightarrow \cdots.$$

By our induction hypothesis we have $H_i(\mathbb{G}') = H_i(\mathbb{G}'') = 0$ for $i > 0$. This implies that $H_i(\mathbb{G}) = 0$ for $i > 1$, and in addition we get the exact sequence

$$0 \rightarrow H_0(\mathbb{G}) \rightarrow H_0(\mathbb{G}') \rightarrow H_0(\mathbb{G}'') \rightarrow H_1(\mathbb{G}) \rightarrow 0.$$

Our induction hypothesis implies that $H_0(\mathbb{G}') = \bigcap_{k=1}^{s-1} J_k$ and that

$$H_0(\mathbb{G}'') = \bigcap_{k=1}^{s-1} (J_k + J_s)/J_s.$$

The map $H_0(\mathbb{G}') \rightarrow H_0(\mathbb{G}'')$ is just the canonical map

$$\bigcap_{k=1}^{s-1} J_k \rightarrow \bigcap_{k=1}^{s-1} (J_k + J_s)/J_s.$$

Since the ideals J_k are all monomial ideals in $T = S/I$ it follows that $\bigcap_{k=1}^{s-1} (J_k + J_s) = (\bigcap_{k=1}^{s-1} J_k) + J_s$. This implies that $H_0(\mathbb{G}') \rightarrow H_0(\mathbb{G}'')$ is surjective, so that $H_1(\mathbb{G}) = 0$, and that $\text{Ker}(H_0(\mathbb{G}') \rightarrow H_0(\mathbb{G}'')) = J$, so that $H_0(\mathbb{G}) = J$, as desired. $\square$

Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^s Q_i$ an irredundant primary decomposition of I , where the Q_i are monomial ideals. Let Q_i be P_i -primary. Then each P_i is a monomial prime ideal and $\text{Ass}(S/I) = \{P_1, \dots, P_s\}$.

According to Lyubeznik [5, Proposition 2] the *size* of I , denoted $\text{size } I$, is the number $v + (n - h) - 1$, where v is the minimum number t such that there exist $j_1 < \cdots < j_t$ with

$$\sqrt{\sum_{k=1}^t Q_{j_k}} = \sqrt{\sum_{j=1}^s Q_j},$$

and where $h = \text{height } \sum_{j=1}^s Q_j$.

Notice that $\sqrt{\sum_{k=1}^t Q_{j_k}} = \sum_{k=1}^t P_{j_k}$ and $\sqrt{\sum_{j=1}^s Q_j} = \sum_{j=1}^s P_j$, so that the size of I depends only on the set of associated prime ideals of S/I .

If in the above definition of size I we replace “there exist $j_1 < \cdots < j_t$ ” by “for all $j_1 < \cdots < j_t$ ”, we obtain the definition of *bigsize* I , introduced by Popescu [7]. Of course $\text{bigsize } I \geq \text{size } I$, and in fact the *bigsize* of I is in general much bigger than the size of I .

To illuminate these concepts, consider the so-called squarefree Veronese ideal $I_{n,d} \subset S$ consisting of all squarefree monomials of degree d . It is known that the minimal prime ideals of $I_{n,d}$ are exactly all the monomial prime ideals of height $n - d + 1$ and that $\text{depth } I_{n,d} = d$. From this information one easily deduces that $\text{size } I_{n,d} = \lceil n/(n - d + 1) \rceil$ and $\text{bigsize } I_{n,d} = d - 1$. Thus we see that

$$\text{bigsize } I_{n,d} + 1 = \text{depth } I_{n,d} \geq \text{size } I_{n,d} + 1.$$

In general $\text{depth } I$ can be bigger or smaller than $\text{bigsize } I + 1$. The inequality $\text{depth } I_{n,d} \geq \text{size } I_{n,d} + 1$ is just a special case of Lyubeznik’s inequality [5, Proposition 2]. We say that I has *minimal depth* if equality holds, i.e. $\text{depth } I = \text{size } I + 1$. In the last section it will be shown that for a monomial ideal one has $\text{sdepth } I \geq 1 + \text{size } I$. Thus if I has minimal depth, then $\text{sdepth } I \geq \text{depth } I$, in which case Stanley’s conjecture holds.

In the next result a sufficient condition is given for a monomial ideal to achieve the lower bound for the depth, as given by Lyubeznik. Recall that a monomial ideal is irreducible if and only if it is generated by powers of subsets of the variables. Each monomial I has a unique presentation $I = \bigcap_{j=1}^s Q_j$ as an intersection of minimal irreducible monomial ideals. Moreover, $\text{Ass}(I) = \{P_1, \dots, P_s\}$, where $P_i = \sqrt{Q_i}$. For example, for $I = (x^2, xy, x^2z^2, xyz^2, y^2z^2)$ we have $I = (x^2, y) \cap (x, y^2) \cap (x, z^2)$ and $\text{Ass}(I) = \{(x, y), (x, z)\}$.

For the proof of the next result it is important to notice that any sum of monomial primary ideals is again primary, and hence in particular, for any subset $U \subset [s]$ we have $\dim S_U = \text{depth } S_U$, where as before, $S_U = S/(\sum_{j \in U} Q_j)$.

Theorem 1.2. *Let $I \subset S$ be a monomial ideal of size t and $I = \bigcap_{j=1}^s Q_j$ be an irredundant primary decomposition of I , and assume without loss of generality that $\sum_{j=1}^s Q_j$ is $\mathfrak{m}$ -primary, where $\mathfrak{m}$ is the graded maximal ideal of S .*

Let $\mathbb{G}$ be the complex attached to $\mathcal{P}$ in S . Let $\mathcal{V} = \{V : |V| = t + 1, \dim S_V = 0\}$, and let $W = \bigoplus_{V \in \mathcal{V}} S_V$. Assume that

- (a) $\text{Im}(G^t \rightarrow G^{t+1}) \cap W \neq (0)$;
- (b) *for all $U \subset [s]$ with $|U| = t$ there exists $V \in \mathcal{V}$ with $U \subset V$.*

Then I has minimal depth. In particular, the conditions of the theorem are satisfied if $\text{bigsize } I = \text{size } I$.

Proof. We may assume that $\sqrt{\sum_{j=1}^s Q_j} = \mathfrak{m}$, where $\mathfrak{m}$ is the graded maximal ideal of S . Let $U_{i+1} = \text{Im}(G^i \rightarrow G^{i+1})$. We are going to show that $\text{depth } U_i = t + 1 - i$. Then, since $U_1 = S/I$, the desired conclusion follows. The module U_{t+1} has a nontrivial intersection with W . Since each element of W is annihilated by a power of $\mathfrak{m}$, it follows from (a) that U_{t+1} contains a nonzero element which is annihilated by $\mathfrak{m}$. This shows that $\text{depth } U_{t+1} = 0$.

Next we will show by induction on i that $\text{depth } G^{t+1-i} \geq i$. (The desired formula for the depth of the U_i is then obviously a consequence of this fact.) For $i = 1$ the assertion follows from the assumption that $\text{size } I = t$. Let us now assume that $\text{depth } G^{t+1-i} \geq i$, but $\text{depth } G^{t+1-(i+1)} < i + 1$. Then there exists $U \subset [s]$ with $|U| = t - i$ and $\dim S_U \leq i$. It follows from condition (b) that there exists $V \subset [s]$ with $|V| = t + 1$ such that $U \subset V$ and $\dim S_V = 0$. Choose such a set V and an element $k \in V \setminus U$. Since $\text{depth } G^{t+1-i} \geq i$ it follows that $\dim S_{U \cup \{k\}} \geq i$. On the other hand, it is clear that $\dim S_U \geq \dim S_{U \cup \{k\}}$. Hence we conclude that $\dim S_U = \dim S_{U \cup \{k\}}$, which is only possible if $Q_k \subset \sqrt{\sum_{j \in U} Q_j}$. It follows that $\sqrt{\sum_{j \in V} Q_j} = \sqrt{\sum_{j \in V \setminus \{k\}} Q_j}$, contradicting the fact that $\text{size } I = t$. $\square$

Examples 1.3. (i) Let $I = (x_1, x_2, x_3) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_3, x_4)$. Then $\text{depth } I = 1 + \text{size } I = 2$, and hence I has minimal depth. In fact, I satisfies the hypotheses of Theorem 1.2. On the other hand, $\text{bigsize } I = 2 > \text{size } I$.

(ii) Let Δ be the simplicial complex on the vertex set $\{1, \dots, 6\}$, associated to the canonical triangulation of the real projective plane $\mathbb{P}^2$, whose facets are

$$\mathcal{F}(\Delta) = \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\}.$$

Then the Stanley-Reisner ideal of Δ (see definition below) is

$$I_\Delta = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

It is known that $\text{depth } I_\Delta = 4$ if $\text{char } K \neq 2$ and $\text{depth } I_\Delta = 3$ if $\text{char } K = 2$. On the other hand, it is easy to see that $\text{size } I_\Delta = 2$ and $\text{bigsize } I_\Delta = 5$. This shows that the property of a monomial ideal to have minimal depth may depend on the characteristic of the base field. It can also be easily checked that condition (b) of Theorem 1.2 is satisfied for the ideal I_Δ . On the other hand, since I does not have minimal depth if $\text{char } K \neq 2$, it follows that in this case condition (a) is not satisfied for I_Δ . In particular, (b) does not imply (a).

(iii) Let $I = (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_1, x_4, x_5, x_6) \cap (x_1, x_3, x_5, x_7) \cap (x_2, x_4, x_6, x_7)$. Then $\text{depth } I = 3$ and $\text{size } I = 1$. The ideal satisfies condition (a) but not condition (b). Thus the examples (ii) and (iii) show that the conditions (a) and (b) in Theorem 1.2 are independent.

By using Alexander duality one easily obtains a statement which is dual to that of Lyubeznik and also dual to that of Theorem 1.2.

In order to describe it, let, as before, $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and Δ a simplicial complex on $[n]$. For each subset $F \subset [n]$ we set

$$\mathbf{x}_F = \prod_{i \in F} x_i.$$

Recall that the Stanley–Reisner ideal of Δ is the ideal I_Δ of S which is generated by those squarefree monomials $\mathbf{x}_F$ with $F \notin \Delta$. One sets $K[\Delta] = S/I_\Delta$.

Then the Alexander dual of Δ is defined to be the simplicial complex

$$\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}.$$

Obviously one has $(\Delta^\vee)^\vee = \Delta$.

We quote the following facts, which for example can be found in [3].

- Let $I_\Delta = P_{F_1} \cap \cdots \cap P_{F_m}$ be the standard primary decomposition of I_Δ . (Here $P_G = (\{x_i\}_i \in G)$ for $G \subset [n]$.) Then $\{x_{F_1}, \dots, x_{F_m}\}$ is the minimal monomial set of generators of $I_{\Delta^\vee}$.
- (Terai) $\text{proj dim } I_\Delta = \text{reg } K[\Delta^\vee]$, where as usual $\text{reg } M$ denotes the regularity of a finitely generated graded S -module M .

Let I be a squarefree monomial ideal minimally generated by the monomials $u_1, \dots, u_m$. Let w be the smallest number t with the property that there exist integers $1 \leq i_1 < i_2 < \cdots < i_t \leq m$ such that

$$\text{lcm}(u_{i_1}, u_{i_2}, \dots, u_{i_t}) = \text{lcm}(u_1, u_2, \dots, u_m).$$

Then we call the number $\deg \text{lcm}(u_1, u_2, \dots, u_m) - w$ the *cosize* of I , denoted $\text{cosize } I$. If in the above definition we replace the words ‘there exist’ by ‘for all’, then we obtain the definition of the *big cosize* of I , denoted $\text{bigcosize } I$.

Now we have

Corollary 1.4. *Let $I \subset S$ be a squarefree monomial ideal. Then:*

- (a) $\text{reg } S/I \leq \text{cosize } I$.
- (b) $\text{reg } S/I = \text{cosize } I$ if $\text{bigcosize } I = \text{cosize } I$.

Proof. Let Δ be the simplicial complex with the property that $I = I_\Delta$. By using the result of Lyubeznik as well as the above facts, we obtain

$$n - \text{reg } K[\Delta] = n - \text{proj dim } I_{\Delta^\vee} = \text{depth } I_{\Delta^\vee} \geq \text{size } I_{\Delta^\vee} + 1,$$

so that $n - \text{reg } K[\Delta] \geq v + (n - h)$. This implies that $\text{reg } K[\Delta] \leq h - v$.

Since $(\Delta^\vee)^\vee = \Delta$, we see that the number v for $I_{\Delta^\vee}$ is equal to the number w for I_Δ , and that the number h for $I_{\Delta^\vee}$ is equal to the number $\deg \text{lcm}(u_1, u_2, \dots, u_m)$ for I_Δ . Thus statement (a) follows.

The assertion (b) is a simple consequence of (a) and Theorem 1.2. $\square$

2. SPLITTING THE VARIABLES TO GET LOWER BOUNDS FOR THE STANLEY DEPTH

In this section we describe and extend a method, introduced in the papers [6] and [7], to decompose a monomial ideal $I \subset S$ into $\mathbb{Z}^n$ -graded subspaces, which allows us to bound from below the Stanley depth of a monomial ideal. The decomposition depends on the choice of a subset Y of the set of variables $X = \{x_1, \dots, x_n\}$, and is also determined by the unique irredundant presentation of I as an intersection $I = \bigcap_{j=1}^s Q_j$ of its minimal irreducible monomial ideals. As before each Q_j is a P_j -primary ideal.

Without loss of generality we may assume that $Y = \{x_1, \dots, x_r\}$ for some number r such that $0 \leq r \leq n$. Then the set of variables splits into the two sets $\{x_1, \dots, x_r\}$ and $\{x_{r+1}, \dots, x_n\}$.

Given a subset $\tau \subset [s]$, we let I_τ be the $\mathbb{Z}^n$ -graded K -vector space spanned by the set of monomials of the form $w = uv$, where u and v are monomials with

$$u \in K[x_1, \dots, x_r] \quad \text{and} \quad u \in \bigcap_{j \notin \tau} Q_j \setminus \sum_{j \in \tau} Q_j,$$

$$v \in K[x_{r+1}, \dots, x_n] \quad \text{and} \quad v \in \bigcap_{j \in \tau} Q_j.$$

The following result extends the corresponding statement shown by Popescu [7] for squarefree monomial ideals.

Proposition 2.1. *With the notation introduced, the ideal I has a decomposition $\mathcal{D}_Y: I = \bigoplus_{\tau \subset [s]} I_\tau$ as a direct sum of $\mathbb{Z}^n$ -graded K -subspaces of I .*

Proof. It is clear from the definition of I_τ that $I_\tau \subset I$, so that $\sum_{\tau \subset [s]} I_\tau \subset I$. Conversely, let $w = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ be a monomial in I . Then $w = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ can be written in a unique way as a product $w = uv$ of monomials with $u \in K[x_1, \dots, x_r]$ and $v \in K[x_{r+1}, \dots, x_n]$. Let $\tau = \{j \in [s] : u \notin Q_j\}$. Then $u \in \bigcap_{j \notin \tau} Q_j$ and $u \notin \sum_{j \in \tau} Q_j$.

Let $j \in \tau$. Since $uv \in I$, it follows that $uv \in Q_j$. Thus, if $Q_j = (x_{i_1}^{b_{i_1}}, \dots, x_{i_k}^{b_{i_k}})$, then there exists an integer ℓ with $a_{i_\ell} \geq b_{i_\ell}$. On the other hand, since $u \notin Q_j$, it follows that $a_{i_t} < b_{i_t}$ for all $i_t \leq r$. This implies that $i_\ell \geq r+1$, and consequently $v \in Q_j$. Hence we see that $v \in \bigcap_{j \in \tau} Q_j$ and conclude that $w \in I_\tau$.

In order to see that the sum is direct assume that $w = uv \in I_\tau \cap I_\sigma$. Suppose that $\tau \neq \sigma$. Then we may assume that $\sigma \setminus \tau \neq \emptyset$. Let $j \in \sigma \setminus \tau$. Then $u \in Q_j$ by the definition of I_τ , and $u \notin Q_j$ by the definition of I_σ , a contradiction. $\square$

The $\mathbb{Z}^n$ -graded K -subspaces I_τ of I have the structure of a $\mathbb{Z}^n$ -graded module over S , and can be interpreted as follows: let

$$S' = K[x_1, \dots, x_r] \quad \text{and} \quad S'' = K[x_{r+1}, \dots, x_n].$$

Let $S \rightarrow S/(x_{r+1}, \dots, x_n) = S'$ be the canonical epimorphism and let Q'_i, P'_i be the images of Q_i , respectively P_i , for $i = 1, \dots, s$. Then we set

$$J_\tau = \bigcap_{j \notin \tau} Q'_j,$$

and let H_τ be the K -vector subspace of S' generated by all the monomials of $J_\tau \setminus (\sum_{j \in \tau} Q'_j)$. Notice that $\sum_{j \in \tau} Q'_j$ is of the form $(x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t})$ with $1 \leq i_1 < i_2 < \cdots < i_t \leq r$ and suitable exponents $a_j > 0$. Thus

$$H_\tau = (J_\tau, x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t}) / (x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t}).$$

Next let $S \rightarrow S/(x_1, \dots, x_r) = S''$ be the canonical epimorphism and let Q''_i be the image of Q_i for $i = 1, \dots, s$. Then we define the monomial ideal $L_\tau \subset S''$ as

$$L_\tau = \bigcap_{j \in \tau} Q''_j.$$

Now I_τ can be written as follows:

$$I_\tau = H_\tau \otimes_K L_\tau.$$

The following example describes the decomposition of I given in Proposition 2.1.

Example 2.2. Let $\mathfrak{m}$ be the maximal graded ideal of $K[x_1, x_2, x_3]$ and $\mathfrak{m}^2 = Q_1 \cap Q_2 \cap Q_3$ be its unique irredundant irreducible decomposition. Then $Q_1 = (x_1^2, x_2, x_3)$, $Q_2 = (x_1, x_2^2, x_3)$ and $Q_3 = (x_1, x_2, x_3^2)$. We choose the set Y to be $\{x_1\}$, which splits the set of variables into the sets $\{x_1\}$ and $\{x_2, x_3\}$. Then $I = \bigoplus_{\tau \subset [3]} I_\tau$, where I_τ is the K -vector space generated by all monomials $w = uv$, where u, v are monomials with

$$\begin{aligned} u \in K[x_1] \quad \text{and} \quad u \in \bigcap_{j \notin \tau} Q_j \setminus \sum_{j \in \tau} Q_j, \\ v \in K[x_2, x_3] \quad \text{and} \quad v \in \bigcap_{j \in \tau} Q_j. \end{aligned}$$

It follows immediately that the only nonzero summands I_τ of I correspond to the following subsets of $\{1, 2, 3\}$:

$$\tau \in \{\emptyset, \{1\}, \{1, 2, 3\}\}.$$

Indeed, since $Q_i + Q_j = \mathfrak{m}$ for all $i \neq j$, we obtain that $Q_k \setminus (Q_i + Q_j) = \emptyset$ for every permutation set $\{i, j, k\}$ of $\{1, 2, 3\}$ and therefore $I_\tau = 0$ for all $\tau \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. For $\tau = \{3\}$ we obtain that $((Q_1 \cap Q_2) \setminus Q_3) \cap K[x_1] = \emptyset$ and consequently $I_{\{3\}} = 0$. Similarly, we obtain for $\tau = \{2\}$ that $I_{\{2\}} = 0$. Now, for the remaining subsets we compute I_τ . If $\tau = \emptyset$, then $I_\emptyset$ is generated as a K -vector space by all monomials $u \cdot v$, where

$$u \in I \cap K[x_1] = (x_1^2)K[x_1] \quad \text{and} \quad v \in K[x_2, x_3].$$

Therefore $I_\emptyset = (x_1^2)K[x_1, x_2, x_3]$. For $\tau = \{1\}$, the K -basis of $I_{\{1\}}$ is given by the monomials $u \cdot v$, where

$$u \in ((Q_2 \cap Q_3) \setminus Q_1) \cap K[x_1] = x_1 K \quad \text{and} \quad v \in Q_1 \cap K[x_2, x_3] = (x_2, x_3)K[x_2, x_3].$$

Consequently we have that $I_{\{1\}} = (x_1 x_2, x_1 x_3)K[x_2, x_3]$. Finally, if $\tau = [3]$ we obtain that $I_{[3]}$ is generated as a K -vector space by all monomials $u \cdot v$, where

$$u \in (S \setminus \mathfrak{m}) \cap K[x_1] = K \quad \text{and} \quad v \in I \cap K[x_2, x_3] = (x_2^2, x_2 x_3, x_3^2)K[x_2, x_3].$$

Therefore $I_{[3]} = (x_2^2, x_2 x_3, x_3^2)K[x_2, x_3]$. Hence we obtain the following decomposition of I into $\mathbb{Z}^n$ -graded K -subspaces of I :

$$I = (x_1^2)K[x_1, x_2, x_3] \oplus (x_1 x_2, x_1 x_3)K[x_2, x_3] \oplus (x_2^2, x_2 x_3, x_3^2)K[x_2, x_3].$$

3. A COMPARISON OF STANLEY DEPTH AND SIZE

As an application of the technique of splitting variables, as introduced in the previous section, we show

Theorem 3.1. *Let I be a monomial ideal of S . Then*

$$\text{sdepth } I \geq 1 + \text{size } I.$$

Proof. Let $I = \bigcap_{j=1}^s Q_j$ be the unique irredundant presentation of I as an intersection of its minimal irreducible monomial ideals. Each of the Q_j is a primary ideal whose associated monomial prime ideal we denote, as before, by P_j .

We may assume that $\sum_{j=1}^s P_j = \mathfrak{m}$. Indeed, let $Z = \{x_i \notin \sum_{j=1}^s P_j\}$, $T = K[X \setminus Z]$ and $J = I \cap T$. Then the sum of the associated prime ideals of J is the graded maximal ideal of T , and

$$\text{sdepth } I = \text{sdepth } J + |Z|, \quad \text{and} \quad \text{size } I = \text{size } J + |Z|.$$

The first equation follows from [4, Lemma 3.6], while the second equation follows from the definition of size.

We choose the splitting set Y to be the set $\{x_i : x_i \in P_1\}$, and we may assume that $Y = \{x_1, \dots, x_r\}$ for some number r such that $1 \leq r \leq n$. If $r = n$, then the desired inequality follows at once since in this case $\text{size } I = 0$, and since for every monomial ideal I we have that $\text{sdepth } I \geq 1$. Therefore from now on we assume that $r < n$. We will prove the assertion of the theorem by induction on s . The case $s = 1$ follows immediately from [10, Theorem 2.4] and [4, Lemma 3.6] since $\text{sdepth } Q_1 = \lceil |Y|/2 \rceil + n - |Y|$ and $\text{size } Q_1 = n - |Y|$.

Assume now that the assertion is proved for all monomial ideals which are intersections of at most $s - 1$ irreducible monomial ideals. Since $Y = \{x_1, \dots, x_r\}$, it follows from the method described before Proposition 2.1 that $I = \bigoplus_{\tau \subset [s]} I_\tau$ with $I_{[s]} = 0$. We obtain from the decomposition of I that

$$\text{sdepth}_S I \geq \min\{\text{sdepth}_S I_\tau : \tau \subset [s] \text{ and } I_\tau \neq 0\}.$$

Hence it remains to prove that for any subset τ of $[s]$ such that $I_\tau \neq 0$ we have that $\text{sdepth}_S I_\tau \geq 1 + \text{size } I$. We will distinguish two cases: $\tau = \emptyset$, or τ is a proper nonempty subset of $[s]$. In both cases we may assume that $I_\tau \neq 0$.

In the first case we have that $I_\emptyset = (I \cap K[x_1, \dots, x_r])S$. Now applying [4] and the fact that the sdepth of any ideal is greater than or equal to 1, we obtain

$$\begin{aligned} \text{sdepth}_S I_\emptyset &= \text{sdepth}_{K[x_1, \dots, x_r]}(I \cap K[x_1, \dots, x_r]) + n - r \geq 1 + \dim S/P_1 \\ &\geq 1 + \text{depth } S/I \geq 1 + \text{size } I. \end{aligned}$$

In the second case we first observe that

$$(1) \quad \text{sdepth}_S I \geq \min_{\tau} \{\text{sdepth}_{S'} H_\tau + \text{sdepth}_{S''} L_\tau\}.$$

Indeed, we noticed already that $\text{sdepth}_S I \geq \min\{\text{sdepth}_S I_\tau : \tau \subset [s]\}$. Since $I_\tau = H_\tau \otimes_K L_\tau$, it follows that $\text{sdepth } I_\tau \geq \text{sdepth}_{S'} H_\tau + \text{sdepth}_{S''} L_\tau$, because if $\bigoplus_i u_i K[Z_i]$ is a Stanley decomposition of H_τ , and $\bigoplus_j v_j K[W_j]$ is a Stanley decomposition of L_τ , then $\bigoplus_{i,j} u_i v_j K[Z_i W_j]$ is a Stanley decomposition of $I_\tau = H_\tau \otimes_K L_\tau$; see [6, Lemma 1.2], where this assertion is shown in the case that H_τ and L_τ are both monomial ideals and [8, Theorem 3.1] in the case that H_τ and L_τ are both quotients of polynomial rings in disjoint sets of variables by monomial ideals. The argument in this slightly more general case is verbatim the same.

In our further discussions, we distinguish whether $P_1 \not\subset \sum_{j \in \tau} P_j$ or $P_1 \subset \sum_{j \in \tau} P_j$.

In the case that $P_1 \not\subset \sum_{j \in \tau} P_j$, one may assume that $H_\tau = J_\tau \setminus (x_k^{a_k}, \dots, x_r^{a_r})$ with $k > 1$. In other words,

$$(2) \quad H_\tau = (J_\tau, x_k^{a_k}, \dots, x_r^{a_r}) / (x_k^{a_k}, \dots, x_r^{a_r}).$$

Thus H_τ is a submodule of $S'/(x_k^{a_k}, \dots, x_r^{a_r})$, from which it follows that $\text{depth } H_\tau > 0$. This in turn implies that $\text{sdepth } H_\tau > 0$; see [2, Theorem 1.4]. Since $|\tau| \leq s - 1$, L_τ is the intersection of at most $s - 1$ irreducible monomial ideals. Thus, applying the induction hypothesis, the inequalities (1) and the subsequent Lemma 3.2, we obtain

$$\text{sdepth}_S I_\tau \geq 1 + \text{sdepth}_{S''} L_\tau \geq 2 + \text{size}_{S''} L_\tau \geq 1 + \text{size}_S I,$$

as desired.

On the other hand, if $P_1 \subset \sum_{j \in \tau} P_j$, then H_τ has a presentation as in (2) but with $k = 1$. Thus in this case, $\text{depth } H_\tau = 0$. Again applying [2, Theorem 1.4] it follows that $\text{sdepth } H_\tau = 0$. Then as before we get

$$\text{sdepth}_S I_\tau \geq \text{sdepth}_{S''} L_\tau \geq 1 + \text{size}_{S''} L_\tau \geq 1 + \text{size}_S I,$$

and we are done. $\square$

Lemma 3.2. *Let $I = \bigcap_{i=1}^s Q_i$ be the unique irredundant presentation of I as the intersection of irreducible monomial ideals, where the Q_i are P_i -primary ideals. Assume that $P_1 = (x_1, \dots, x_r)$ is one of the associated monomial prime ideals of I . Let $\tau \subset [s]$ such that $L_\tau \neq 0$. Then*

$$\text{size}_{S''}(L_\tau) + 1 \geq \text{size}_S I.$$

Moreover, if $P_1 \subset \sum_{j \in \tau} P_j$ we even have

$$\text{size}_{S''}(L_\tau) \geq \text{size}_S I.$$

Proof. We may assume as in Theorem 3.1 that $\sum_{j=1}^s P_j = \mathfrak{m}$. Let c be the minimum number t such that there exist $j_1 < \dots < j_t$ in τ with

$$\sum_{k=1}^t (P_{j_k} \cap S'') = \sum_{j \in \tau} (P_j \cap S'').$$

We have to analyze two cases: $\sum_{j \in \tau} (P_j \cap S'') = (x_{r+1}, \dots, x_n)$ or $\sum_{j \in \tau} (P_j \cap S'')$ is properly contained in $(x_{r+1}, \dots, x_n)$. In the first case we have that $\text{size}_{S''}(L_\tau) = c - 1$ and $P_1 + \sum_{k=1}^c P_{j_k} = P_1 + \sum_{j \in \tau} P_j = \mathfrak{m}$. This yields the first inequality. In particular, if $P_1 \subset \sum_{j \in \tau} P_j$, then $\sum_{k=1}^c P_{j_k} = P_1 + \sum_{k=1}^c P_{j_k} = \mathfrak{m}$ and therefore we have the second inequality.

In the second case let $\{x_{i_1}, \dots, x_{i_d}\}$ be the variables from S'' that do not belong to $\sum_{j \in \tau} (P_j \cap S'')$. Then we have $\text{size}_{S''}(L_\tau) = c - 1 + d$. Since, by our assumptions we have $P_1 = (x_1, \dots, x_r)$ and $\sum_{j=1}^s P_j = \mathfrak{m}$, it follows that for each k with $1 \leq k \leq d$ there exists an integer $l_k \in \{2, \dots, s\} \setminus \tau$ such that $x_{i_k} \in P_{l_k}$. Then we have

$$\sum_{k=1}^c (P_{j_k} \cap S'') + \sum_{k=1}^d (P_{l_k} \cap S'') = (x_{r+1}, \dots, x_n),$$

and consequently

$$P_1 + \sum_{k=1}^c P_{j_k} + \sum_{k=1}^d P_{l_k} = \mathfrak{m}.$$

Hence we obtain that $\text{size } I \leq c + d$, which is the desired first inequality. In particular, if $P_1 \subset \sum_{j \in \tau} P_j$, then

$$\sum_{k=1}^c P_{j_k} + \sum_{k=1}^d P_{l_k} = P_1 + \sum_{k=1}^c P_{j_k} + \sum_{k=1}^d P_{l_k} = \mathfrak{m},$$

which yields the second inequality. $\square$

The reader may wonder why in the proof of Theorem 3.1 we have chosen the set Y as the set of generators of one of the associated prime ideals of I . This was chosen to make sure that $I_{[s]} = 0$. Indeed, if $I_{[s]} \neq 0$, then this would be a summand in the decomposition of I , which may have sdepth less than or equal to the size of I , as the following example shows.

Example 3.3. Let $I = (x_1, x_2, x_3, x_6) \cap (x_2, x_3, x_4, x_6) \cap (x_2, x_3, x_5, x_6)$ be a monomial ideal of $K[x_1, \dots, x_6]$. One can easily see that $\text{size } I = 2$. If we now choose Y to be the set $\{x_1, \dots, x_5\}$, then the set of variables splits into the sets $\{x_1, \dots, x_5\}$ and $\{x_6\}$. Then, for $\tau = [3]$ we have that $I_{[3]}$ is the K -vector space whose basis consists of the monomials $w = uv$, where u, v are monomials with

$$u \in (S \setminus \mathfrak{m}) \cap K[x_1, \dots, x_5] = K \quad \text{and} \quad v \in I \cap K[x_6] = (x_6)K[x_6].$$

Therefore $I_{[3]} = (x_6)K[x_6]$ and consequently $\text{sdepth } I_{[3]} = 1 < \text{size } I$.

Dual to the Lyubeznik inequality, $\text{depth } I \geq \text{size } I + 1$, we have $\text{reg } I \leq \text{cosize } I + 1$, as we have seen in Section 1. Similarly there is an inequality dual to $\text{sdepth } I \geq \text{size } I + 1$, as we shall now see. For its proof we have to recall a few results.

- (γ) Alexander duality can be extended to finitely generated squarefree $\mathbb{Z}^n$ -graded modules M ; see [9] and [12]. Then one obtains a functor $M \mapsto M^\vee$ from the category of $\mathbb{Z}^n$ -graded modules into itself with the property that $(M^\vee)^\vee = M$. Moreover one has $(I_\Delta)^\vee = K[\Delta^\vee]$.
- (δ) Let $\mathcal{D}: M = \bigoplus_{i=1}^m u_i K[Z_i]$ be a Stanley decomposition of M . Then

$$\text{sreg } \mathcal{D} = \max\{\deg u_i : i = 1, \dots, m\}$$

is called the Stanley regularity of $\mathcal{D}$, and

$$\text{sreg } M = \min\{\text{sreg } \mathcal{D} : \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the *Stanley regularity* of M . The crucial fact that we need has been shown by Soleyman Jahan [11, Theorem 3.9]: namely, $\text{sreg } M = n - \text{sdepth } M^\vee$.

Corollary 3.4. Let $I \subset S$ be a squarefree monomial ideal. Then $\text{sreg } S/I \leq \text{cosize } I$, and equality holds if $\text{bigcosize } I = \text{cosize } I$.

Proof. Let Δ be the simplicial complex with $I = I_\Delta$. Then, by using (γ) and (δ) as well as Theorem 3.1, we obtain

$$\text{sreg } S/I_\Delta = n - \text{sdepth } I_{\Delta^\vee} \leq n - (\text{size } I_{\Delta^\vee} + 1) = \text{cosize } I_\Delta. \quad \square$$

REFERENCES

- [1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Revised edition. Cambridge University Press (1998). MR1251956 (95h:13020)
- [2] M. Cimpoeaş, Some remarks on the Stanley depth for multigraded modules. *Le Matematiche* **LXIII** (2008), 165–171 (2009). MR2531658 (2010d:13032)
- [3] J. Herzog and T. Hibi, *Monomial Ideals*. Graduate Texts in Math. 260. Springer-Verlag, 2011. MR2724673
- [4] J. Herzog, M. Vladoiu and X. Zheng, How to compute the Stanley depth of a monomial ideal. *J. Algebra* **322**, 3151–3169 (2009). MR2567414 (2010k:13036)
- [5] G. Lyubeznik, On the arithmetical rank of monomial ideals. *J. Algebra* **112**, 86–89 (1988). MR921965 (89b:13020)
- [6] A. Popescu, Special Stanley Decompositions. *Bull. Math. Soc. Sci. Math. Roumanie* **53** (101), 361–372 (2010).
- [7] D. Popescu, Stanley conjecture on intersections of four monomial prime ideals. *arXiv.AC/1009.5646*.
- [8] A. Rauf, Depth and Stanley Depth of Multigraded Modules. *Comm. in Algebra* **38**, 773–784 (2010). MR2598911
- [9] T. Römer, Generalized Alexander duality and applications. *Osaka J. Math.* **38**, 469–485 (2001). MR1833633 (2002c:13029)
- [10] Y. Shen, Stanley depth of complete intersection monomial ideals and upper-discrete partitions. *J. Algebra* **321**, 1285–1292 (2009). MR2489900 (2009k:13044)

- [11] A. Soleyman Jahan, Prime filtrations and Stanley decompositions of squarefree modules and Alexander duality. *Manuscripta Math.* **130**, 533–550 (2009). MR2563149 (2010j:13042)
- [12] K. Yanagawa, Alexander duality for Stanley-Reisner rings and squarefree $\mathbb{N}^n$ -graded modules. *J. Algebra* **225**, 630–645 (2000). MR1741555 (2000m:13036)

FACHBEREICH MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY

E-mail address: `juergen.herzog@uni-essen.de`

INSTITUTE OF MATHEMATICS “SIMION STOILOW” OF THE ROMANIAN ACADEMY, UNIVERSITY OF BUCHAREST, P. O. BOX 1-764, BUCHAREST 014700, ROMANIA

E-mail address: `dorin.popescu@imar.ro`

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BUCHAREST, STR. ACADEMIEI 14, BUCHAREST, RO-010014, ROMANIA

E-mail address: `vladoiu@gtamath.unibuc.ro`