A METRIC DISCREPANCY RESULT FOR LACUNARY SEQUENCES

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ABSTRACT. We prove that every value greater than or equal to $1/2$ can be a constant appearing in the law of the iterated logarithm for discrepancies of a lacunary sequence satisfying the Hadamard gap condition.

1. INTRODUCTION

In the theory of the uniform distribution, we use the following discrepancies of a sequence $\{a_k\}$:

$$D_N\{a_k\} = \sup_{0 \leq a' < a < 1} \left| \frac{1}{N} \# \{k \leq N \mid \langle a_k \rangle \in [a', a)\} - (a - a') \right|,$$

$$D_N^*\{a_k\} = \sup_{0 \leq a < 1} \left| \frac{1}{N} \# \{k \leq N \mid \langle a_k \rangle \in [0, a)\} - a \right|,$$

where $\langle x \rangle$ denotes the fractional part $x - [x]$ of x .

One of the most typical results on the asymptotic behavior of discrepancies is celebrated in the Chung-Smirnov theorem, which asserts the law of the iterated logarithm for discrepancies of uniformly distributed i.i.d. $\{U_k\}$:

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{U_k\}}{\sqrt{2N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{U_k\}}{\sqrt{2N \log \log N}} = \frac{1}{2}, \quad \text{a.s.}$$

We have similar phenomena without assuming the independence of the sequence of random variables. For a sequence $\{n_k\}$ satisfying the Hadamard gap condition,

$$(1.1) \quad \inf_{k \in \mathbf{N}} n_{k+1}/n_k > 1,$$

Philipp [9, 10] proved the bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} \leq \Sigma^*\{n_k x\} := \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}} \leq \Sigma\{n_k x\} := \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} \leq C$$

for almost every x , where C is a constant depending only on the infimum in (1.1). Recently it became possible to calculate concrete values of $\Sigma\{n_k x\}$ and $\Sigma^*\{n_k x\}$.

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It is proved in [5] that for all real numbers $\theta > 1$ there exists a constant Σ_θ such that

$$(1.2) \quad \Sigma^*\{\theta^k x\} = \Sigma\{\theta^k x\} = \Sigma_\theta, \quad \text{a.e.}$$

We have $\Sigma_\theta = 1/2$ if θ satisfies the condition $\theta^r \notin \mathbf{Q}$ for all $r \in \mathbf{N}$. Otherwise let us express θ by $\theta = \sqrt[r]{p/q}$ where $r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\}$ and $\gcd(p, q) = 1$. In this case Σ_θ does not depend on r and is evaluated in the following cases:

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2 & \text{if } p \text{ and } q \text{ are odd,} \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2 & \text{if } p \geq 4 \text{ is even and } q = 1, \\ \sqrt{42}/9 & \text{if } p = 2 \text{ and } q = 1, \\ \sqrt{22}/9 & \text{if } p = 5 \text{ and } q = 2. \end{cases}$$

It is also proved that $\max_{\theta > 1} \Sigma_\theta = \Sigma_2 = \sqrt{42}/9$ (cf. [7]). Aistleitner [1] gave a nearly optimal Diophantine condition on the sequence $\{n_k\}$ to have $\Sigma^*\{n_k x\} = \Sigma\{n_k x\} = 1/2$ a.e., which coincides with the case of uniformly distributed i.i.d.

In [8], it is proved that $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$ are equal to a constant if $\{n_k\}$ is a subsequence of $\{\theta^k\}$ and the set of possible values of constants coincides with the interval $[1/2, \Sigma_\theta]$. Therefore every value in $[1/2, \sqrt{42}/9]$ is proved to be a possible value of $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$ for some $\{n_k\}$ satisfying the Hadamard gap condition.

It is natural to ask whether values greater than $\sqrt{42}/9$ are also possible values of $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$.

Related to this question, we can find the following result by Berkes and Philipp [3]. For any real number L , there exists a sequence $\{n_k\}$ satisfying the Hadamard gap condition (1.1) such that $\Sigma\{n_k x\} \geq L$ a.e. Unfortunately, we cannot know by this result if $\Sigma\{n_k x\}$ is constant a.e. or not.

So far, the only known greater constants are those for the so-called Hardy-Littlewood-Pólya sequences defined as below: Let $q_1, \dots, q_\tau \geq 2$ be relatively prime integers and let $\{n_k\}$ be an arrangement in increasing order of $\{q_1^{i_1} \dots q_\tau^{i_\tau} \mid i_1, \dots, i_\tau = 0, 1, 2, \dots\}$. Then $\{n_k\}$ is said to be the Hardy-Littlewood-Pólya sequence generated by $q_1, \dots, q_\tau$. As to this sequence, $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$ are equal to a constant a.e. (cf. [6]), and when the set of generators consists of odd integers, then

$$\Sigma^*\{n_k x\} = \Sigma\{n_k x\} = \frac{1}{2} \left(\prod_{i=1}^{\tau} \frac{q_i + 1}{q_i - 1} \right)^{1/2}, \quad \text{a.e.}$$

See also [2] to find a detailed study for permutations of these sequences.

The last value becomes arbitrarily large if we choose generators properly. Unfortunately, when $\tau \geq 2$, Hardy-Littlewood-Pólya sequences do not satisfy the Hadamard gap condition. But the method of approximating the Hardy-Littlewood-Pólya sequence by subsequences satisfying the Hadamard gap condition can be found in [4]. We adopt this method together with randomization techniques to solve the above problem.

Now we are in a position to state our result.

Theorem 1.1. *For all $\sigma \geq 1/2$, there exists a sequence $\{n_k\}$ of positive integers satisfying the Hadamard gap condition (1.1) such that*

$$(1.3) \quad \Sigma\{n_k x\} = \Sigma^*\{n_k x\} = \sigma, \quad \text{a.e.}$$

For $0 < \sigma < 1/2$, there exists a sequence $\{n_k\}$ with bounded gaps $n_{k+1} - n_k = O(1)$ such that (1.3) holds (cf. [7]). It is an open question whether we can take such $\{n_k\}$ satisfying the Hadamard gap condition.

2. PROOF

We first note that discrepancies are written by

$$D_N\{a_k\} = \sup_{0 \leq a < b < 1} \left| \frac{1}{N} \sum_{k=1}^N \tilde{\mathbf{1}}_{a,b}(a_k) \right|, \quad D_N^*\{a_k\} = \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N \tilde{\mathbf{1}}_{0,a}(a_k) \right|,$$

where $\tilde{\mathbf{1}}_{a,b}(x) = \mathbf{1}_{[a,b)}(\langle x \rangle) - (b - a)$ and $\mathbf{1}_{[a,b)}$ denotes the indicator function of $[a, b)$.

Denote the sequence of all odd prime numbers by $3 = q_1 < q_2 < \dots$. Put $\phi(x) = x \vee 0$ and $\psi(x) = (-x) \vee 0$. We have $\phi(x) + \psi(x) = |x|$. Denote

$$\Phi_{I;j_1, \dots, j_\tau} = \frac{\phi(I - |j_1|)}{I} \dots \frac{\phi(I - |j_\tau|)}{I} \quad \text{and} \quad \Psi(x) = \sqrt{2x \log \log x}.$$

We use the next proposition.

Proposition 2.1. *Let $0 < p < 1$ and let $\{X_n\}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$. Let us regard the set $\{q_{\tau+1}^i \mid X_i(\omega) = 1\}$ as an increasing sequence and denote it by $\{n_k^\circ(\omega)\}$. Then*

$$\mathbb{P}\left(\#\{n_k^\circ\} = \infty, \overline{\lim}_{N \rightarrow \infty} \Psi^{-1}(N) \left| \sum_{k=1}^N f(n_k^\circ x) \right| = \sigma(f, p) \text{ a.e. } x, \text{ for all } f \in BV_0\right) = 1,$$

where BV_0 denotes the class of functions f of bounded variation with period 1 satisfying $\int_0^1 f(t) dt = 0$ and $\sigma^2(f, p)$ is given by

$$\sigma^2(f, p) = \int_0^1 f^2(x) dx + 2p \sum_{k=1}^{\infty} \int_0^1 f(q_{\tau+1}^k x) f(x) dx.$$

In [8] the authors proved a similar statement given by replacing ‘for all $f \in BV_0$ ’ with ‘for all $\tilde{\mathbf{1}}_{a,b}$ with $0 \leq a < b < 1$ ’ and $q_{\tau+1}$ with 3. Proposition 2.1 can be proved completely in the same way. We apply it for

$$f_{a,b}(x) = \frac{1}{I^{\tau/2}} \sum_{i_1=0}^{I-1} \dots \sum_{i_\tau=0}^{I-1} \tilde{\mathbf{1}}_{a,b}(q_1^{i_1} \dots q_\tau^{i_\tau} x).$$

Let us evaluate $\sigma^2(f_{a,b}, p)$. By changing the variable $y = q^{i \wedge i'} x$ and by noting $i - i \wedge i' = \phi(i - i')$ and $i' - i \wedge i' = \psi(i - i')$, we have

$$\frac{1}{I} \sum_{i=0}^{I-1} \sum_{i'=0}^{I-1} \int_0^1 g(q' q^i x) g(q'' q^{i'} x) dx = \sum_{j \in \mathbf{Z}} \frac{\phi(I - |j|)}{I} \int_0^1 g(q' q^{\phi(j)} y) g(q'' q^{\psi(j)} y) dy.$$

Here we used the fact that $\#\{(i, i') \in \{0, 1, \dots, I-1\}^2 \mid j = i - i'\} = \phi(I - |j|)$. Hence we have

$$\begin{aligned} & \int_0^1 f_{a,b}(q_{\tau+1}^{\phi(k)} x) f_{a,b}(q_{\tau+1}^{\psi(k)} x) dx \\ &= \frac{1}{I^\tau} \sum_{i_1, i'_1=0}^{I-1} \cdots \sum_{i_\tau, i'_\tau=0}^{I-1} \int_0^1 \tilde{\mathbf{1}}_{a,b}(q_1^{i_1} \cdots q_\tau^{i_\tau} q_{\tau+1}^{\phi(k)} x) \tilde{\mathbf{1}}_{a,b}(q_1^{i'_1} \cdots q_\tau^{i'_\tau} q_{\tau+1}^{\psi(k)} x) dx \\ &= \sum_{j_1, \dots, j_\tau \in \mathbf{Z}} \Phi_{I; j_1, \dots, j_\tau} \int_0^1 \tilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)} \cdots q_\tau^{\phi(j_\tau)} q_{\tau+1}^{\phi(k)} x) \tilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)} \cdots q_\tau^{\psi(j_\tau)} q_{\tau+1}^{\psi(k)} x) dx. \end{aligned}$$

Therefore we have

$$\begin{aligned} \sigma^2(f_{a,b}, 0) &= \int_0^1 f_{a,b}^2(x) dx \\ &= \sum_{j_1, \dots, j_\tau \in \mathbf{Z}} \Phi_{I; j_1, \dots, j_\tau} \int_0^1 \tilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)} \cdots q_\tau^{\phi(j_\tau)} x) \tilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)} \cdots q_\tau^{\psi(j_\tau)} x) dx, \end{aligned}$$

and by noting

$$\sigma^2(f_{a,b}, 1) = \sum_{k \in \mathbf{Z}} \int_0^1 f_{a,b}(q_{\tau+1}^{\phi(k)} x) f_{a,b}(q_{\tau+1}^{\psi(k)} x) dx,$$

we see that $\sigma^2(f_{a,b}, 1)$ equals

$$\sum_{k, j_1, \dots, j_\tau \in \mathbf{Z}} \Phi_{I; j_1, \dots, j_\tau} \int_0^1 \tilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)} \cdots q_\tau^{\phi(j_\tau)} q_{\tau+1}^{\phi(k)} x) \tilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)} \cdots q_\tau^{\psi(j_\tau)} q_{\tau+1}^{\psi(k)} x) dx.$$

To evaluate the integrals above, we use the next lemma, which is proved in [5].

Lemma 2.2. *For $x, y, \xi, \eta \in [0, 1]$, put $V(x, \xi) = x \wedge \xi - x\xi$ and $\tilde{V}(x, y, \xi, \eta) = V(x, \xi) + V(y, \eta) - V(x, \eta) - V(y, \xi)$. For any positive integers P and Q with $\gcd(P, Q) = 1$, we have*

$$\begin{aligned} \int_0^1 \tilde{\mathbf{1}}_{a,b}(Px) \tilde{\mathbf{1}}_{a,b}(Qx) dx &= \frac{1}{PQ} \tilde{V}(\langle Pa \rangle, \langle Pb \rangle, \langle Qa \rangle, \langle Qb \rangle), \\ \tilde{V}(\langle Pa \rangle, \langle Pb \rangle, \langle Qa \rangle, \langle Qb \rangle) &\leq V(\langle P(b-a) \rangle, \langle Q(b-a) \rangle) \leq 1/4. \end{aligned}$$

If P and Q are odd and if $a = 0, b = 1/2$, then $\tilde{V}(\langle Pa \rangle, \langle Pb \rangle, \langle Qa \rangle, \langle Qb \rangle) = 1/4$.

By applying this lemma and by noting that $\phi(x) + \psi(x) = |x|$, we have

$$\int_0^1 \tilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)} \cdots q_\tau^{\phi(j_\tau)} q_{\tau+1}^{\phi(k)} x) \tilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)} \cdots q_\tau^{\psi(j_\tau)} q_{\tau+1}^{\psi(k)} x) dx \leq \frac{1}{4q_1^{|j_1|} \cdots q_\tau^{|j_\tau|} q_{\tau+1}^{|k|}},$$

and we see that the equality holds if $a = 0$ and $b = 1/2$. Put

$$Q_\tau = \frac{1}{2} \left(\prod_{i=1}^{\tau} \frac{q_i + 1}{q_i - 1} \right)^{1/2} \quad \text{and} \quad Q_{\tau, I} = \frac{1}{2} \left(\sum_{j_1, \dots, j_\tau \in \mathbf{Z}} \frac{\Phi_{I; j_1, \dots, j_\tau}}{q_1^{|j_1|} \cdots q_\tau^{|j_\tau|}} \right)^{1/2}.$$

Hence we see

$$\begin{aligned}\sigma^2(f_{a,b}, 0) &\leq \frac{1}{4} \sum_{j_1, \dots, j_\tau \in \mathbf{Z}} \frac{\Phi_{I; j_1, \dots, j_\tau}}{q_1^{|j_1|} \dots q_\tau^{|j_\tau|}} = Q_{\tau, I}^2 = \sigma^2(f_{0,1/2}, 0), \\ \sigma^2(f_{a,b}, 1) &\leq \frac{1}{4} \sum_{k, j_1, \dots, j_\tau \in \mathbf{Z}} \frac{\Phi_{I; j_1, \dots, j_\tau}}{q_1^{|j_1|} \dots q_\tau^{|j_\tau|} q_{\tau+1}^{|k|}} = Q_{\tau, I}^2 \frac{q_{\tau+1} + 1}{q_{\tau+1} - 1} = \sigma^2(f_{0,1/2}, 1),\end{aligned}$$

which imply

$$\begin{aligned}\sup_{S \ni a < b \in S} \sigma(f_{a,b}, 0) &= \sup_{a \in S} \sigma(f_{0,a}, 0) = \sigma(f_{0,1/2}, 0) = Q_{\tau, I}, \\ \sup_{S \ni a < b \in S} \sigma(f_{a,b}, 1) &= \sup_{a \in S} \sigma(f_{0,a}, 1) = \sigma(f_{0,1/2}, 1) = Q_{\tau, I} \left(\frac{q_{\tau+1} + 1}{q_{\tau+1} - 1} \right)^{1/2}\end{aligned}$$

where $S = [0, 1) \cap \mathbf{Q}$. By $\sigma^2(f_{a,b}, p) = (1-p)\sigma^2(f_{a,b}, 0) + p\sigma^2(f_{a,b}, 1)$, we also have

$$\sup_{S \ni a < b \in S} \sigma(f_{a,b}, p) = \sup_{a \in S} \sigma(f_{0,a}, p) = \sigma(f_{0,1/2}, p) = \left(1 - p + p \frac{q_{\tau+1} + 1}{q_{\tau+1} - 1} \right)^{1/2} Q_{\tau, I}.$$

Let $\{n_k\}$ be an arrangement in increasing order of

$$\{n_k^\circ q_1^{i_1} \dots q_\tau^{i_\tau} : k = 0, 1, \dots; i_1, \dots, i_\tau = 0, 1, \dots, I-1\}.$$

It is clear that $\{n_k\}$ satisfies the Hadamard gap condition. Let us take $\rho(k)$ such that $n_{\rho(k)} = n_k^\circ q_1^{I-1} \dots q_\tau^{I-1}$. If $n_k^\circ q_1^{i_1'} \dots q_\tau^{i_\tau'} \leq n_k^\circ q_1^{I-1} \dots q_\tau^{I-1}$, then $n_{k'}/n_k^\circ \leq q_1^{I-1} \dots q_\tau^{I-1}$. In case $k' \geq k$, we have $3^{k'-k} \leq n_{k'}/n_k^\circ$ and thereby $k' \leq k + \log_3 q_1^{I-1} \dots q_\tau^{I-1}$. The last inequality is also valid when $k' < k$. Hence we see

$$\begin{aligned}\{n_k^\circ q_1^{i_1} \dots q_\tau^{i_\tau} \mid k \leq K, i_1, \dots, i_\tau < I\} &\subset \{n_k \mid k \leq \rho(K)\} \\ &\subset \{n_k^\circ q_1^{i_1} \dots q_\tau^{i_\tau} \mid k \leq K + \log_3 q_1^{I-1} \dots q_\tau^{I-1}, i_1, \dots, i_\tau < I\}.\end{aligned}$$

Therefore we have $KI^\tau \leq \rho(K) \leq KI^\tau + I^\tau \log_3 q_1^{I-1} \dots q_\tau^{I-1}$ and $\rho(K) \sim KI^\tau$.

For given N , take the largest K such that $KI^\tau \leq N$. Then $N < KI^\tau + I^\tau$. It implies $|N - \rho(K)| \leq I^\tau + I^\tau \log_3 q_1^{I-1} \dots q_\tau^{I-1}$. Therefore we have

$$\left| \sum_{k=1}^N \tilde{\mathbf{1}}_{a,b}(n_k x) - \sum_{k=1}^K \sum_{i_1, \dots, i_\tau < I} \tilde{\mathbf{1}}_{a,b}(n_k^\circ q_1^{i_1} \dots q_\tau^{i_\tau} x) \right| = O(1),$$

and thereby we have

$$\sigma(f_{a,b}, p) = \overline{\lim}_{K \rightarrow \infty} \Psi(K)^{-1} \sum_{k=1}^K f_{a,b}(n_k^\circ x) = \overline{\lim}_{N \rightarrow \infty} \Psi(N)^{-1} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{a,b}(n_k x) \right|, \quad \text{a.e.}$$

We use the fundamental relation below, which can be found in [7].

Lemma 2.3. *For any countable dense subset S of $[0, 1)$ and for any sequence $\{n_k\}$ of positive real numbers satisfying the Hadamard gap condition, we have*

$$\begin{aligned}\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} &= \sup_{S \ni a < b \in S} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{a,b}(n_k x) \right|, \quad \text{a.e.}, \\ \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}} &= \sup_{a \in S} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{0,a}(n_k x) \right|, \quad \text{a.e.}\end{aligned}$$

By applying the lemma, we have

$$\Sigma\{n_k x\} = \Sigma^*\{n_k x\} = \left(1 - p + p \frac{q_{\tau+1} + 1}{q_{\tau+1} - 1}\right)^{1/2} Q_{\tau, I}, \quad \text{a.e.}$$

Therefore, for every $\sigma \in (Q_{\tau, I}, (\frac{q_{\tau+1}+1}{q_{\tau+1}-1})^{1/2} Q_{\tau, I})$, there exists a $p \in (0, 1)$ such that (1.3) holds. Because $Q_{\tau, I} < Q_\tau$, $Q_{\tau, I} \uparrow Q_\tau$ as $I \rightarrow \infty$, and $(\frac{q_{\tau+1}+1}{q_{\tau+1}-1})^{1/2} Q_\tau = Q_{\tau+1}$, we see that for every $\sigma \in [Q_\tau, Q_{\tau+1})$ there exists a sequence $\{n_k\}$ satisfying (1.1) and (1.3). Because $Q_1 = 1/\sqrt{2}$ and $Q_\tau \rightarrow \infty$ as $\tau \rightarrow \infty$, we can conclude that for every $\sigma > 1/\sqrt{2}$ there exists a sequence $\{n_k\}$ satisfying (1.1) and (1.3). On the other hand, for any $\sigma \in [1/2, 1/\sqrt{2}]$ there exists a subsequence $\{n_k\}$ of $\{3^k\}$ such that (1.1) and (1.3) hold (cf. [8]).

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