INTEGRAL REPRESENTATIONS FOR NEUMANN-TYPE SERIES OF BESSEL FUNCTIONS I_ν , Y_ν AND K_ν

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ABSTRACT. Recently Pogány and Süli [*Proc. Amer. Math. Soc.* **137**(7) (2009), 2363–2368] derived a closed-form integral expression for Neumann series of Bessel functions of the first kind J_ν . In this paper our aim is to establish analogous integral representations for the Neumann-type series of modified Bessel functions of the first kind I_ν and for Bessel functions of the second kind Y_ν, K_ν , and to give links for the same question for the Hankel functions $H_\nu^{(1)}, H_\nu^{(2)}$.

1. INTRODUCTION

The problem of computing the coefficients of the Neumann series of Bessel functions of the first kind,

$$(1.1) \quad \mathfrak{N}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+n}(z), \quad z \in \mathbb{C},$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν , has been considered in a number of publications in the mathematical literature; see, e.g., the extensive reference list in [5, 8].

Recently, an integral representation for (1.1) has been given by Pogány and Süli in [8]. The main result of [8] is the following theorem.

Theorem A. *Let $\alpha \in C^1(\mathbb{R}_+)$ and let $\alpha|_{\mathbb{N}} = \{\alpha_n\}_{n \in \mathbb{N}}$. Then, for all x, ν such that*

$$x \in \left(0, 2 \min \left(1, \left(e \limsup_{n \rightarrow \infty} n^{-1} |\alpha_n|^{1/n}\right)^{-1}\right)\right) =: \mathcal{I}_\alpha, \quad \nu > -3/2,$$

we have

$$\mathfrak{N}_\nu(x) = - \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} (\Gamma(\nu + u + 1/2) J_{\nu+u}(x)) \mathfrak{d}_s \left(\frac{\alpha(s)}{\Gamma(\nu + s + 1/2)} \right) du ds,$$

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where

$$\mathfrak{d}_x := 1 + \{x\} \frac{d}{dx},$$

and for $a \in \mathbb{R}$, $[a]$ and $\{a\} = a - [a]$ denote the integer and fractional part of a , respectively.

In this paper we pose the problem of integral representation for another Neumann-type series of Bessel functions when J_ν is replaced in (1.1) by modified Bessel function of the first kind I_ν , Bessel functions of the second kind Y_ν, K_ν (called Basset–Neumann and MacDonald functions respectively), Hankel functions $H_\nu^{(1)}, H_\nu^{(2)}$ (or Bessel functions of the third kind), of which precise definitions can be found in [9].

According to the established nomenclatures in the sequel we will distinguish Neumann series of the first, second and third kind depending on the Bessel functions which build this series. So, the *first kind Neumann series* are

$$(1.2) \quad \mathfrak{N}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+n}(z), \quad \mathfrak{M}_\nu(z) := \sum_{n=1}^{\infty} \beta_n I_{\nu+n}(z).$$

The *second kind Neumann series* we introduce as

$$(1.3) \quad \mathfrak{J}_\nu(z) := \sum_{n=1}^{\infty} \delta_n K_{\nu+n}(z), \quad \mathfrak{X}_\nu(z) := \sum_{n=1}^{\infty} \gamma_n Y_{\nu+n}(z).$$

In the next section our aim is to present closed-form expressions for these Neumann series occurring in (1.2) and (1.3). Our main tools include the Laplace-integral form of a Dirichlet series [6], the condensed form of the Euler–Maclaurin summation formula [8, p. 2365] and certain bounding inequalities for I_ν and K_ν ; see [2].

Throughout this paper, by convention, $[a]$ and $\{a\} = a - [a]$ denote the integer and fractional parts of some a , respectively.

2. MAIN RESULTS

First, we present an integral representation for the first kind Neumann series $\mathfrak{M}_\nu(x)$, where I_ν is the modified Bessel function of the first kind of order ν , defined by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)n!}, \quad z, \nu \in \mathbb{C}.$$

Theorem 2.1. *Let $\beta \in C^1(\mathbb{R}_+)$, $\beta|_{\mathbb{N}} = \{\beta_n\}_{n \in \mathbb{N}}$ and assume that $\sum_{n=1}^{\infty} \beta_n$ is absolutely convergent. Then, for all*

$$x \in \left(0, 2 \min \left(1, \left(e \limsup_{n \rightarrow \infty} n^{-1} |\beta_n|^{1/n}\right)^{-1}\right)\right) =: \mathcal{I}_\beta, \quad \nu > -3/2,$$

we have the integral representation

$$\mathfrak{M}_\nu(x) = - \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} (\Gamma(\nu+u+1/2) I_{\nu+u}(x)) \cdot \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu+s+1/2)} \right) du ds.$$

Proof. First, we establish the convergence conditions of the first kind Neumann series $\mathfrak{M}_\nu(x)$. By virtue of the bounding inequality [2, p. 583],

$$I_\mu(x) < \frac{\left(\frac{x}{2}\right)^\mu}{\Gamma(\mu+1)} e^{\frac{x^2}{4(\mu+1)}}, \quad x > 0, \mu+1 > 0,$$

and having in mind that $\mathcal{I}_\beta \subseteq (0, 2)$, we conclude that

$$|\mathfrak{M}_\nu(x)| < \max_{n \in \mathbb{N}} \frac{\left(\frac{x}{2}\right)^{\nu+n}}{\Gamma(\nu+n+1)} e^{\frac{x^2}{4(\nu+n+1)}} \sum_{n=1}^{\infty} |\beta_n| = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma(\nu+2)} e^{\frac{x^2}{4(\nu+2)}} \sum_{n=1}^{\infty} |\beta_n|.$$

Thus, the absolute convergence of $\sum_{n=1}^{\infty} \beta_n$ suffices for the finiteness of $\mathfrak{M}_\nu(x)$ on $\mathcal{I}_\beta$. Here we used tacitly that for $x \in \mathcal{I}_\beta$ and $\nu > -1$ fixed, the function

$$\alpha \mapsto f(\alpha) = \frac{\left(\frac{x}{2}\right)^{\nu+\alpha}}{\Gamma(\nu+\alpha+1)} e^{\frac{x^2}{4(\nu+\alpha+1)}}$$

is decreasing on $[\alpha_0, \infty)$, where $\alpha_0 = 1.46163$ denotes the abscissa of the minimum of Γ , because Γ is increasing on $[\alpha_0, \infty)$ and then

$$\frac{f'(\alpha)}{f(\alpha)} = \ln\left(\frac{x}{2}\right) - \frac{x^2}{4(\nu+\alpha+1)^2} - \frac{\Gamma'(\nu+\alpha+1)}{\Gamma(\nu+\alpha+1)} \leq 0.$$

Consequently, for all $n \in \{2, 3, \dots\}$ we have $f(n) \leq f(2)$. Moreover, by using the inequality $e^x \geq 1 + x$, it can be shown easily that $f(1) \geq f(2)$ for all $x > 0$ and $\nu > -1$. These in turn imply that indeed $\max_{n \in \mathbb{N}} f(n) = f(1)$, i.e.

$$\max_{n \in \mathbb{N}} \frac{\left(\frac{x}{2}\right)^{\nu+n}}{\Gamma(\nu+n+1)} e^{\frac{x^2}{4(\nu+n+1)}} = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma(\nu+2)} e^{\frac{x^2}{4(\nu+2)}},$$

as we required.

Now, recall the following integral representation [9, p. 79]:

$$(2.1) \quad I_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi} \Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \cosh(zt) dt, \quad z \in \mathbb{C}, \operatorname{Re}(\nu) > -1/2,$$

which will be used in the sequel. Since (2.1) is valid only for $\nu > -1/2$, in what follows for the Neumann series $\mathfrak{M}_\nu(x)$ we suppose that $\nu > -3/2$. Setting (2.1) into the right-hand series in (1.2) we have

$$(2.2) \quad \mathfrak{M}_\nu(x) = \sqrt{\frac{2x}{\pi}} \int_0^1 \cosh(xt) \left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2} \mathcal{D}_\beta(t) dt, \quad x > 0,$$

with the Dirichlet series

$$(2.3) \quad \mathcal{D}_\beta(t) := \sum_{n=1}^{\infty} \frac{\beta_n}{\Gamma(n+\nu+1/2)} \exp\left(-n \ln \frac{2}{x(1-t^2)}\right).$$

Following the lines of the proof of [8, Theorem] we deduce that the x -domain is

$$0 < x < 2 \min \left(1, \left(e \limsup_{n \rightarrow \infty} n^{-1} \sqrt[n]{|\beta_n|} \right)^{-1} \right).$$

For such an x , the convergent Dirichlet series (2.3) possesses a Laplace-integral form

$$(2.4) \quad \mathcal{D}_\beta(t) = \ln \frac{2}{x(1-t^2)} \int_0^\infty \left(\frac{x(1-t^2)}{2}\right)^u \left(\sum_{j=1}^{[u]} \frac{\beta_j}{\Gamma(j+\nu+1/2)}\right) du.$$

Expressing (2.4) *via* the condensed Euler–Maclaurin summation formula [8, p. 2365], we get

$$(2.5) \quad \mathcal{D}_\beta(t) = \ln \frac{2}{x(1-t^2)} \int_0^\infty \int_0^{[u]} \left(\frac{x(1-t^2)}{2}\right)^u \cdot \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu+s+1/2)}\right) du ds.$$

Substituting (2.5) into (2.2) we get

$$(2.6) \quad \mathfrak{M}_\nu(x) = -\sqrt{\frac{2x}{\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu + s + 1/2)} \right) \cdot \left(\int_0^1 \cosh(xt) \left(\frac{x(1-t^2)}{2} \right)^{\nu+u-1/2} \ln \frac{x(1-t^2)}{2} dt \right) du ds.$$

Now, let us simplify the t -integral in (2.6):

$$(2.7) \quad \mathcal{J}_x(w) := \int_0^1 \cosh(xt) \cdot \left(\frac{x(1-t^2)}{2} \right)^w \ln \frac{x(1-t^2)}{2} dt, \quad w := \nu + u - 1/2.$$

Indefinite integration under the sign of the integral in (2.7) results in

$$\int \mathcal{J}_x(w) dw = \left(\frac{x}{2} \right)^w \int_0^1 \cosh(xt) (1-t^2)^w dt = \sqrt{\frac{\pi}{2x}} \Gamma(w+1) I_{w+1/2}(x).$$

Now, observing that $dw = du$, we get

$$\mathcal{J}_x(\nu + u - 1/2) = \sqrt{\frac{\pi}{2x}} \frac{\partial}{\partial u} (\Gamma(\nu + u + 1/2) I_{\nu+u}(x)).$$

From (2.6) and (2.7), we immediately get the proof of the theorem, with the assertion that the integration domain $\mathbb{R}_+$ changes to $[1, \infty)$ because $[u]$ is equal to zero for all $u \in [0, 1)$. $\square$

Below, we present an integral representation for the Neumann-type series $\mathfrak{J}_\nu(x)$.

Theorem 2.2. *Let $\delta \in C^1(\mathbb{R}_+)$ and let $\delta|_{\mathbb{N}} = \{\delta_n\}_{n \in \mathbb{N}}$. Then for all $\nu > 0$ and*

$$x \in \mathcal{I}_\delta := \left(\frac{2}{e} \limsup_{n \rightarrow \infty} n |\delta_n|^{1/n}, +\infty \right),$$

we have the integral representation

$$\mathfrak{J}_\nu(x) = - \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} K_{\nu+u}(x) \cdot \mathfrak{d}_s \delta(s) du ds.$$

Proof. We begin by establishing first the convergence conditions for $\mathfrak{J}_\nu(x)$. To this aim let us consider the integral representation referred to as Basset in [9, p. 172]:

$$K_\nu(x) = \frac{2^\nu \Gamma(\nu + 1/2)}{x^\nu \sqrt{\pi}} \int_0^\infty \frac{\cos(xt)}{(1+t^2)^{\nu+1/2}} dt, \quad \operatorname{Re}(\nu) > -1/2, \operatorname{Re}(x) > 0.$$

Consequently, for all $\operatorname{Re}(\nu) > 0$, $x > 0$, we have

$$(2.8) \quad K_\nu(x) \leq \frac{2^\nu \Gamma(\nu + 1/2)}{x^\nu \sqrt{\pi}} \int_0^\infty \frac{dt}{(1+t^2)^{\nu+1/2}} = \frac{1}{2} \left(\frac{2}{x} \right)^\nu \Gamma(\nu).$$

Now, recalling that $\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} (1 + \mathcal{O}(s^{-1}))$, $|s| \rightarrow \infty$, we have

$$\begin{aligned} |\mathfrak{J}_\nu(x)| &\leq \frac{1}{2} \left(\frac{2}{x} \right)^\nu \sum_{n=1}^\infty |\delta_n| \Gamma(\nu + n) \left(\frac{2}{x} \right)^n \\ &\sim \sqrt{\frac{\pi}{2}} \left(\frac{2}{ex} \right)^\nu \sum_{n=1}^\infty (\nu + n)^{\nu+n-1/2} |\delta_n| \left(\frac{2}{ex} \right)^n, \end{aligned}$$

where the last series converges uniformly for all $\nu > 0$ and $x \in \mathcal{I}_\delta$. Note that a more convenient integral representation for the modified Bessel function of the second kind is [9, p. 183]

$$(2.9) \quad K_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\nu \int_0^\infty t^{-\nu-1} e^{-t-\frac{x^2}{4t}} dt, \quad |\arg(x)| < \pi/2, \quad \operatorname{Re}(\nu) > 0.$$

Thus, combining the right-hand equality in (1.3) and (2.9) we get

$$(2.10) \quad \mathfrak{J}_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\nu \int_0^\infty t^{-\nu-1} e^{-t-\frac{x^2}{4t}} \cdot \mathcal{D}_\delta(t) dt, \quad x \in \mathcal{I}_\delta,$$

where $\mathcal{D}_\delta(t)$ is the Dirichlet series

$$(2.11) \quad \mathcal{D}_\delta(t) = \sum_{n=1}^\infty \delta_n \left(\frac{x}{2t}\right)^n = \sum_{n=1}^\infty \delta_n \exp\left(-n \ln \frac{2t}{x}\right).$$

The Dirichlet series' parameter is necessarily positive; therefore (2.11) converges for all $x \in \mathcal{I}_\delta$. Now, the related Laplace-integral and the Euler–Maclaurin summation formula give us:

$$(2.12) \quad \mathcal{D}_\delta(t) = \ln \frac{2t}{x} \int_0^\infty \int_0^{[u]} \left(\frac{x}{2t}\right)^u \cdot \mathfrak{d}_s \delta(s) du ds.$$

Substituting (2.12) into (2.10) we get

$$(2.13) \quad \mathfrak{J}_\nu(x) = -\frac{x^\nu}{2^{\nu+1}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \delta(s) \left(\int_0^\infty \left(\frac{x}{2t}\right)^u \ln \left(\frac{x}{2t}\right) t^{-\nu-1} e^{-t-\frac{x^2}{4t}} dt \right) du ds.$$

Denoting

$$\mathcal{I}_x(u) := \int_0^\infty \left(\frac{x}{2t}\right)^u \ln \left(\frac{x}{2t}\right) t^{-\nu-1} e^{-t-\frac{x^2}{4t}} dt,$$

we obtain

$$\int \mathcal{I}_x(u) du = \left(\frac{x}{2}\right)^u \int_0^\infty t^{-(\nu+u)-1} e^{-t-\frac{x^2}{4t}} dt = 2 \left(\frac{2}{x}\right)^\nu K_{\nu+u}(x).$$

Therefore

$$(2.14) \quad \mathcal{I}_x(u) = 2 \left(\frac{2}{x}\right)^\nu \frac{\partial}{\partial u} K_{\nu+u}(x).$$

Finally, by using (2.13) and (2.14) the proof of this theorem is done. $\square$

Remark 2.3. It should be mentioned here that by using Alzer's sharp inequality [1],

$$\Gamma(x+y) \leq \Gamma(x+1) \cdot \Gamma(y+1), \quad \min(x, y) \geq 1,$$

and combining this with (2.8) we obtain that

$$|\mathfrak{J}_\nu(x)| \leq \frac{1}{2} \left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty |\delta_n| \Gamma(\nu+n) \left(\frac{2}{x}\right)^n \leq \frac{\Gamma(\nu+1)}{2} \left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty n! |\delta_n| \left(\frac{2}{x}\right)^n,$$

and the resulting power series converges in $\mathcal{I}_\delta$ for $\nu \geq 1$.

On the other hand, it is worthwhile to note that, since $[x^\nu K_\nu(x)]' = -x^\nu K_{\nu-1}(x)$, the function $x \mapsto x^\nu K_\nu(x)$ is decreasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$, and because of the asymptotic relation $x^\nu K_\nu(x) \sim 2^{\nu-1} \Gamma(\nu)$, where $\nu > 0$ and $x \rightarrow 0$, we obtain again

the inequality (2.8). This inequality is actually the counterpart of the inequality (see [4, 7])

$$x^\nu e^x K_\nu(x) > 2^{\nu-1} \Gamma(\nu),$$

valid for all $\nu > 1/2$ and $x > 0$. Moreover, by using the classical Čebyšev integral inequality, it can be shown that (see [3]) the above lower bound can be improved as follows:

$$(2.15) \quad x^{\nu-1} K_\nu(x) \geq 2^{\nu-1} \Gamma(\nu) K_1(x),$$

where $\nu \geq 1$ and $x > 0$. Summarizing, for all $x > 0$ and $\nu \geq 1$, we have the following chain of inequalities:

$$\frac{1}{2} \left(\frac{2}{x} \right)^{\nu-1} \Gamma(\nu) e^{-x} < \left(\frac{2}{x} \right)^{\nu-1} \Gamma(\nu) K_1(x) \leq K_\nu(x) \leq \frac{1}{2} \left(\frac{2}{x} \right)^\nu \Gamma(\nu).$$

Finally, observe that (see [3]) the inequality (2.15) is reversed when $0 < \nu \leq 1$, and this reversed inequality is actually better than (2.8) for $0 < \nu \leq 1$; that is, we have

$$x^\nu K_\nu(x) \leq 2^{\nu-1} \Gamma(\nu) x K_1(x) \leq 2^{\nu-1} \Gamma(\nu),$$

where in the last inequality we used (2.8) for $\nu = 1$.

Now, we are going to deduce a closed integral expression for the Neumann series $\mathfrak{X}_\nu(x)$ by using the Struve function $\mathbf{H}_\nu$.

Theorem 2.4. *Let $\gamma \in C^1(\mathbb{R}_+)$ and let $\gamma|_{\mathbb{N}} = \{\gamma_n\}_{n \in \mathbb{N}}$. Then for all*

$$(2.16) \quad x \in \mathcal{I}_\gamma = \begin{cases} (0, 2(e\ell)^{-1}), & \text{if } -1/2 < \nu \leq 1/2, \\ (2Le^{-1}, 2(e\ell)^{-1}), & \text{if } 1/2 < \nu \leq 3/2, \\ (4Le^{-1}, (e\ell)^{-1}), & \text{if } \nu > 3/2, \end{cases}$$

where

$$\ell := \limsup_{n \rightarrow \infty} n^{-1} |\gamma_n|^{1/n}, \quad L := \limsup_{n \rightarrow \infty} n |\gamma_n|^{1/n},$$

it follows that

$$(2.17) \quad \begin{aligned} \mathfrak{X}_\nu(x) = & \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} ((\Gamma(\nu + u + 1/2) - \Gamma(\nu + u - 1/2)) \mathbf{H}_{\nu+u}(x) \\ & + \Gamma(\nu + u - 1/2) Y_{\nu+u}(x)) \cdot \mathfrak{d}_s \left(\frac{\gamma(s)}{\Gamma(\nu + s + 1/2)} \right) \mathrm{d}u \mathrm{d}s \end{aligned}$$

for Neumann series of the second kind $\mathfrak{X}_\nu(x)$ with coefficients $\{\gamma_n\}_{n \in \mathbb{N}}$ satisfying

$$(2.18) \quad \ell > \begin{cases} e^{-1}, & \text{if } \nu \in (-1/2, 3/2], \\ (2e)^{-1}, & \text{if } \nu > 3/2, \end{cases} \quad L \in \begin{cases} (e^{-1}, 1), & \text{if } \nu \in (-1/2, 3/2], \\ ((2e)^{-1}, 1/2), & \text{if } \nu > 3/2. \end{cases}$$

Proof. First we establish the convergence region and related parameter constraints upon ν for $\mathfrak{X}_\nu(x)$. The Gubler–Weber formula [9, p. 165]

$$(2.19) \quad Y_\nu(z) = \frac{2 \left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1/2) \sqrt{\pi}} \left(\int_0^1 \sin(zt) (1-t^2)^{\nu-1/2} \mathrm{d}t + \int_0^\infty e^{-zt} (1+t^2)^{\nu-1/2} \mathrm{d}t \right),$$

where $\operatorname{Re}(z) > 0$ and $\nu > -1/2$, enables the derivation of an integral expression for the Neumann series of the second kind $\mathfrak{X}_\nu(x)$ by following the lines of derivation for $\mathfrak{J}_\nu(x)$. From (2.19), by means of the well-known inequality

$$(1+t^2)^{\nu-1/2} \leq C_\nu(1+t^{2\nu-1}), \text{ where } C_\nu = \begin{cases} 1, & \text{if } 1/2 < \nu \leq 3/2, \\ 2^{\nu-3/2}, & \text{if } \nu > 3/2, \end{cases}$$

we distinguish the following two cases.

Assuming $\nu \in (1/2, 3/2]$ we have

$$\begin{aligned} Y_\nu(x) &\leq \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \left(\int_0^1 (1-t^2)^{\nu-1/2} dt + \int_0^\infty e^{-xt} (1+t^{2\nu-1}) dt \right) \\ &= \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \left(\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{2\Gamma(\nu+1)} + x^{-1} + \frac{\Gamma(2\nu)}{x^{2\nu}} \right) \\ &= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu + \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)} \left(\frac{x}{2}\right)^{\nu-1} + \frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu. \end{aligned}$$

Hence

$$\begin{aligned} |\mathfrak{X}_\nu(x)| &\leq \left(\frac{x}{2}\right)^\nu \sum_{n=1}^\infty \frac{|\gamma_n|}{\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^n + \frac{1}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{\nu-1} \sum_{n=1}^\infty \frac{|\gamma_n|}{\Gamma(\nu+n+1/2)} \left(\frac{x}{2}\right)^n \\ &\quad + \frac{1}{\pi} \left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty |\gamma_n| \Gamma(\nu+n) \left(\frac{2}{x}\right)^n. \end{aligned}$$

The first two series converge uniformly in $(0, 2(e\ell)^{-1})$, and the third one is uniformly convergent in $(2Le^{-1}, \infty)$. Consequently the interval of convergence becomes $\mathcal{I}_\gamma = (2Le^{-1}, 2(e\ell)^{-1})$, and then the coefficients γ_n satisfy the condition $\ell \cdot L < 1$. This implies that the necessary condition for convergence of $\mathfrak{X}_\nu(x)$ is $\limsup_{n \rightarrow \infty} |\gamma_n|^{1/n} < 1$.

In the case $\nu > 3/2$ we have

$$\begin{aligned} Y_\nu(x) &\leq \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \left(\int_0^1 (1-t^2)^{\nu-1/2} dt + 2^{\nu-3/2} \int_0^\infty e^{-xt} (1+t^{2\nu-1}) dt \right) \\ &= \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \left(\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{2\Gamma(\nu+1)} + 2^{\nu-3/2} \left(x^{-1} + \frac{\Gamma(2\nu)}{x^{2\nu}} \right) \right) \\ &= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu + \frac{x^{\nu-1}}{\sqrt{2\pi}\Gamma(\nu+1/2)} + \frac{2^{2\nu-3/2}\Gamma(\nu)}{\pi x^\nu}. \end{aligned}$$

Therefore

$$\begin{aligned} |\mathfrak{X}_\nu(x)| &\leq \left(\frac{x}{2}\right)^\nu \sum_{n=1}^\infty \frac{|\gamma_n|}{\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^n + \frac{x^{\nu-1}}{\sqrt{2\pi}} \sum_{n=1}^\infty \frac{|\gamma_n| x^n}{\Gamma(\nu+n+1/2)} \\ &\quad + \frac{1}{2\pi\sqrt{2}} \left(\frac{4}{x}\right)^\nu \sum_{n=1}^\infty |\gamma_n| \Gamma(\nu+n) \left(\frac{4}{x}\right)^n. \end{aligned}$$

The first two series converge in $(0, 2(e\ell)^{-1})$, $(0, (e\ell)^{-1})$ respectively, while the third series converges uniformly for all $x > 4L/e$. This yields the interval of convergence $\mathcal{I}_\gamma = (4Le^{-1}, (e\ell)^{-1})$. In this case the coefficients γ_n satisfy the constraint $4\ell L < 1$, and then the necessary condition for convergence of $\mathfrak{X}_\nu(x)$ is $\limsup_{n \rightarrow \infty} |\gamma_n|^{1/n} < 1/2$.

It remains the case that $-1/2 < \nu \leq 1/2$. Then, because of $(1+t^2)^{\nu-1/2} \leq 1$, we conclude that

$$Y_\nu(x) \leq \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu + \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}} \left(\frac{x}{2}\right)^{\nu-1},$$

and consequently $\mathcal{I}_\gamma = (0, 2(e\ell)^{-1})$. Collecting these cases we get (2.16) and (2.18).

Now, let us focus on the integral representation for $\mathfrak{X}_\nu(x)$, where $x \in \mathcal{I}_\gamma$. By the Gubler–Weber formula (2.19) we have

$$(2.20) \quad \begin{aligned} \mathfrak{X}_\nu(x) &= \frac{2}{\sqrt{\pi}} \left(\frac{x}{2}\right)^\nu \sum_{n=1}^{\infty} \frac{\gamma_n}{\Gamma(\nu+n+1/2)} \left(\frac{x}{2}\right)^n \\ &\cdot \left(\int_0^1 \sin(xt)(1-t^2)^{\nu+n-1/2} dt + \int_0^\infty e^{-xt}(1+t^2)^{\nu+n-1/2} dt \right). \end{aligned}$$

The first expression in (2.20) can be rewritten as

$$\begin{aligned} \Sigma_1(x) &= \frac{2}{\sqrt{\pi}} \left(\frac{x}{2}\right)^\nu \sum_{n=1}^{\infty} \frac{\gamma_n \left(\frac{x}{2}\right)^n}{\Gamma(\nu+n+1/2)} \int_0^1 \sin(xt)(1-t^2)^{\nu+n-1/2} dt \\ &= \sqrt{\frac{2x}{\pi}} \int_0^1 \sin(xt) \left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2} \mathcal{D}_\gamma(t) dt, \end{aligned}$$

where

$$\mathcal{D}_\gamma(t) := \sum_{n=1}^{\infty} \frac{\gamma_n}{\Gamma(n+\nu+1/2)} \exp\left(-n \ln \frac{2}{x(1-t^2)}\right)$$

is the Dirichlet series analogous to the one in (2.3). It is easy to see that in view of (2.18) for all $x \in \mathcal{I}_\gamma$ and $t \in (0, 1)$ we have

$$\ln \frac{2}{x(1-t^2)} > 0.$$

More precisely, if $-1/2 < \nu \leq 3/2$, then $x < 2(e\ell)^{-1}$ and

$$\frac{2}{x(1-t^2)} > \frac{e\ell}{1-t^2} > e\ell > 1.$$

Similarly, if $\nu > 3/2$, then $x < (e\ell)^{-1}$ and

$$\frac{2}{x(1-t^2)} > \frac{2e\ell}{1-t^2} > 2e\ell > 1.$$

Thus, the Dirichlet series' parameter is necessarily positive, and therefore $\mathcal{D}_\gamma(t)$ converges for all $x \in \mathcal{I}_\gamma$. Consequently, following the same lines as in the proof of Theorem 2.1 we deduce that

$$(2.21) \quad \Sigma_1(x) = - \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)} \right) \frac{\partial}{\partial u} (\Gamma(\nu+u+1/2) \mathbf{H}_{\nu+u}(x)) \, du ds,$$

where $\mathbf{H}_\nu$ stands for the familiar Struve function.

Below, we will simplify the second expression in (2.20):

$$\begin{aligned}\Sigma_2(x) &= \frac{2}{\sqrt{\pi}} \left(\frac{x}{2}\right)^\nu \sum_{n=1}^{\infty} \frac{\gamma_n\left(\frac{x}{2}\right)^n}{\Gamma(\nu+n+1/2)} \int_0^\infty e^{-xt} (1+t^2)^{\nu+n-1/2} dt \\ &= \sqrt{\frac{2x}{\pi}} \int_0^\infty e^{-xt} \left(\frac{x(1+t^2)}{2}\right)^{\nu-1/2} \tilde{\mathcal{D}}_\gamma(t) dt,\end{aligned}$$

where $\tilde{\mathcal{D}}_\gamma(t) = \mathcal{D}_\gamma(it)$. Thus,

$$\begin{aligned}\Sigma_2(x) &= -\sqrt{\frac{2x}{\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)} \right) \\ &\quad \cdot \left(\int_0^\infty e^{-xt} \left(\frac{x(1+t^2)}{2} \right)^{\nu+u-1/2} \ln \frac{x(1+t^2)}{2} dt \right) du ds \\ &= -\pi \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)} \right) \cdot \frac{\partial}{\partial u} \frac{1}{\Gamma(1/2-\nu-u)} \\ &\quad \cdot \left(\frac{2J_{-\nu-u}(x)}{\sin 2\pi(\nu+u)} - \frac{J_{\nu+u}(x)}{\sin \pi(\nu+u)} + \frac{\mathbf{H}_{\nu+u}(x)}{\cos \pi(\nu+u)} \right) du ds \\ &= \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} \Gamma(\nu+u-1/2) (Y_{\nu+u}(x) - \mathbf{H}_{\nu+u}(x)) \\ &\quad \cdot \mathfrak{d}_s \left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)} \right) du ds.\end{aligned}\tag{2.22}$$

Here we applied the Euler reflection formula $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec}(\pi z)$ and the well-known property of the Bessel functions that

$$Y_\nu(x) = \operatorname{cosec}(\pi\nu) (\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)).$$

Summing (2.21) and (2.22) we have the desired integral representation (2.17). $\square$

Remark 2.5. Another two linearly independent solutions of the Bessel homogeneous differential equation are the *Hankel functions* $H_\nu^{(1)}$ and $H_\nu^{(2)}$, which can be expressed as [9, p. 73]

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-\nu\pi i} J_\nu(x)}{i \sin(\nu\pi)},\tag{2.23}$$

$$H_\nu^{(2)}(x) = \frac{J_{-\nu}(x) - e^{\nu\pi i} J_\nu(x)}{-i \sin(\nu\pi)},\tag{2.24}$$

which build the *third kind Neumann series*:

$$\mathfrak{G}_\nu^{(1)}(z) := \sum_{n=1}^{\infty} \epsilon_n H_{\nu+n}^{(1)}(z), \quad \mathfrak{G}_\nu^{(2)}(z) := \sum_{n=1}^{\infty} \varphi_n H_{\nu+n}^{(2)}(z).$$

Using the formulae (2.23), (2.24) we see that integral expressions for third kind Neumann series are linear combinations of similar fashion integrals achieved for $\mathfrak{N}_\nu(x)$ in Theorem A.

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