WIENER TYPE THEOREMS FOR JACOBI SERIES WITH NONNEGATIVE COEFFICIENTS

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ABSTRACT. This paper gives three theorems regarding functions integrable on $[-1, 1]$ with respect to Jacobi weights and having nonnegative coefficients in their (Fourier-)Jacobi expansions. We show that the L^p -integrability (with respect to the Jacobi weight) on an interval near 1 implies the L^p -integrability on the whole interval if p is an even integer. The Jacobi expansion of a function locally in L^∞ near 1 is shown to converge uniformly and absolutely on $[-1, 1]$; in particular, such a function is shown to be continuous on $[-1, 1]$. Similar results are obtained for functions in local Besov approximation spaces.

1. INTRODUCTION

A well-known theorem by Norbert Wiener states (see e.g. [5, pp. 242, 250]) that if f is a 2π -periodic function in $L^1(-\pi, \pi)$ with nonnegative Fourier coefficients $c_n(f) \geq 0$ and $f \in L^2(-\delta, \delta)$ for some $\delta > 0$, then $f \in L^2(-\pi, \pi)$. Since the Fourier coefficients of $|f|^2$ are given formally by $\sum_{k \in \mathbb{Z}} c_{n-k}(f) \overline{c_k(f)}$, it is easy to deduce that a similar conclusion holds also when the L^2 norms are replaced by L^p norms, $p = 2, 4, 6, \dots$. For $1 \leq p \leq \infty$, let

$$L_{\text{loc},+}^p := \{f \in L^1(-\pi, \pi) : f \sim \sum_{n \in \mathbb{Z}} c_n(f) e^{ino}, \ c_n(f) \geq 0 \text{ for every } n \in \mathbb{Z}, \\ \text{and there exists } \delta > 0 \text{ such that } f \in L^p(-\delta, \delta)\}.$$

Wiener's theorem can now be formulated more concisely as follows: For even positive integer values of p , $L_{\text{loc},+}^p \subset L^p(-\pi, \pi)$. Wainger [16] and Shapiro [12] have given counterexamples to show that such an inclusion is not true if p is not an even positive integer.

Even though the inclusion $L_{\text{loc},+}^p \subset L^p(-\pi, \pi)$ does not hold in general, the variation stated in the following Theorem 1.1 was proved in [1]. A subspace $X \subset L^1(-\pi, \pi)$ is called *solid* if $f, g \in L^1(-\pi, \pi)$, $|c_n(f)| \leq c_n(g)$, $n \in \mathbb{Z}$, and $g \in X$ imply that $f \in X$. (Some authors use the phrase “with upper majorant property”

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instead of solid.) For example, if $1 < p \leq 2$ and p' is its conjugate exponent, then the Hausdorff–Young inequality [17, Chapter XII, (2.3)] implies that

$$L^p(-\pi, \pi) \subset \left\{ f \in L^1(-\pi, \pi) : \sum_{n \in \mathbb{Z}} |c_n(f)|^{p'} < \infty \right\} =: \ell^{p'}.$$

Similarly, a result of Hardy and Littlewood (cf. [17, Chapter XII, (3.19)]) shows that if $1 < p \leq 2$, then

$$L^p(-\pi, \pi) \subset \left\{ f \in L^1(-\pi, \pi) : \sum_{n \in \mathbb{Z}} (|n| + 1)^{p-2} |c_n(f)|^p < \infty \right\} =: H L^p.$$

Clearly, both $\ell^{p'}$ and $H L^p$ are solid spaces. Other similar examples can be found in [1].

Theorem 1.1. *Let X be a solid space, $L^p(-\pi, \pi) \subseteq X$. Then*

$$(1.1) \quad L_{\text{loc},+}^p \subseteq X.$$

In the case when p is an even positive integer, the space $L^p(-\pi, \pi)$ is itself a solid space, and hence Theorem 1.1 is a generalization of Wiener's theorem.

In the case when $p = \infty$, Paley [11] observed that if $f \in L_{\text{loc},+}^\infty$ and f is an even function, then f is continuous on $[-\pi, \pi]$ and its Fourier series converges uniformly and absolutely. In fact, certain smoothness properties of such functions can be *characterized* in terms of their Fourier coefficients. For example, a theorem of Lorentz [9, Section 4] implies immediately the following result: Let $f(x) \sim \sum_{n \geq 0} c_n \cos nx$, $c_n \geq 0$, and assume further that $(c_n)_{n \geq 0}$ is a monotone sequence. Let $E_n(f)$ be the best approximation of f (in the supremum norm) by trigonometric polynomials of order n , and let $\gamma \in (0, 1)$. Then $\limsup_{n \rightarrow \infty} n^\gamma E_n(f) < \infty$ if and only if $\limsup_{n \rightarrow \infty} n^{\gamma+1} c_n < \infty$. Extensions of this result to the case of Besov and Besov–Nikolskii spaces have been given in [7, 2, 14], so as to include the case of L^p , $1 < p < \infty$, as well.

Our main goal in this paper is to present analogues of the above results for functions on $[-1, 1]$ and their Jacobi polynomial expansions. We must defer the precise statements to Section 3 after the necessary notation and definitions are given, but we will describe here briefly the results and ideas to the extent we can. We will consider a class $\mathbb{P}$ of functions with nonnegative Jacobi coefficients, where the parameters of the Jacobi polynomials are assumed to be $\geq -1/2$. The role of 0 in the definition of local spaces in the context of trigonometric series will now be played by 1. In Theorem 3.1, we show that if X is a solid space (defined based on Jacobi coefficients) and if $f \in \mathbb{P}$ locally in X in a neighborhood of 1, then $f \in X$. In Theorem 3.4, we will prove that the Jacobi expansion of a function in $\mathbb{P}$ that is locally essentially bounded near 1 is absolutely and uniformly convergent on $[-1, 1]$, so that the function is not just essentially bounded but continuous on $[-1, 1]$. In Theorem 3.6, we state a refinement of this result where Besov spaces and their local analogues are considered rather than mere continuity. Our results make extensive use of the product and linearization formula for Jacobi polynomials ([3, 8]) as well as the summability and frame operators for Jacobi expansions ([10]).

In Section 2, we review some necessary facts regarding Jacobi polynomials and introduce some notation. The main theorems are stated and proved in Section 3.

In order to avoid unnecessarily complicated notation, the symbols used in this introduction will have different meanings in the rest of this paper.

2. JACOBI POLYNOMIALS

We will denote the set of all nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. If $n \in \mathbb{N}_0$, we will denote by Π_n (respectively, $\Pi_{n/2}$) the class of all polynomials of degree at most n (respectively, $n/2$). Let $\alpha, \beta > -1$, and let

$$w_{\alpha, \beta}(x) := \begin{cases} (1-x)^\alpha(1+x)^\beta, & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq p < \infty$ the space $L^p = L^p(\alpha, \beta)$ is defined as the space of (equivalence classes of) functions f with

$$\|f\|_{\alpha, \beta; p} := \left(\int_{-1}^1 |f(x)|^p w_{\alpha, \beta}(x) dx \right)^{1/p} < \infty.$$

The space of all continuous real-valued functions on $[-1, 1]$, equipped with the supremum norm, will be denoted by C , and the supremum norm of $f \in C$ will be denoted by $\|f\|_\infty$. The space of all infinitely often differentiable $f : [-1, 1] \rightarrow \mathbb{C}$ will be denoted by C^∞ .

There exists a unique system of (Jacobi) polynomials $(R_k^{(\alpha, \beta)} \in \Pi_k)_{k \in \mathbb{N}_0}$ such that for integer $k, \ell \in \mathbb{N}_0$, $R_k^{(\alpha, \beta)}(1) = 1$ and

$$(2.1) \quad \int_{-1}^1 R_k^{(\alpha, \beta)}(x) R_\ell^{(\alpha, \beta)}(x) w_{\alpha, \beta}(x) dx = \begin{cases} \rho_k, & \text{if } k = \ell, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$(2.2) \quad \rho_k = \|R_k\|_{\alpha, \beta; 2}^2 = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+1)^2}{2k + \alpha + \beta + 1} \frac{\Gamma(k+1) \Gamma(k+\beta+1)}{\Gamma(k+\alpha+1) \Gamma(k+\alpha+\beta+1)}.$$

The uniqueness of the system implies that $R_k^{(\beta, \alpha)}(x) = R_k^{(\alpha, \beta)}(-x)/R_k^{(\alpha, \beta)}(-1)$, $x \in \mathbb{R}$, $k \in \mathbb{N}_0$. Therefore, we may assume in the sequel that $\alpha \geq \beta$. We will assume also that $\alpha \geq \beta \geq -1/2$. Note that the latter assumption will be used further (see formula (2.5)).

For $f \in L^1$, we may define the Jacobi coefficients by

$$\hat{f}(k) := \hat{f}(\alpha, \beta; k) := \rho_k^{-1} \int_{-1}^1 f(y) R_k^{(\alpha, \beta)}(y) w_{\alpha, \beta}(y) dy, \quad k \in \mathbb{N}_0.$$

Then the formal Jacobi expansion of f has the form $\sum_{k=0}^{\infty} \hat{f}(k) R_k(x)$.

Next, we need formulas for functions $f * g$ such that $\widehat{f * g}(k) = \hat{f}(k) \hat{g}(k)$, $k \in \mathbb{N}_0$, and for the Jacobi coefficients of the product fg . Following Koornwinder [8], if $x, y \in [-1, 1]$, $r \in [0, 1]$, and $\psi \in [0, \pi]$, let

$$(2.3) \quad Z(x, y; r, \psi) = \frac{1}{2}(1+x)(1+y) + \sqrt{1-x^2} \sqrt{1-y^2} r \cos \psi + \frac{1}{2}(1-x)(1-y) r^2 - 1.$$

We observe that

$$\begin{aligned} Z(x, y; r, \psi) &= \frac{1}{2} \left(\sqrt{(1+x)(1+y)} - r \sqrt{(1-x)(1-y)} \right)^2 \\ &\quad + r \sqrt{1-x^2} \sqrt{1-y^2} (1 + \cos \psi) - 1 \geq -1. \end{aligned}$$

If $\theta, \varphi \in [0, \pi]$ such that $x = \cos \theta, y = \cos \varphi$, then

$$\begin{aligned} Z(x, y; r, \psi) &= \frac{1}{2}(1+x)(1+y) + \frac{1}{2}(1-x)(1-y) + \sqrt{1-x^2}\sqrt{1-y^2} \\ &\quad + \frac{1}{2}(1-x)(1-y)(r^2-1) + \sqrt{1-x^2}\sqrt{1-y^2}(r \cos \psi - 1) - 1 \\ &= xy + \sqrt{1-x^2}\sqrt{1-y^2} + \frac{1}{2}(1-x)(1-y)(r^2-1) + \sqrt{1-x^2}\sqrt{1-y^2}(r \cos \psi - 1), \end{aligned}$$

and we have

$$\begin{aligned} 1 - Z(x, y; r, \psi) &= 1 - (xy + \sqrt{1-x^2}\sqrt{1-y^2}) + \frac{1}{2}(1-x)(1-y)(1-r^2) \\ &\quad + (1-r \cos \psi)\sqrt{1-x^2}\sqrt{1-y^2} \\ (2.4) \quad &\geq 1 - (xy + \sqrt{1-x^2}\sqrt{1-y^2}) = 1 - \cos(\theta - \varphi) \geq 0. \end{aligned}$$

Thus, $Z(x, y; r, \psi) \in [-1, 1]$ for all $x, y \in [-1, 1]$, $r \in [0, 1]$ and $\psi \in [0, \pi]$.

For $\alpha \geq \beta \geq -1/2$, Koornwinder [8] has proved that there exists a probability measure $\nu^{(\alpha, \beta)}$ on $[0, 1] \times [0, \pi]$ such that we have

$$(2.5) \quad R_n^{(\alpha, \beta)}(x) R_n^{(\alpha, \beta)}(y) = \int_0^\pi \int_0^1 R_n^{(\alpha, \beta)}(Z(x, y; r, \psi)) d\nu^{(\alpha, \beta)}(r, \psi), \quad n \in \mathbb{N}_0.$$

An interesting consequence of (2.5) is the following. For almost all $x, y \in [-1, 1]$ and $f \in L^1$, let

$$(2.6) \quad \mathcal{T}_y f(x) = \int_0^\pi \int_0^1 f(Z(x, y; r, \psi)) d\nu^{(\alpha, \beta)}(r, \psi).$$

Then it is clear that for $x, y \in [-1, 1]$, $\mathcal{T}_y f(x) = \mathcal{T}_x f(y)$ and

$$(2.7) \quad \widehat{\mathcal{T}_y f}(k) = \hat{f}(k) R_k^{(\alpha, \beta)}(y).$$

The corresponding convolution operator is defined by

$$(f * g)(x) := \int_{-1}^1 f(y) \mathcal{T}_y g(x) w_{\alpha, \beta}(y) dy, \quad f, g \in L^1.$$

We have

$$(2.8) \quad \widehat{f * g}(k) = \hat{f}(k) \hat{g}(k), \quad k \in \mathbb{N}_0, \quad f, g \in L^1.$$

Dual to the product formula (2.5) is a linearization formula (see [3]):

$$(2.9) \quad R_n^{(\alpha, \beta)}(x) R_m^{(\alpha, \beta)}(x) = \sum_{k=|n-m|}^{n+m} g^{(\alpha, \beta)}(n, m; k) R_k^{(\alpha, \beta)}(x)$$

for all $n, m \in \mathbb{N}_0, x \in [-1, 1]$. The coefficients $g^{(\alpha, \beta)}(n, m; k)$ are nonnegative for all $k, n, m \in \mathbb{N}_0$ and, moreover,

$$(2.10) \quad \sum_{k=|n-m|}^{n+m} g^{(\alpha, \beta)}(n, m; k) = 1.$$

Thus, we may think of $(g^{(\alpha, \beta)}(n, m; k))_k$ as a probability distribution on a subset of $\mathbb{N}_0$. We note also that

$$g^{(\alpha, \beta)}(n, m; k) = \rho_k^{-1} \int_{-1}^1 R_n^{(\alpha, \beta)}(y) R_m^{(\alpha, \beta)}(y) R_k^{(\alpha, \beta)}(y) w_{\alpha, \beta}(y) dy.$$

In particular,

$$(2.11) \quad g^{(\alpha, \beta)}(k, 0; k) = \rho_k^{-1} \int_{-1}^1 \left\{ R_k^{(\alpha, \beta)}(y) \right\}^2 w_{\alpha, \beta}(y) dy = 1.$$

Just as (2.5) can be used via (2.6) to define a generalized convolution of two functions, (2.9) can be used to define a convolution of sequences. If $\mathbf{a} = (a_k)_{k \in \mathbb{N}_0}$ and $\mathbf{b} = (b_k)_{k \in \mathbb{N}_0}$, we define formally

$$(2.12) \quad (\mathbf{a} * \mathbf{b})(k) = \sum_{n, m=0}^{\infty} g^{(\alpha, \beta)}(n, m; k) a_n b_m.$$

Analogous to (2.8) and the classical Cauchy formula for the products of power series, we have the following formula for products of formal Jacobi expansions:

$$(2.13) \quad \left(\sum_{n=0}^{\infty} a_n R_n^{(\alpha, \beta)}(x) \right) \left(\sum_{m=0}^{\infty} b_m R_m^{(\alpha, \beta)}(x) \right) = \sum_{k=0}^{\infty} (\mathbf{a} * \mathbf{b})(k) R_k^{(\alpha, \beta)}(x), \quad x \in [-1, 1].$$

3. MAIN RESULTS

In the sequel, we consider $\alpha \geq \beta \geq -1/2$ to be fixed parameters. Let $\mathbb{P}$ be the class of all $f \in L^1$ such that $\hat{f}(k) \geq 0$ for $k \in \mathbb{N}_0$. It is well known that in the theory of Jacobi expansions, the point 1 plays the role of 0 in the theory of Fourier series. Thus, if T is a trigonometric polynomial with nonnegative Fourier coefficients, then $\max_{x \in [-\pi, \pi]} |T(x)| = T(0)$. Similarly, in view of [13, Theorem 7.32.1],

$$(3.1) \quad \max_{x \in [-1, 1]} |R_k^{(\alpha, \beta)}(x)| = R_k^{(\alpha, \beta)}(1) = 1.$$

Therefore, it is easy to verify that for every $P \in \mathbb{P} \cap \left(\bigcup_{n=0}^{\infty} \Pi_n \right)$ and $a \in [-1, 1)$,

$$(3.2) \quad \max_{x \in [-1, 1]} |P(x)| = P(1) = \max_{x \in [a, 1]} |P(x)|.$$

Accordingly, we define the analogues of local spaces as follows. If Y is any subspace of L^1 , then Y_{loc} is the class of all $f \in L^1$ with the following property: there exists a nondegenerate interval $I \subseteq [-1, 1]$ with $1 \in I$ such that for any $\phi \in C^\infty$ supported on I , $f\phi \in Y$. If $f, g \in L^1$, we will write $f \preceq g$ if $|\hat{f}(k)| \leq \hat{g}(k)$ for all $k \in \mathbb{N}_0$. A subspace $X \subset L^1$ is called *solid* if $f, g \in L^1$, $f \preceq g$, and $g \in X$ imply that $f \in X$. For example, using (2.13), (2.12), and the fact that $g^{(\alpha, \beta)}(n, m; k) \geq 0$ for all $n, m, k \in \mathbb{N}_0$, it is easy to conclude that when $f \preceq g$, then $|f|^2 \preceq |g|^2$. Hence, if p is an even positive integer, then L^p is a solid space. Thus, Theorem 3.1 below is the analogue of Wiener's theorem for Jacobi expansions.

Our first main result is the analogue of Theorem 1.1.

Theorem 3.1. *Let $X \subset L^1$ be a solid space. Then $X_{\text{loc}} \cap \mathbb{P} = X \cap \mathbb{P}$. In particular, if $1 < p \leq \infty$ and $L^p \subseteq X$, then $L_{\text{loc}}^p \cap \mathbb{P} \subseteq X$.*

In the case when p is not a positive even integer, we note that using the counterexamples of Wainger and Shapiro, we can always construct a cosine trigonometric series with positive coefficients such that its sum is not p -th power integrable on $[0, \pi]$. Since $R_k^{(-1/2, -1/2)}$ are Chebyshev polynomials of the first kind, these counterexamples also demonstrate that with $\alpha = \beta = -1/2$, $L_{\text{loc}}^p \cap \mathbb{P} \not\subseteq L^p$ when p is not a positive even integer.

The proof of Theorem 3.1 relies upon the following two lemmas. If $f \in C$, we define

$$(3.3) \quad E_n(f) := \min_{P \in \Pi_n} \|f - P\|_\infty.$$

Lemma 3.2. *Let $f \in C^\infty$ and $x \in [-1, 1]$. Then the function $x \mapsto \mathcal{T}_x f$ is in C^∞ .*

Proof. The well-known direct theorem of approximation theory [15, Section 5.1.5, (22)] implies that for any integer $S \geq 1$, $E_n(f) \leq c_1(f, S)n^{-S}$ for some positive constant $c_1(f, S)$ independent of n . Let $\ell \geq 0$ be an integer. We choose $S > 2\ell + 2\alpha + 3$. In view of (2.1) and (3.1), we obtain for $k = 2, 3, \dots$ and any $P \in \Pi_{k-1}$,

$$\begin{aligned} |\hat{f}(k)| &= \rho_k^{-1} \left| \int_{-1}^1 f(t) R_k(t) w_{\alpha, \beta}(t) dt \right| = \rho_k^{-1} \left| \int_{-1}^1 (f(t) - P(t)) R_k(t) w_{\alpha, \beta}(t) dt \right| \\ &\leq \rho_k^{-1} \rho_0 \|f - P\|_\infty. \end{aligned}$$

Thus, (2.2) leads to the estimate $|\hat{f}(k)| \leq c_2(S, \alpha, \beta, f) k^{2\alpha+1-S}$. Moreover, Markov's inequality [15, Section 4.8.62, (32)] implies that $\|R_k^{(\ell)}\|_\infty \leq k^{2\ell}$. Since $S > 2\ell + 2\alpha + 3$, it follows that

$$\sum_{k=2}^{\infty} |\hat{f}(k)| \|R_k\|_\infty \|R_k^{(\ell)}\|_\infty \leq c_2(S, \alpha, \beta, f) \sum_{k=2}^{\infty} k^{2\alpha+1-S+2\ell} < \infty.$$

This completes the proof. $\square$

Lemma 3.3. *Let $\delta \in (0, 1)$. There exists $\phi_\delta : [-1, 1] \rightarrow [0, \infty)$ such that $\phi_\delta \in C^\infty$, $\phi_\delta(t) = 0$ if $-1 \leq t \leq 1 - \delta$, $\phi_\delta \in \mathbb{P}$, and $\hat{\phi}_\delta(0) = 1$. If $f \in \mathbb{P}$, then $f \preceq f\phi_\delta$.*

Proof. In this proof only, let $\theta_0 \in (0, \pi/4)$ be chosen so that $1 - \delta = \cos(2\theta_0)$, and let $g = g_\delta : [-1, 1] \rightarrow [0, \infty)$ be a function in C^∞ , supported on $[\cos \theta_0, 1]$, such that $\hat{g}(0) = 1$. Let

$$\begin{aligned} \phi_\delta(x) &:= (g * g)(x) = \int_{-1}^1 g(y) \mathcal{T}_y g(x) w_{\alpha, \beta}(y) dy \\ &= \int_{-1}^1 g(y) \mathcal{T}_x g(y) w_{\alpha, \beta}(y) dy, \quad x \in [-1, 1]. \end{aligned}$$

In view of Lemma 3.2, $\phi_\delta : [-1, 1] \rightarrow [0, \infty)$ is in C^∞ . Also, in view of (2.8), $\hat{\phi}_\delta(k) = (\hat{g}(k))^2 \geq 0$, $k \in \mathbb{N}_0$, and in particular, $\hat{\phi}_\delta(0) = (\hat{g}(0))^2 = 1$.

If $|\theta - \varphi| \geq \theta_0$, then (2.4) implies that $Z(x, y; r, \psi) \leq \cos(\theta - \varphi) \leq \cos \theta_0$. Therefore, $g(Z(x, y; r, \psi)) = 0$ and hence, $\mathcal{T}_y g(x) = 0$. Hence, if J denotes the set $\{\cos \varphi : |\theta - \varphi| \leq \theta_0\}$, then

$$\phi_\delta(x) = \int_J g(y) \mathcal{T}_y g(x) w_{\alpha, \beta}(y) dy.$$

If $\theta \geq 2\theta_0$, then for $y \in J$, $0 < \theta_0 \leq \theta - \theta_0 \leq \varphi$; i.e., $y = \cos \varphi \leq \cos \theta_0$, and $g(y) = 0$. Thus, ϕ_δ is supported on $[\cos(2\theta_0), 1] = [1 - \delta, 1]$.

Next, let $f \in \mathbb{P}$. Since $\hat{f}(n) \geq 0$ for all $n \in \mathbb{N}_0$, we obtain in view of (2.13), (2.12), (2.11) that for each $k \in \mathbb{N}_0$,

$$\begin{aligned} \widehat{f\phi_\delta}(k) &= \sum_{n, m=0}^{\infty} g^{(\alpha, \beta)}(n, m; k) \hat{f}(n) \hat{\phi}_\delta(m) \\ (3.4) \quad &\geq g^{(\alpha, \beta)}(k, 0; k) \hat{f}(k) \hat{\phi}_\delta(0) = \hat{f}(k). \end{aligned}$$

Thus, $f \preceq f\phi_\delta$. $\square$

Proof of Theorem 3.1. Let $f \in X_{\text{loc}} \cap \mathbb{P}$ and $\delta \in (0, 1)$ be such that $f\phi_\delta \in X$, where ϕ_δ is as in Lemma 3.3. Since X is solid, $f \preceq f\phi_\delta$, and $f\phi_\delta \in X$, it follows that $f \in X$ as well. $\square$

Next, we consider the case $p = \infty$. First, we state an analogue of the result of Paley.

Theorem 3.4. *If $f \in L_{\text{loc}}^\infty \cap \mathbb{P}$, then $\sum_{k=0}^\infty \hat{f}(k)|R_k(x)| < \infty$, $x \in [-1, 1]$, with the series converging uniformly on $[-1, 1]$. In particular, $L_{\text{loc}}^\infty \cap \mathbb{P} = C \cap \mathbb{P}$.*

The proof of this theorem depends upon the following lemma, which is formulated by many authors; a proof can be found, for example, in [10, Lemma 4.2]. First, we define the de la Vallée-Poussin type operators. Let $h : [0, \infty) \rightarrow [0, 1]$ be a C^∞ , nonincreasing function such that $h(x) = 1$ for $[0, 1/2]$ and $h(x) = 0$ for $[1, \infty)$. For $f \in L^1$, $n \geq 0$, $x \in [-1, 1]$, let

$$(3.5) \quad \sigma_n(h, f, x) = \sum_{k=0}^{n-1} h\left(\frac{k}{n}\right) \hat{f}(k) R_k(x)$$

and

$$(3.6) \quad \tau_n(h, f) = \sigma_{2n}(h, f) - \sigma_n(h, f).$$

Lemma 3.5. *Let $f \in C$. Then*

$$(3.7) \quad \|\sigma_n(h, f)\|_\infty \leq c\|f\|_\infty,$$

where c is a positive constant depending only on α , β , and h . Moreover, $\sigma_n(h, P) = P$ if $P \in \Pi_{n/2}$, and hence

$$(3.8) \quad E_n(f) \leq \|f - \sigma_n(h, f)\|_\infty \leq (1 + c)E_{n/2}(f).$$

Proof of Theorem 3.4. Let $f \in L_{\text{loc}}^\infty \cap \mathbb{P}$ and $\delta \in (0, 1)$ be such that $f\phi_\delta \in L^\infty$, where ϕ_δ is as in Lemma 3.3. Let $n \geq 1$ be an integer. Since h is a nonnegative function, $f \in \mathbb{P}$, and $f \preceq f\phi_\delta$, using (3.1) we see that

$$h\left(\frac{k}{2n}\right) \hat{f}(k) \|R_k\|_\infty = h\left(\frac{k}{2n}\right) \hat{f}(k) \leq h\left(\frac{k}{2n}\right) \widehat{f\phi_\delta}(k), \quad k \in \mathbb{N}_0.$$

Since $h(t) = 1$ for $0 \leq t \leq 1/2$, we obtain using (3.2) and (3.7) that for every integer $n \geq 1$,

$$\begin{aligned} \sum_{k=0}^n \hat{f}(k) \|R_k\|_\infty &\leq \sum_{k=0}^\infty h\left(\frac{k}{2n}\right) \hat{f}(k) \|R_k\|_\infty \leq \sum_{k=0}^\infty h\left(\frac{k}{2n}\right) \widehat{f\phi_\delta}(k) \\ &= \|\sigma_{2n}(h, f\phi_\delta)\|_\infty \leq c\|f\phi_\delta\|_\infty < \infty. \end{aligned}$$

This completes the proof. $\square$

Finally, we discuss a variant for Besov spaces, which we now define. Let $0 < \rho \leq \infty$, $\gamma > 0$, and $\mathbf{a} = (a_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers. We define a sequence of space as

$$(3.9) \quad \|\mathbf{a}\|_{\rho, \gamma} := \begin{cases} \left[\sum_{n=0}^\infty 2^{n\gamma\rho} |a_n|^\rho \right]^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \sup_{n \geq 0} 2^{n\gamma} |a_n|, & \text{if } \rho = \infty. \end{cases}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\rho,\gamma} < \infty$ will be denoted by $\mathbf{b}_{\rho,\gamma}$. For $0 < \rho \leq \infty$, $\gamma > 0$, the Besov space $B_{\rho,\gamma}$ consists of functions $f \in L^\infty$ for which the sequence $(E_{2^n}(f))_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$. We have proved in [10, Theorem 2.1] that $f \in B_{\rho,\gamma}$ if and only if $(\|\tau_{2^n}(h, f)\|_\infty)_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$.

The following theorem is a refinement of Theorem 3.4. Let $S_n(f)$ be the n -th partial sum of the Jacobi expansion of f , i.e., $S_n(f) = \sum_{k=0}^{n-1} \hat{f}(k) R_k$.

Theorem 3.6. *Let $0 < \rho \leq \infty$, $\gamma > 0$. For $f \in \mathbb{P}$ the following conditions are equivalent:*

- (a) $f \in (B_{\rho,\gamma})_{\text{loc}}$,
- (b) $f \in B_{\rho,\gamma}$,
- (c) $\left(\|S_{2^{n+1}} - S_{2^n}(f)\|_\infty = \sum_{k=2^n}^{2^{n+1}-1} \hat{f}(k) \right)_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$.

Proof. To prove (a) $\Rightarrow$ (b), we need to show that $(B_{\rho,\gamma})_{\text{loc}} \cap \mathbb{P} \subset B_{\rho,\gamma}$. Let $f \in (B_{\rho,\gamma})_{\text{loc}} \cap \mathbb{P}$ and $\delta \in (0, 1)$ be such that $f\phi_\delta \in B_{\rho,\gamma} \cap \mathbb{P}$, where ϕ_δ is as in Lemma 3.3. Then $(\|\tau_{2^n}(h, f\phi_\delta)\|_\infty)_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$. Since h is nonincreasing, we have

$$g_{k,n} := h(k/2^{n+1}) - h(k/2^n) \geq 0, \quad k, n \in \mathbb{N}_0.$$

Since $f \preceq f\phi_\delta$, this implies that

$$0 \leq g_{k,n} \hat{f}(k) \leq g_{k,n} \widehat{f\phi_\delta}(k), \quad k, n \in \mathbb{N}_0.$$

So, using (3.2), we conclude that for $n \in \mathbb{N}_0$,

$$\begin{aligned} \|\tau_{2^n}(h, f)\|_\infty &= \tau_{2^n}(h, f, 1) = \sum_{k=0}^{\infty} g_{k,n} \hat{f}(k) \\ &\leq \sum_{k=0}^{\infty} g_{k,n} \widehat{f\phi_\delta}(k) = \|\tau_{2^n}(h, f\phi_\delta)\|_\infty. \end{aligned}$$

Since $(\|\tau_{2^n}(h, f\phi_\delta)\|_\infty)_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$, this implies that $(\|\tau_{2^n}(h, f)\|_\infty)_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$; i.e., $f \in B_{\rho,\gamma}$.

The implication (b) $\Rightarrow$ (a) is clear.

We prove that (a) (alternatively (b)) $\Rightarrow$ (c). Since (a) implies in particular that $f \in L_{\text{loc}}^\infty \cap \mathbb{P}$, Theorem 3.4 implies that the series $\sum \hat{f}(k) R_k$ converges uniformly and absolutely to f . Then, by (3.1) and (3.8), we have

$$\begin{aligned} E_n(f) &\leq \|f - S_n(f)\|_\infty \leq \sum_{k=n}^{\infty} \hat{f}(k) \leq \sum_{k=0}^{\infty} (1 - h(k/n)) \hat{f}(k) = f(1) - \sigma_n(h, f, 1) \\ &\leq \|f - \sigma_n(h, f)\|_\infty \leq (1 + c) E_{n/2}(f). \end{aligned}$$

Thus, $f \in B_{\rho,\gamma}$ if and only if $(\|f - S_{2^n}(f)\|_\infty)_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$, and secondly, $f \in B_{\rho,\gamma}$ if and only if $(\sum_{k=2^n}^{\infty} \hat{f}(k))_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$. In light of the discrete Hardy inequality [6, Lemma 3.4, p. 27], the latter is equivalent to $(\sum_{k=2^n}^{2^{n+1}-1} \hat{f}(k))_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho,\gamma}$ for any positive ρ . $\square$

Remark 1. Assuming additional conditions on the coefficients, we obtain an analogue of Lorentz's result: under the conditions of Theorem 3.6, conditions (a) and (b) are equivalent to the condition $(\hat{f}(2^n))_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho, \gamma}$ if $(\hat{f}(n))_{n \in \mathbb{N}_0}$ is lacunary or $(\hat{f}(2^n))_{n \in \mathbb{N}_0} \in \mathbf{b}_{\rho, \gamma+1}$ if $(\hat{f}(n))_{n \in \mathbb{N}_0}$ is monotone.

Remark 2. Let $a \geq b \geq -1/2$. In [4], Askey and Gasper have given sufficient conditions in order that the *connection coefficients* $\gamma_{j,k}$ in the expansion below are all nonnegative:

$$(3.10) \quad R_k^{(\alpha, \beta)}(x) = \sum_{j=0}^k \gamma_{j,k} R_j^{(a,b)}(x), \quad k \in \mathbb{N}_0, \quad x \in \mathbb{R}.$$

If $f \in C$ and $\hat{f}(\alpha, \beta; k) \geq 0$ for $k \in \mathbb{N}_0$, then (3.8) can be used to deduce easily that

$$\hat{f}(a, b; j) = \lim_{n \rightarrow \infty} \sum_{k=j}^n \gamma_{j,k} h(k/n) \hat{f}(\alpha, \beta; k) \geq 0.$$

Thus, Theorem 3.4 implies that for $f \in L_{\text{loc}}^\infty \cap \mathbb{P}$, the series $\sum |\hat{f}(a, b; k)| \|R_j^{(a,b)}\|_\infty$ converges for all $a \geq b \geq -1/2$ for which the connection coefficients in (3.10) are all nonnegative. A similar conclusion can also be made about Theorem 3.6 and Remark 1 above.

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