PRIME END ROTATION NUMBERS OF INVARIANT SEPARATING CONTINUA OF ANNULAR HOMEOMORPHISMS

SHIGENORI MATSUMOTO

(Communicated by Yingfei Yi)

ABSTRACT. Let f be a homeomorphism of the closed annulus A isotopic to the identity, and let $X \subset \text{Int}A$ be an f -invariant continuum which separates A into two domains, the upper domain U_+ and the lower domain U_- . Fixing a lift of f to the universal cover of A , one defines the rotation set $\tilde{\rho}(X)$ of X by means of the invariant probabilities on X , as well as the prime end rotation number $\tilde{\rho}_\pm$ of $U_\pm$. The purpose of this paper is to show that $\tilde{\rho}_\pm$ belongs to $\tilde{\rho}(X)$ for any separating invariant continuum X .

1. INTRODUCTION

Let f be a homeomorphism of the closed annulus $A = S^1 \times [-1, 1]$, isotopic to the identity; i.e. f preserves the orientation and each of the boundary components $\partial_\pm A = S^1 \times \{\pm 1\}$. Suppose there is an f -invariant partition of A ; $A = U_- \cup X \cup U_+$, where $U_\pm$ is a connected open set containing the boundary component $\partial_\pm A$ and X is a connected compact set. Let

$$\pi : \tilde{A} = \mathbb{R} \times [-1, 1] \rightarrow S^1 \times [-1, 1]$$

be the universal covering map and $T : \tilde{A} \rightarrow \tilde{A}$ a generator of the covering transformation group, $T(\xi, \eta) = (\xi + 1, \eta)$. Denote by $p : \tilde{A} \rightarrow \mathbb{R}$ the projection onto the first factor.

Fix once and for all a lift $\tilde{f} : \tilde{A} \rightarrow \tilde{A}$ of f . Then the function $p \circ \tilde{f} - p$ is T -invariant and can be looked upon as a function on the annulus A . Define the *rotation set* $\tilde{\rho}(X)$ as the set of values $\mu(p \circ \tilde{f} - p)$, where μ ranges over the f -invariant probability measures supported on X . The rotation set is a compact interval (maybe one point) in $\mathbb{R}$, which depends upon the choice of the lift $\tilde{f}$ of f .

The first example of an invariant continuum X such that the frontiers of $U_\pm$ satisfy $\text{Fr}(U_+) = \text{Fr}(U_-) = X$ and that the rotation set $\tilde{\rho}(X)$ is not a singleton was constructed by G. D. Birkhoff in his 1932 paper [B] and is referred to as a *Birkhoff attractor*. It turns out that the Birkhoff attractor is an indecomposable continuum ([C, L2]). Furthermore it is shown by P. Le Calvez ([L1]) that any rational number between the two prime end rotation numbers is realized by a corresponding periodic point of $\tilde{f}$.

Received by the editors December 5, 2010.

2010 *Mathematics Subject Classification*. Primary 37E30; Secondary 37E45.

Key words and phrases. Continuum, rotation set, prime end rotation number, Brouwer line, foliations.

The author was partially supported by Grant-in-Aid for Scientific Research (C) No. 20540096.

©2011 American Mathematical Society
 Reverts to public domain 28 years from publication

Let $\hat{U}_\pm = U_\pm \cup \partial_\infty U_\pm$ be the prime end compactification of $U_\pm$, where $\partial_\infty U_\pm$ is the space of the prime ends ([E, M, MN]). The space $\partial_\infty U_\pm$ is homeomorphic to the circle and $\hat{U}_\pm$ to the closed annulus. As is well known, the homeomorphism f restricted to $U_\pm$ extends to a homeomorphism $\hat{f}_\pm : \hat{U}_\pm \rightarrow \hat{U}_\pm$. Denoting $I_+ = [0, 1]$ and $I_- = [-1, 0]$, define a homeomorphism

$$\Psi_\pm : \hat{U}_\pm \rightarrow S^1 \times I_\pm$$

such that $\Psi_\pm(\partial_\infty U_\pm) = S^1 \times 0$. By some abuse of notation denote by $\pi : \check{U}_\pm \rightarrow \hat{U}_\pm$ the universal covering map. Thus $\pi^{-1}(U_\pm)$ is considered to be a subspace of both $\check{A}$ and $\check{U}_\pm$. Let $\check{f}_\pm : \check{U}_\pm \rightarrow \check{U}_\pm$ be the lift of $\hat{f}_\pm$ such that $\check{f}_\pm = \hat{f}$ on $\pi^{-1}(U_\pm)$. The rotation number of the restriction of $\check{f}_\pm$ to $\pi^{-1}(\partial_\infty U_\pm)$, denoted by $\check{\rho}_\pm$, is called the *prime end rotation number* of $U_\pm$.

The purpose of this paper is to show the following.

Theorem 1. *The prime end rotation number $\check{\rho}_\pm$ belongs to $\tilde{\rho}(X)$.*

This result is already known for $X = \text{Fr}(U_-) = \text{Fr}(U_+)$ ([BG]), and for any X if the homeomorphism f is area preserving (Lemma 5.4, [FL]).

It is shown in Theorem 2.2 of [F] that any rational number in $\tilde{\rho}(X)$ is realized by a periodic point if X consists of nonwandering points. Notice then that X , consisting of chain recurrent points, is chain transitive since it is connected, and thus satisfies the condition of Theorem 2.2. As a corollary we have

Corollary 2. *If X consists of nonwandering points and if p/q lies in the closed interval bounded by $\check{\rho}_-$ and $\check{\rho}_+$, then there is a point $x \in \pi^{-1}(X)$ such that $\tilde{f}^q(x) = T^p(x)$.*

In what follows we also use the following notation. Let

$$\check{\Psi}_\pm : \check{U}_\pm \rightarrow \mathbb{R} \times I_\pm$$

be a lift of $\Psi_\pm$, and define $\check{p}_\pm : \check{U}_\pm \rightarrow \mathbb{R}$ by $\check{p}_\pm = p \circ \check{\Psi}_\pm$. The projection $\check{p}_\pm$ is within a bounded error of p on $\pi^{-1}(C)$ for a compact domain C of $U_\pm$. However, they may be quite different on the whole $\pi^{-1}(U_\pm)$.

2. PROOF

First of all let us state a deep and quite useful theorem of P. Le Calvez ([L3]) which plays a key role in the proof. A fixed point free and orientation preserving homeomorphism F of the plane $\mathbb{R}^2$ is called a *Brouwer homeomorphism*. A proper oriented simple curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is called a *Brouwer line* for F if $F(\gamma) \subset R(\gamma)$ and $F^{-1}(\gamma) \subset L(\gamma)$, where $R(\gamma)$ (resp. $L(\gamma)$) is the right (resp. left) side complementary domain of γ , which is decided by the orientation of γ .

Theorem 2.1. *Let F be a Brouwer homeomorphism commuting with the elements of a group Γ which acts on $\mathbb{R}^2$ freely and properly discontinuously. Then there is a Γ -invariant oriented topological foliation of $\mathbb{R}^2$ whose leaves are Brouwer lines of F .*

The proof of Theorem 1 is by absurdity. Assume by way of contradiction that $\check{\rho}_- < p/q < \inf \tilde{\rho}(X)$. Considering $\tilde{f}^q T^{-p}$ instead of $\tilde{f}$, it suffices to deduce a contradiction under the following assumption.

Assumption 2.2. $\check{\rho}_- < 0 < \inf \tilde{\rho}(X)$.

Since $\inf \tilde{\rho}(X) > 0$, the map $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(X)$. The overall strategy of the proof is to modify the homeomorphism f away from X to a new one g without creating fixed points in A such that the restrictions of $\tilde{g}$ to the lifts of both boundary circles $\pi^{-1}(\partial_{\pm} A)$ are nontrivial rigid translations by the same translation number. Then by glueing the two boundary circles we obtain a torus T^2 and a homeomorphism on T^2 . Now we can apply Theorem 2.1 to the lift of the homeomorphism to the universal covering space. This yields a topological foliation on T^2 , which has long been well understood. The proof will be done by analyzing the foliation. We first prepare a lemma which is necessary for the desired modification. We do not presume Assumption 2.2 in the following.

Lemma 2.3. *Assume $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(X)$. Then the prime end rotation number $\tilde{\rho}_{\pm}$ is nonzero.*

Proof. Consider the mapping $\tilde{f} - \text{Id}$ defined on $\tilde{A}$. Since it is T -invariant, it yields a mapping from A , still denoted by the same letter. Then since there is no fixed point of $\tilde{f}$ in X , we have $(\tilde{f} - \text{Id})(X) \subset \mathbb{R}^2 \setminus \{0\}$. Therefore there is an annular open neighbourhood V of X for which we get a mapping

$$\tilde{f} - \text{Id} : V \rightarrow \mathbb{R}^2 \setminus \{0\}.$$

Clearly for any positively oriented essential simple closed curve γ in V , the degree of the map

$$\tilde{f} - \text{Id} : \gamma \rightarrow \mathbb{R}^2 \setminus \{0\}$$

must be the same. If the curve γ is contained in $U_{\pm}$, then the degree can be studied by considering the map $\tilde{f}_{\pm}$ defined on the lift $\tilde{U}_{\pm}$ of the prime end compactification $\hat{U}_{\pm}$. If the prime end rotation number $\tilde{\rho}_{\pm}$ is nonzero, the degree is clearly 0. Notice that our definition of the degree differs from the usual definition of the index.

To analyze the case $\tilde{\rho}_{\pm} = 0$, we need the following form of the Cartwright-Littlewood theorem [CL].

Theorem 2.4. *If $\tilde{\rho}_{+} = 0$ and if $\text{Fix}(\tilde{f}) \cap \pi^{-1}(X) = \emptyset$, then the map $\hat{f}_{+}$ on $\partial_{\infty} U_{+}$ is Morse Smale and the attractors (resp. repellers) of $\hat{f}_{+}|_{\partial_{\infty} U_{+}}$ are attractors (resp. repellers) of the whole map $\hat{f}_{+}$.*

This is slightly stronger than the usual version in which it is assumed that $\text{Fix}(f) \cap X = \emptyset$. However, the proof works as well under the assumption of Theorem 2.4. See e.g. Section 3 of [MN].

Let us complete the proof of Lemma 2.3. Theorem 2.4 enables us to compute the degree of the curve δ in $U_{\pm}$ when $\tilde{\rho}_{\pm} = 0$. The degree is n if $\delta \subset U_{-}$ and $-n$ if $\delta \subset U_{+}$, where n is the number of the attractors. Since the degree must be the same in U_{-} and U_{+} , the conclusion follows.

Now we have $\tilde{\rho}_{-} < 0$ and $\tilde{\rho}_{+} \neq 0$ by Assumption 2.2 and Lemma 2.3. Let us start the modification of f .

Lemma 2.5. *Under Assumption 2.2, there exists a homeomorphism g of A such that:*

- (1) $g = f$ in some neighbourhood of X ,
- (2) $\tilde{g}$ does not admit a fixed point in $\tilde{A}$, where $\tilde{g}$ is the lift of g such that $\tilde{g} = \tilde{f}$ on $\pi^{-1}(X)$,

- (3) $\tilde{g}$ is a negative rigid translation by the same translation number on $\pi^{-1}(\partial_{\pm}A)$,
 and
 (4) $\tilde{p}_- \circ \tilde{g}_- - \tilde{p}_- \leq -c$ on $\hat{U}_-$ for some positive number c .

Proof. The modification in U_- will be done in the following way. We identify $\hat{U}_-$ with $S^1 \times [-1, 0]$ by the homeomorphism Ψ_- and the universal covering space $\check{U}_-$ with $\mathbb{R} \times [-1, 0]$. Thus $\tilde{p}_-$ is just the projection onto the first factor, $\tilde{p}_-(\xi, \eta) = \xi$. Since $\tilde{\rho}_- < 0$, the lift

$$\check{f}_- : \mathbb{R} \times [-1, 0] \rightarrow \mathbb{R} \times [-1, 0]$$

of $\hat{f}_-$ satisfies that $\tilde{p}_- \circ \check{f}_-(\xi, 0) < \xi - 2c$ for some $c > 0$. Therefore changing the coordinates of $[-1, 0]$ if necessary, one may assume that $\tilde{p}_- \circ \check{f}_-(\xi, \eta) \leq \xi - c$ if $(\xi, \eta) \in \mathbb{R} \times [-1/2, 0]$. Define a homeomorphism h of $S^1 \times [-1, 0]$ by

$$h(\xi, \eta) = (\xi + \varphi(\eta) \bmod 1, \eta),$$

where $\varphi : [-1, 0] \rightarrow (-\infty, 0]$ is a continuous function such that $\varphi([-1/2, 0]) = 0$ and

$$\varphi(\eta) \leq -\sup\{(\tilde{p}_- \circ \check{f}_- - \tilde{p}_-)(\xi, \eta) \mid \xi \in S^1\} - c.$$

Define $g = f \circ h$. Then its lift $\check{g}_-$ satisfies

$$\check{p}_- \circ \check{g}_- - \check{p}_- \leq -c$$

on $\check{U}_- = \mathbb{R} \times [-1, 0]$. Clearly condition (3) for $\pi^{-1}(\partial_-A)$ can be established by a further obvious modification.

Now to modify f in U_+ , we do the same thing as in U_- . If the prime end rotation number $\tilde{\rho}_+$ is negative, then with an auxiliary modification we are done. If it is positive, insert a time one map of the Reeb flow. $\square$

Consider the torus T^2 which is obtained from A by glueing the two boundary curves ∂_-A and ∂_+A . Then condition (3) above shows that g induces a homeomorphism of T^2 , again denoted by g . The universal cover of T^2 is $\mathbb{R}^2$, and $\tilde{A} = \mathbb{R} \times [-1, 1]$ is a subset of $\mathbb{R}^2$. The lift $\tilde{g} : \tilde{A} \rightarrow \tilde{A}$ can be extended uniquely to a lift $\tilde{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of $g : T^2 \rightarrow T^2$. The covering transformation group Γ is isomorphic to $\mathbb{Z}^2$, generated by the horizontal translation T and the vertical translation by 2, denoted by S . Since $\tilde{g}$ is a Brouwer homeomorphism which commutes with Γ , there is a Γ -invariant oriented foliation on $\mathbb{R}^2$ whose leaves are Brouwer lines for $\tilde{g}$. This yields an oriented foliation $\mathcal{F}$ on the torus T^2 . The proof is divided into several cases according to the topological type of the foliation $\mathcal{F}$. We are going to deduce a contradiction in each case. However, before going into detail we need another lemma.

Lemma 2.6. *For any $C > 0$ there is $n > 0$ such that $p \circ \tilde{g}^n - p \geq C$ on X .*

Proof. If not, there would be a point $x_n \in X$ for any $n > 0$ such that

$$(p \circ \tilde{g}^n - p)(x_n) = \sum_{j=0}^{n-1} (p \circ \tilde{g} - p)(g^j(x_n)) < C$$

for some $C > 0$, and the averages of Dirac masses

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} g_*^j \delta_{x_n}$$

would satisfy $\mu_n(p \circ \tilde{g} - p) < C/n$. Therefore an accumulation point μ of μ_n would have the property that $\mu(p \circ \tilde{g} - p) \leq 0$, contradicting the assumption $\inf \tilde{\rho}(X) > 0$. $\square$

Case 1. The foliation $\mathcal{F}$ does not admit a compact leaf. Then $\mathcal{F}$ is conjugate either to a linear foliation or to a Denjoy foliation, both of irrational slope. The lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the open annulus $\mathbb{R}^2/\langle T \rangle$ is conjugate to a foliation by vertical lines. The space of leaves of $\tilde{\mathcal{F}}$ is homeomorphic to S^1 , and there is a projection from $\mathbb{R}^2/\langle T \rangle$ to S^1 along the leaves of the foliation. This lifts to a projection $q : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Now q restricted to $\tilde{A}$ is within a bounded error of the first factor projection $p : \tilde{A} \rightarrow \mathbb{R}$ that we have used for the definition of the rotation set $\tilde{\rho}(X)$. In fact both p and q are lifts of degree one maps from $\mathbb{R}^2/\langle T \rangle$ to S^1 , and their difference is bounded on the preimage $\tilde{A} = \pi^{-1}(A)$ of a compact subset A . Thus Lemma 2.6 shows that $q \circ \tilde{g}^n(x) \rightarrow \infty$ ($n \rightarrow \infty$) for $x \in \pi^{-1}(X)$. That is, the foliation $\tilde{\mathcal{F}}$ is oriented upward. But this shows that $q \circ \tilde{g}(x) > q(x)$ even for a point $x \in \pi^{-1}(\partial_- A)$. On the other hand, by condition (3) of Lemma 2.5, $\tilde{g}$ is a negative translation on $\pi^{-1}(\partial_- A)$, a contradiction.

Case 2.1. The foliation $\mathcal{F}$ admits a compact leaf L of nonzero slope and does not admit a Reeb component. In this case the lifted foliation $\tilde{\mathcal{F}}$ is also conjugate to the vertical foliation, and the argument of Case 1 applies.

Case 2.2. The foliation $\mathcal{F}$ admits a Reeb component R of nonzero slope. The Brouwer property of leaves implies that $g(R) \subset \text{Int}(R)$ or $g^{-1}(R) \subset \text{Int}(R)$. That is, a point of the boundary of R is wandering under g . Therefore $\partial_- A$, consisting of nonwandering points of g according to (3) of Lemma 2.5, cannot intersect the boundary of R , which is however impossible since the slope of R is nonzero.

Case 2.3. The foliation $\mathcal{F}$ admits a compact leaf of slope 0. Hereafter we only consider the dynamics and the foliation on the open annulus $\mathbb{R}^2/\langle T \rangle$. Recall that A is a subset of $\mathbb{R}^2/\langle T \rangle$, and the homeomorphism g on A is extended to the whole $\mathbb{R}^2/\langle T \rangle$, again denoted by g , in such a way that g commutes with the vertical translation S , while the foliation is denoted by $\tilde{\mathcal{F}}$ as before.

Now the foliation $\tilde{\mathcal{F}}$ yields a partition $\mathcal{P}$ of the open annulus $\mathbb{R}^2/\langle T \rangle$ into compact leaves, interiors of Reeb components and foliated I -bundles. The set $\mathcal{P}$ is totally ordered by the height. The minimal element which intersects X cannot be a compact leaf by the Brouwer line property. Let R be the closure of the minimal element. Thus R is either a Reeb component or a foliated I -bundle such that $\text{Int}(R) \cap X \neq \emptyset$ and $\partial_- R \cap X = \emptyset$, where $\partial_- R$ is the lower boundary curve of R .

Assume for a while that $\partial_- R$ is oriented from the right to the left. Thus the homeomorphism g carries $\partial_- R$ into the upper complement of $\partial_- R$.

Case 2.3.1. R is a Reeb component. First notice that $g(R) \subset \text{Int } R$ and that the interior leaves of R are oriented upwards by the assumption $\inf \tilde{\rho}(X) > 0$ and the fact that $g(X \cap R) \subset X \cap R$. Choose a simple arc

$$\alpha : [0, 1] \rightarrow \pi^{-1}(R)$$

such that $\alpha(0) \in \pi^{-1}(\partial_- R)$, $\alpha(1) = \tilde{g}(\alpha(0))$, and $\alpha((0, 1)) \subset \text{Int}(\pi^{-1}(R)) \setminus \tilde{g}(\pi^{-1}(R))$. Since $g^{-1}(\pi(\alpha))$ is below $\text{Int } R$, $\tilde{g}^{-1}(\alpha)$, and hence α , is contained in $\pi^{-1}(U_-)$.

Concatenating nonnegative iterates of α , we obtain a simple path $\gamma : [0, \infty) \rightarrow \pi^{-1}(R \cap U_-)$ such that $\tilde{g} \circ \gamma(t) = \gamma(t+1)$ for any $t \geq 0$. Let $q : \pi^{-1}(\text{Int}(R)) \rightarrow \mathbb{R}$ be the lift of the projection along the leaves. Since $\gamma([1, \infty))$ is contained in the lift of a compact subset $\tilde{g}(R) \subset \text{Int}(R)$ and the leaves in $\text{Int}(R)$ are oriented upward, we have $q \circ \gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$. We also have $p \circ \gamma(t) \rightarrow \infty$ because q is within bounded error of p on $\gamma([1, \infty))$.

On the other hand, by condition (4) of Lemma 2.5, we have $\tilde{p} \circ \gamma(t) \rightarrow -\infty$ as $t \rightarrow \infty$. In particular, the curve γ is proper both in $\tilde{A}$ and in $\tilde{U}_-$ pointing toward the opposite direction. By joining the point $\gamma(0)$ to an appropriate point in $\pi^{-1}(\partial_- A)$, we obtain a simple curve δ in $\pi^{-1}(U_-)$ starting at a point on $\pi^{-1}(\partial_- A)$ which extends γ .

Notice that there is a point of $\pi^{-1}(X)$ on the left of a proper oriented curve δ in $\tilde{A}$ because the map p is bounded from below on δ and a high iterate of T^{-1} carries a point in $\pi^{-1}(X)$ beyond that bound. (There might be a point of $\pi^{-1}(X)$ on the right of δ however.)

Let x be a point in $\pi^{-1}(\partial_- A)$ left to the initial point of δ . Then there is a simple path $\beta : [0, \infty) \rightarrow \pi^{-1}(U_-)$ such that $\beta(0) = x$, $\lim_{t \rightarrow \infty} \beta(t) \in \pi^{-1}(X)$, and β is disjoint from δ . The path β , extendable in $\pi^{-1}(A)$, is also extendable in $\tilde{U}_-$, the lift of the prime end compactification. (See e.g. Lemma 2.5 of [MN].) This implies that β defines a simple path in $\tilde{U}_-$ joining x to a prime end in $\pi^{-1}(\partial_\infty U_-)$ without intersecting δ , which is impossible since $\pi^{-1}(\partial_\infty U_-)$ is contained on the right side of the proper path δ in $\tilde{U}_-$ since $\tilde{p} \circ \delta(t) \rightarrow -\infty$, while x is on the left side, a contradiction.

Case 2.3.2. R is a foliated I -bundle. Thus the upper boundary curve $\partial_+ R$ of R is also oriented from the right to the left, and its image by g lies on the upper complement of R . The interior leaves of R are oriented upward.

Recall that the boundary component $\partial_- A$ consisting of nonwandering points cannot intersect a compact leaf. Moreover, $\partial_- A$ lies in a Reeb component or a foliated I -bundle whose interior leaves are oriented downward since $p\tilde{g}^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$ for $x \in \pi^{-1}(\partial_- A)$. Let C be the annulus in $\mathbb{R}^2/\langle T \rangle$ bounded by $\partial_- A$ and $\partial_+ R$, the upper boundary curve of R . Notice that $\text{Int}(C)$ contains $\partial_- R$.

Case 2.3.2.1. The intersection $X \cap C$ has a component which separates $\partial_- A$ from $\partial_+ A$. One can derive a contradiction by the same argument as in Case 2.3.1, since the like defined path γ cannot evade R .

Case 2.3.2.2. There is a simple path in U_- joining a point in $\partial_- A$ with a point in $\partial_+ R$. Notice first of all that $g^{-1}(C) \subset C$. Let $\mathcal{Y}$ be the family of the connected components of $\pi^{-1}(X \cap C)$. Then any element $Y \in \mathcal{Y}$ is compact and intersects $\pi^{-1}(\partial_+ R)$ since otherwise Y would be a connected component of $\pi^{-1}(X)$ itself.

Choose a simple curve $\gamma : [0, 1] \rightarrow \pi^{-1}(C)$ such that

- (1) $\gamma(0) \in \pi^{-1}(\partial_- A)$,
- (2) $\gamma(1) \in \pi^{-1}(X \cap C)$, and
- (3) $\gamma([0, 1)) \subset \pi^{-1}(U_- \cap C)$.

Let Y be an element of $\mathcal{Y}$ which contains $\gamma(1)$. Then there are two unbounded connected components of the complement $\pi^{-1}(C) \setminus (Y \cup \gamma)$, one $L(Y \cup \gamma)$ on the left and the other $R(Y \cup \gamma)$ on the right.

Notice that for any $n > 0$, $\tilde{g}^{-n}\gamma$ is a path in C , and that $p\tilde{g}^{-n}(\gamma(1)) \rightarrow -\infty$ and $p\tilde{g}^{-n}(\gamma(0)) \rightarrow \infty$ as $n \rightarrow \infty$. That is, for any large n , $\tilde{g}^{-n}(\gamma(1)) \in L(Y \cup \gamma)$

and $\tilde{g}^{-n}(\gamma(0)) \in R(Y \cup \gamma)$, showing that $\tilde{g}^{-n}(\gamma)$ intersects γ . On the other hand, in $\tilde{U}_-$, γ defines a curve from a point in $\pi^{-1}(\partial_- A)$ to a prime end in $\pi^{-1}(\partial_\infty U_-)$. However, by condition (4) of Lemma 2.5, γ cannot intersect $\tilde{g}^{-n}(\gamma)$ for any large n , a contradiction.

Finally the case where $\partial_- R$ is oriented from the left to the right can be dealt with similarly by reversing the time. This completes the proof of Theorem 1.

REFERENCES

- [B] G. D. Birkhoff, *Sur quelques courbes fermées remarquables*, Bull. Soc. Math. France **60**(1932) 1-26; also in *Collected Mathematical Papers of G. D. Birkhoff*, vol. II, New York: Dover, 1968, pp. 444-461. MR1504983
- [BG] M. Barge and R. M. Gillete, *Rotation and periodicity in plane separating continua*, Ergod. Th. Dyn. Sys. **11**(1991) 619-631. MR1145613 (92m:54070)
- [C] M. Charpentier, *Sur quelques propriétés des courbes de M. Birkhoff*, Bull. Soc. Math. France **62**(1934) 193-224. MR1505024
- [CL] M. L. Cartwright and J. E. Littlewood, *Some fixed point theorems*, Ann. Math. **54**(1951) 1-37. MR0042690 (13:148f)
- [E] D. B. A. Epstein, *Prime ends*, Proc. London Math. Soc. **42**(1981) 385-414. MR614728 (83c:30025)
- [F] J. Franks, *Recurrence and fixed points of surface homeomorphisms*, Ergod. Th. Dyn. Sys. **8**(1988) 99-107. MR967632 (90d:58124)
- [FL] J. Franks and P. Le Calvez, *Regions of instability for non-twist maps*, Ergod. Th. Dyn. Sys. **23**(2003), 111-141. MR1971199 (2003m:37053)
- [L1] P. Le Calvez, *Existence d'orbites quasi-périodiques dans les attracteurs de Birkhoff*, Commun. Math. Phys. **106**(1986) 383-394. MR859817 (88a:58124)
- [L2] P. Le Calvez, *Propriétés des attracteurs de Birkhoff*, Ergod. Th. Dyn. Sys. **8**(1987) 241-310. MR951271 (90a:58103)
- [L3] P. Le Calvez, *Une version feuilletée équivariante du théorème de translation de Brouwer*, Publ. Math. I. H. E. S. **102**(2005) 1-98. MR2217051 (2007m:37100)
- [M] J. Mather, *Topological proofs of some purely topological consequences of Carathéodory's theory of prime ends*, in: Th. M. Rassias, G. M. Rassias, eds., *Selected Studies*, North-Holland (1982) 225-255. MR662863 (84k:57004)
- [MN] S. Matsumoto and H. Nakayama, *Continua as minimal sets of homeomorphisms of S^2* , arXiv:1005.0360.

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY, NIHON UNIVERSITY,
1-8-14 KANDA, SURUGADAI, CHIYODA-KU, TOKYO, 101-8308 JAPAN
E-mail address: matsumo@math.cst.nihon-u.ac.jp