ON A RESULT OF BREZIS AND MAWHIN

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ABSTRACT. Brezis and Mawhin proved the existence of at least one T periodic solution for differential equations of the form

$$(0.1) \quad (\phi(u'))' - g(t, u) = h(t)$$

when $\phi : (-a, a) \rightarrow \mathbb{R}$, $0 < a < \infty$, is an increasing homeomorphism with $\phi(0) = 0$, g is a Carathéodory function T periodic with respect to t , 2π periodic with respect to u , of mean value zero with respect to u , and $h \in L^1_{loc}(\mathbb{R})$ is T periodic and has mean value zero. Their proof was partly variational. First it was shown that the corresponding action integral had a minimum at some point u_0 in a closed convex subset $\mathcal{K}$ of the space of T periodic Lipschitz functions. However, u_0 may not be an interior point of $\mathcal{K}$, so it may not be a critical point of the action integral. The authors used an ingenious argument based on variational inequalities and uniqueness of a T periodic solution to (0.1) when $g(t, u) = u$ to show that u_0 is indeed a T periodic solution of (0.1). Here we make full use of the variational structure of the problem to obtain Brezis and Mawhin's result.

1. INTRODUCTION

Recently Brezis and Mawhin [1] proved the existence of at least one T periodic solution for differential equations of the form

$$(1.1) \quad (\phi(u'))' - g(t, u) = h(t)$$

when $\phi : (-a, a) \rightarrow \mathbb{R}$, $0 < a < \infty$, is an increasing homeomorphism with $\phi(0) = 0$, g is a Carathéodory function T periodic with respect to t , 2π periodic with respect to u , of mean value zero with respect to u , and $h \in L^1_{loc}(\mathbb{R})$ is T periodic and has mean value zero. A special case of interest is the relativistic forced pendulum equation

$$(1.2) \quad \left(\frac{u'}{\sqrt{1-u'^2}} \right)' + A \sin(u) = h(t).$$

The proof in [1] was partly variational. First they showed that the corresponding action integral had a minimum at some point u_0 in a closed convex subset $\mathcal{K}$ of the space of T periodic Lipschitz functions. However, u_0 may not be an interior point of $\mathcal{K}$, so it may not be a critical point of the action integral. The authors then used an ingenious argument based on variational inequalities and uniqueness of a solution

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to (1.1) when $g(t, u) = u$ to show that u_0 is indeed a T periodic solution of (1.1). Here we make full use of the variational structure of the problem to obtain Brezis and Mawhin's result. We rewrite the problem as a two dimensional Hamiltonian system and obtain a solution by applying the now classical saddle point theorem of Rabinowitz [3] to a sequence of approximating problems, thereby obtaining a sequence of critical points. A subsequence of these critical points converges to a solution.

The reader is referred to [1] and its bibliography for a history of the T periodic problem for equations (1.1), (1.2) and for the classical forced pendulum.

We first rewrite (1.1) as a system in Hamiltonian form. Let $\psi := \phi^{-1}$ so that ψ is an increasing homeomorphism from $\mathbb{R}$ onto $(-a, a)$. Equation (1.1) is equivalent to the system

$$(1.3) \quad \begin{aligned} \dot{u} &= \psi(v), \\ \dot{v} &= g(t, u) + h(t), \end{aligned}$$

where $\dot{u} = du/dt$, $\dot{v} := dv/dt$. Let $\Psi(v) := \int_0^v \psi(s)ds$, $G(t, u) := \int_0^u g(t, s)ds$, $w := (u, v)$ and

$$H(t, w) := -\Psi(v) + G(t, u) + h(t)u.$$

System (1.3) is equivalent to

$$(1.4) \quad \dot{w} = J \nabla_w H(t, w),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will show that (1.4), and hence (1.1), has a T periodic solution. We make the following assumptions:

- (H_ϕ) $\phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, with inverse $\psi := \phi^{-1}$.
- (H_g) $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, u) \mapsto g(t, u)$, is a bounded Carathéodory function, so g is continuous in u for a.a. t and is Lebesgue measurable in t for all u . Additionally, g is T periodic in t for all u and 2π periodic in u for a.a. $t \in \mathbb{R}$ and is of mean value zero with respect to u . It follows there is a constant c_1 such that both $|g(t, u)| \leq c_1$ and $|G(t, u)| \leq c_1$ for a.a. t and all u .

Let L_T^1 denote the set of all $h : \mathbb{R} \rightarrow \mathbb{R}$, locally integrable and T periodic. For $h \in L_T^1$ let $\bar{h} := \frac{1}{T} \int_0^T h(s)ds$ denote the mean value of h .

By a (classical) T periodic solution to (1.1) is meant a T periodic function $u \in C^1(\mathbb{R}, \mathbb{R})$ such that $\phi(u')$ is absolutely continuous and (1.1) is satisfied a.e. on $\mathbb{R}$.

Theorem 1.1. *Let (H_ϕ) and (H_g) hold. Then (1.1) has a T periodic solution for any $h \in L_T^1$ with $\bar{h} = 0$.*

In the next section we describe in detail our approach. In the third section we prove Theorem 1.1.

2. PRELIMINARIES

Let $E := W_T^{1,2}(\mathbb{R}, \mathbb{R}^2)$ denote the Hilbert space of T periodic $\mathbb{R}^2$ valued functions w such that the components of w are absolutely continuous and $w' \in L^2((0, T), \mathbb{R}^2)$ with inner product

$$(2.1) \quad \langle w, z \rangle := \int_0^T w \cdot z \, dt + \int_0^T \dot{w} \cdot \dot{z} \, dt,$$

where $\dot{w} = dw/dt$, $\dot{z} = dz/dt$, and $\cdot$ denotes the dot product in $\mathbb{R}^2$. We will denote the associated norm for $w \in E$ by $\|w\|_1$. An equivalent norm is $\|w\|_0 := |w|_0 + \left(\int_0^T |\dot{w}|^2 \, dt \right)^{1/2}$, where $|w|_0$ is the maximum norm in C_T^0 , the space of continuous T periodic functions. Here, for $V = (v_1, v_2) \in \mathbb{R}^2$, $|V|^2 = V \cdot V$. For $w = (w_1, w_2)$

$$|w|_0 := \max_{t \in [0, T]} |w_1(t)| + \max_{t \in [0, T]} |w_2(t)|.$$

It is known that E is compactly imbedded in C_T^0 .

For functions $w = (u, v)$, $z = (p, q) \in E$ let

$$(2.2) \quad \langle Lw, z \rangle := \int_0^T -J\dot{w} \cdot z \, dt = \int_0^T \dot{v}p - \dot{u}q \, dt.$$

One can show that there is a constant C such that for all such $w, z \in E$,

$$\left| \int_0^T \dot{v}p \, dt \right| \leq C \|w\|_1 \|z\|_1 \quad \text{and} \quad \left| \int_0^T \dot{u}q \, dt \right| \leq C \|w\|_1 \|z\|_1,$$

and hence (2.2) defines a bounded bilinear form on E and thus defines L as a bounded linear operator on E . The operator L is self-adjoint on E .

The space E can be written as the direct sum $E^+ \oplus E^0 \oplus E^-$ of the three subspaces E^+ , E^- , and E^0 associated with the positive, negative, and zero eigenvalues, respectively, of L . These eigenvalues are also the eigenvalues of the problem

$$\dot{w} = \lambda Jw, \quad w(0) = w(T).$$

The eigenvalues are the set $\{\lambda_n = 2n\pi/T | n \in \mathbb{Z}\}$. The operator L is positive definite on E^+ , negative definite on E^- , and 0 on E^0 . Note that E^0 consists of the constant vectors. Moreover, if we write $w \in E$ as $w = w^+ + w^0 + w^-$, where $w^s \in E^s$, $s \in \{+, 0, -\}$, then

$$(2.3) \quad \langle Lw, w \rangle = \langle Lw^+, w^+ \rangle + \langle Lw^-, w^- \rangle,$$

and a norm on E equivalent to $\|\cdot\|_0$ and $\|\cdot\|_1$ is given by $\|w\|^2 = |w^0|^2 + \langle Lw^+, w^+ \rangle - \langle Lw^-, w^- \rangle$. Thus we may write for $w \in E$

$$(2.4) \quad \|w\|^2 = |w^0|^2 + \|w^+\|^2 + \|w^-\|^2 \quad \text{and} \quad \langle Lw, w \rangle = \|w^+\|^2 - \|w^-\|^2,$$

since the subspaces are orthogonal under the bilinear form (2.2); indeed, they are also orthogonal under the original inner product (2.1). Moreover, there are constants m_1 and m_2 such that

$$m_1 \|w\| \leq \|w\|_0 \leq m_2 \|w\|.$$

For $w \in E$ let

$$(2.5) \quad \mathcal{H}(w) := \int_0^T H(t, w) dt$$

and

$$(2.6) \quad I(w) := \frac{1}{2} \langle Lw, w \rangle - \mathcal{H}(w).$$

We shall show, under the assumptions above, that I has a critical point in E . This critical point will be a T periodic solution to (1.1). We will need the following generalized Palais-Smale condition:

$(PS)_c$ A Palais-Smale sequence at level $c \in \mathbb{R}$ for I is a sequence $\{y_n\} \subset E$ such that

$$I(y_n) \rightarrow c \text{ and } I'(y_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The function I satisfies the local Palais-Smale $(PS)_c$ condition at c if the existence of a Palais-Smale sequence at level c for I implies that c is a critical value for I .

We will use the following saddle point theorem, due to Rabinowitz [3] (see also [4]). Originally proven with the standard Palais-Smale condition on I , it also holds with only $(PS)_c$ assumed (see [2], Theorem 4.7).

Theorem 2.1. *Let E be a real Banach space such that $E = E_1 \oplus E_2$ with E_1 finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$ and*

- (1) *There is a real number α such that $I|_{E_2} \geq \alpha$.*
- (2) *There is a real number $\beta < \alpha$ and a bounded neighborhood Ω of 0 in E_1 such that $I|_{\partial\Omega} \leq \beta$.*

Let $S := \{g \in C(\overline{\Omega}, E) | g(w) = w \text{ for } w \in \partial\Omega\}$, and let $c = \inf_{g \in S} \max_{w \in \overline{\Omega}} I(g(w))$.

Suppose

- (3) *I satisfies the local Palais-Smale condition, $(PS)_c$, at c . Then c is a critical value for I .*

If the conditions of Theorem 2.1 are satisfied, it follows from finite dimensional degree theory that $0 \in g(\Omega)$ for every $g \in S$. Thus for every $g \in S$, $\max_{w \in \overline{\Omega}} I(g(w)) \geq I(0) \geq \alpha$ (since $0 \in E_2$). Hence $c \geq \alpha$ and c is finite.

3. PROOF OF THEOREM 1.1

The functional I in (2.6) is strongly indefinite, and Theorem 2.1 does not apply to it. We will prove Theorem 1.1 by using a modified sequence of functionals on a sequence of subspaces of E , to which Theorem 2.1 does apply, thus obtaining a sequence of critical points. We then show that this sequence of critical points has a convergent subsequence, the limit of which is then necessarily a weak periodic solution of (1.4). Recall that a weak periodic solution of (1.4) is a function $w = (u, v) \in E$ such that for all $z = (p, q) \in E$ one has

$$\int_0^T [\dot{v}p - \dot{u}q] dt + \int_0^T [\psi(v)q - g(t, u)p - h(t)p] dt = 0.$$

Standard regularity arguments show that a weak periodic solution of (1.4) is a classical periodic solution of (1.4). We proceed.

Let E_N^- be the subspace of E^- spanned by the eigenfunctions of L associated with the first N negative eigenvalues $\{\lambda_n : -N \leq n \leq -1\}$. Let $E_N := E_N^- \oplus E^0 \oplus E^+$. The space E_N is a Hilbert space, with the same inner product as E . Let P_N be the natural orthogonal projection of E onto E_N (note that for $w \in E$, $P_N w$ is represented by the truncated Fourier series of w , the terms associated with the eigenvalues $\{\lambda_n | n \leq -N - 1\}$ omitted). Define I_N by

$$I_N(w) := \frac{1}{2} \langle P_N L w, w \rangle - \mathcal{H}(w)$$

for $w \in E_N$. Thus

$$I_N(w) = \frac{1}{2} \|w^+\|^2 - \frac{1}{2} \|w_N^-\|^2 + \int_0^T \Psi(v) - G(t, u) - h(t)u \, dt,$$

where $w = (u, v)$, written as a pair, and $w = w_N^- + w_0 + w^+$, where $w_N^- \in E_N^-$, $w_0 \in E^0$, and $w^+ \in E^+$.

We first show that conditions (1) and (2) in Theorem 2.1 hold on some subspaces E_1, E_2 with $E_N = E_1 \oplus E_2$. We let $E_1 := E_N^-$, and $E_2 := E^0 \oplus E^+$ so that E_1 is finite dimensional and $E_N = E_1 \oplus E_2$. Let $w_2 = (u, v) \in E_2$ with $w_2 = w^0 + w^+$. We have, since $\Psi \geq 0$ and $\bar{h} = 0$,

$$\begin{aligned} I_N(w_2) &= \frac{1}{2} \|w^+\|^2 + \int_0^T \Psi(v^0 + v^+) - G(t, u^0 + u^+) - h(t)u^+ \, dt \\ &\geq \frac{1}{2} \|w^+\|^2 - c_1 T - |h|_{L^1} |u^+|_0 \\ &\geq \frac{1}{2} \|w^+\|^2 - c_1 T - |h|_{L^1} m_2 \|w^+\| \\ &\geq -\frac{1}{2} |h|_{L^1}^2 m_2^2 - c_1 T := \alpha, \end{aligned}$$

where m_2 is a constant such that $\|w\|_0 \leq m_2 \|w\|$ for any $w = (u, v) \in E$. Recall that $|\cdot|_0$ denotes the maximum norm on $[0, T]$. For $w = w_N^- = (u_N^-, v_N^-) \in E_1$ we have

$$I_N(w_N^-) = -\frac{1}{2} \|w_N^-\|^2 + \int_0^T \Psi(v_N^-) - G(t, u_N^-) - h(t)u_N^- \, dt.$$

Now $|\psi(v)| \leq a$ for all $v \in \mathbb{R}$, so

$$\Psi(v) = \int_0^v \psi(s) \, ds \leq \left| \int_0^v |\psi(s)| \, ds \right| \leq a |v|.$$

Therefore for any $\beta < \alpha$,

$$\begin{aligned} I_N(w_N^-) &\leq -\frac{1}{2} \|w_N^-\|^2 + a \int_0^T |v_N^-(t)| - G(t, u_N^-(t)) - h(t)u_N^-(t) \, dt \\ &\leq -\frac{1}{2} \|w_N^-\|^2 + aT |v_N^-|_0 + c_1 T + |h|_{L^1} |u_N^-|_0 \\ &\leq -\frac{1}{2} \|w_N^-\|^2 + aT m_2 \|w_N^-\| + c_1 T + |h|_{L^1} m_2 \|w_N^-\| \\ &= -\frac{1}{2} \|w_N^-\|^2 + (aT m_2 + |h|_{L^1} m_2) \|w_N^-\| + c_1 T \\ &\leq \beta < \alpha, \end{aligned}$$

provided $\|w_N^-\| = R$ for R sufficiently large (the minimum value needed for R is independent of N and is easily calculated as a function of β, a, c_1, m_2, T , and $|h|_{L^1}$). This shows that (2) of Theorem 2.1 holds, with Ω the open ball in E_1 of radius R , centered at 0.

We now proceed to show that I_N satisfies $(PS)_c$ at every value of $c \in \mathbb{R}$. Let $c \in \mathbb{R}$ be fixed. Suppose $\{w_k\} \subset E_N$, $I_N(w_k) \rightarrow c$ and $I'_N(w_k) \rightarrow 0$ as $k \rightarrow \infty$. Now for $w = (u, v), z = (p, q) \in E_N$, for simplicity of notation, we will omit the subscript N in $w_{N,k}, w_{N,k}^+$, etc., and just write $w_k, w_k^+, \dots$, etc., respectively. We have

$$\begin{aligned} I_N(w_k) &= \frac{1}{2} \langle P_N L w_k, w_k \rangle - \mathcal{H}(w_k) \\ &= \frac{1}{2} \|w_k^+\|^2 - \frac{1}{2} \|w_k^-\|^2 + \int_0^T \Psi(v_k) - G(t, u_k) - h(t)u_k dt, \end{aligned}$$

and for $z \in E_N, z = (p, q)$, we get

$$I'_N(w_k)z = \langle P_N L w_k, z \rangle + \int_0^T \psi(v_k)q - g(t, u_k)p - h(t)p dt.$$

Thus letting $w_k = (u_k, v_k), w_k^+ = (u_k^+, v_k^+)$, and $w_k^- = (u_k^-, v_k^-)$, we have

$$\begin{aligned} I'_N(w_k)w_k^+ &= \langle P_N L w_k, w_k^+ \rangle + \int_0^T \psi(v_k)v_k^+ - g(t, u_k)u_k^+ - h(t)u_k^+ dt \\ &= \|w_k^+\|^2 + \int_0^T \psi(v_k)v_k^+ - g(t, u_k)u_k^+ - h(t)u_k^+ dt \\ &\geq \|w_k^+\|^2 - a \int_0^T |v_k^+| dt - c_1 \int_0^T |u_k^+| dt - \|h\|_{L^1} |u_k^+|_0 \\ &\geq \|w_k^+\|^2 - C_1 \|w_k^+\| - C_2 \|w_k^+\| - C_3 \|w_k^+\| \\ &= \|w_k^+\|^2 - C_4 \|w_k^+\| \end{aligned}$$

since $|\psi(v)| \leq a$ and $|g(t, u)| \leq c_1$, and where $C_i, i = 1, 2, 3$ are constants involving imbedding and other constants and $C_4 = C_1 + C_2 + C_3$. Since $I'_N(w_k) \rightarrow 0$ we may assume $\|I'_N(w_k)\| \leq 1$, and hence we get

$$\|w_k^+\| \geq |I'_N(w_k)w_k^+| \geq I'_N(w_k)w_k^+ \geq \|w_k^+\|^2 - C_4 \|w_k^+\|,$$

from which we deduce that there is a constant C_5 such that

$$\|w_k^+\| \leq C_5 \text{ for all } k \geq 1.$$

In a similar manner one can see that there is a constant C_6 such that

$$\|w_k^-\| \leq C_6 \text{ for all } k \geq 1.$$

Now we have to deal with $\{w_k^0\}$. Let $w_k^0 = (u_k^0, v_k^0) \in \mathbb{R}^2$. Then since $\bar{h} = 0$ we have

$$(3.1) \quad \begin{aligned} I_N(w_k) &= \frac{1}{2} \|w_k^+\|^2 - \frac{1}{2} \|w_k^-\|^2 + \int_0^T \Psi(v_k^0 + v_k^+ + v_k^-) - G(t, u_k) - h(t)(u_k^+ + u_k^-) dt \\ &\geq \int_0^T \Psi(v_k^0 + v_k^+ + v_k^-) dt - C \end{aligned}$$

since $\|w_k^\pm\|$ are bounded and $|G(t, u)| \leq c_1$. Now $\Psi(v) = \int_0^v \psi(s) ds$ and ψ is increasing with $\psi(s)s > 0$ for $s \neq 0$, so

$$\lim_{|v| \rightarrow \infty} \Psi(v) = \infty;$$

and hence if $\{v_k^0\}$ is not a bounded sequence, then since $\|v_k^+\|$ and $\|v_k^-\|$ are bounded, we have

$$\limsup_{n \rightarrow \infty} \int_0^T \Psi(v_k^0 + v_k^+ + v_k^-) dt = \infty,$$

which contradicts (3.1) and the boundedness of $I_N(w_k)$. Therefore $\{v_k^0\}$ is bounded. Now consider $\{u_k^0\}$. These terms appear in (3.1) only in G . Now $G(t, u + 2j\pi) = G(t, u)$ for all u and all $j \in \mathbb{Z}$. Thus we can choose $\hat{u}_k^0 \in [0, 2\pi)$ with $u_k^0 - \hat{u}_k^0 = 2j\pi$ for some $j \in \mathbb{Z}$, and $G(t, u_k^0 + u_k^+(t) + u_k^-(t)) = G(t, \hat{u}_k^0 + u_k^+(t) + u_k^-(t))$ for all $k \in \mathbb{N}$. Now let $y_k^0 := (\hat{u}_k^0, v_k^0)$, $y_k^+ := w_k^+$, $y_k^- := w_k^-$ and $y_k := y_k^0 + y_k^+ + y_k^-$. Then $\{y_k\}$ is bounded in E_N and $I_N(y_k) = I_N(w_k)$, $I'_N(y_k) = I'_N(w_k)$ for all $k \in \mathbb{N}$. A subsequence of $\{y_k\}$ converges weakly in E_N to some $y^* \in E_N$. Standard arguments (see, e.g., [2], [4]) now show that the subsequence actually converges strongly in E_N to y^* . Relabel the convergent subsequence as $\{y_k\}$. We have $I(y_k) \rightarrow c$, $I'(y_k) \rightarrow 0$ as $k \rightarrow \infty$, and by the C^1 continuity of I_N , $I_N(y^*) = c$ and $I'_N(y^*) = 0$. Thus I_N satisfies $(PS)_c$ at every $c \in \mathbb{R}$. By Theorem 2.1, I_N has a critical point $w_N \in E_N$. We now need to show that the set of critical points $\{w_N : N \in \mathbb{N}\} \subset E$ has a convergent subsequence, converging to a weak solution of (1.3).

Let $w_N = (u_N, v_N)$ be a critical point for I_N . We also write $w_N = w_N^- + w_N^0 + w_N^+$. Then for any $z \in E$, with $z_N = P_N z$, $z_N = (p_N, q_N)$, we have

$$0 = I'_N(w_N)P_N z = \langle Lw_N, P_N z \rangle + \int_0^T [\psi(v_N)q_N - g(t, u_N)p_N - h(t)p_N] dt.$$

Hence

$$0 = I'_N(w_N)w_N^+ = \langle Lw_N, w_N^+ \rangle + \int_0^T [\psi(v_N)v_N^+ - g(t, u_N)u_N^+ - h(t)u_N^+] dt,$$

and hence

$$\begin{aligned} \|w_N^+\|^2 + \int_0^T [\psi(v_N)v_N^+ - g(t, u_N)u_N^+ - h(t)u_N^+] dt &= 0, \\ \|w_N^+\|^2 &\leq aT |v_N^+|_0 + c_1 T |u_N^+|_0 + |h|_{L^1} |u_N^+|_0 \\ &\leq (aT + c_1 T + |h|_{L^1}) \|w_N^+\|. \end{aligned}$$

Therefore for all $N \in \mathbb{N}$

$$\|w_N^+\| \leq (aT + c_1 T + |h|_{L^1}).$$

Similarly

$$\|w_N^-\| \leq (aT + c_1 T + |h|_{L^1}).$$

As for $w_N^0 = (u_N^0, v_N^0)$, we may take $u_N^0 \in [0, 2\pi)$, so the sequence $\{u_N^0\}$ is bounded, and

$$(3.2) \quad \begin{aligned} 0 &= I'_N(w_N)w_N^0 = \int_0^T [\psi(v_N)v_N^0 - g(t, u_N)u_N^0 - h(t)u_N^0] dt \\ &= \int_0^T [\psi(v_N^0 + v_N^+ + v_n^-)v_N^0 - g(t, u_N)u_N^0] dt. \end{aligned}$$

Since v_N^+ and v_n^- are bounded, if a subsequence $\{v_{N_k}^0\}$ converges to $+\infty$, we have $\psi(v_{N_k}^0 + v_{N_k}^+(t) + v_{N_k}^-(t)) \geq a/2$ for all k sufficiently large and all t , and then $\psi(v_{N_k}^0 + v_{N_k}^+(t) + v_{N_k}^-(t))v_{N_k}^0 \rightarrow \infty$ as $k \rightarrow \infty$, uniformly on $[0, T]$, contradicting (3.2). In the same manner, no subsequence of $\{v_N^0\}$ can diverge to $-\infty$. Thus $\{v_N^0\}$ is bounded. It follows that there is a uniform bound in E on the sequence $\{w_N\}$. This sequence thus has a weakly convergent subsequence in E , strongly convergent in $C([0, T], \mathbb{R}^2)$, to some $w^* \in E$. We relabel this weakly convergent subsequence as $\{w_N\}$. For $z = (p, q) \in E$

$$(3.3) \quad \langle Lw_N, P_N z \rangle = - \int_0^T [\psi(v_N)q_N - g(t, u_N)p_N - h(t)p_N] dt.$$

Now L is self-adjoint, so $\langle Lw_N, P_N z \rangle = \langle w_N, LP_N z \rangle = \langle w_N, P_N Lz \rangle = \langle w_N, Lz \rangle$ since L maps each E_N into itself. Since $w_N \rightarrow w^*$ weakly, we have

$$\lim_{N \rightarrow \infty} \langle Lw_N, P_N z \rangle = \lim_{N \rightarrow \infty} \langle w_N, Lz \rangle = \langle w^*, Lz \rangle = \langle Lw^*, z \rangle.$$

By (3.3) we have for each $N \in \mathbb{N}$

$$\langle w_N, Lz \rangle = \langle Lw_N, P_N z \rangle = - \int_0^T [\psi(v_N)q_N - g(t, u_N)p_N - h(t)p_N] dt.$$

Note that $P_N z$ is represented by the Fourier series of z truncated to omit the terms associated with the eigenfunctions for the eigenvalues $\{\lambda_n | n \leq -N - 1\}$. The Fourier series converges to z in $L^2((0, T), \mathbb{R}^2)$, so $q_N \rightarrow q, p_N \rightarrow p$ in $L^2((0, T), \mathbb{R})$. Thus

$$\lim_{N \rightarrow \infty} \int_0^T [\psi(v_N)q_N - g(t, u_N)p_N - h(t)p_N] dt = \int_0^T [\psi(v^*)q - g(t, u^*)p - h(t)p] dt.$$

Combining the above, we have

$$\langle Lw^*, z \rangle + \int_0^T [\psi(v^*)q - g(t, u^*)p - h(t)p] dt = 0$$

for all $z \in E$, and $w^* \in E$ is a weak T periodic solution to (1.3) and, as noted earlier, it is therefore a classical T periodic solution to (1.3) and provides a classical solution to (1.1).

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