CANONICAL CURVES ON SURFACES OF VERY LOW DEGREE

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ABSTRACT. Let C be a non-hyperelliptic curve of genus g . We recall some facts about curves endowed with a base-point-free g_4^1 . Then we prove that if the minimal degree of a surface containing the canonical model of C in $\check{\mathbb{P}}_k^{g-1}$ is g , then $7 \leq g \leq 12$ and C carries exactly one g_4^1 . As a by-product, we deduce that if the canonical model of C in $\check{\mathbb{P}}_k^{g-1}$ is contained in a surface of degree at most g , then C is either trigonal or tetragonal or isomorphic to a plane sextic.

1. INTRODUCTION AND NOTATION

Let k be an algebraically closed field and denote by $\text{char}(k)$ its characteristic. Let C be a curve (i.e. a smooth, connected, projective scheme of dimension 1) and denote by $g \geq 2$ its genus. The canonical system on C gives rise to a natural morphism $\phi: C \rightarrow \check{\mathbb{P}}_k^{g-1}$. The map ϕ is either the double cover of a rational normal curve or it is an isomorphism. In the former case C is said to be *hyperelliptic* and in the latter $C_{\text{can}} := \phi(C)$ is called a *canonical model of C* .

The study of the intrinsic properties of a non-hyperelliptic curve C is closely related to the description of the embedding $C_{\text{can}} \subseteq \check{\mathbb{P}}_k^{g-1}$.

For example, one could inspect some properties of surfaces $S \subseteq \check{\mathbb{P}}_k^{g-1}$ (i.e. integral, projective schemes of dimension 2) containing C_{can} . Notice that such surfaces must be obviously non-degenerate, i.e. not contained in any hyperplane of $\check{\mathbb{P}}_k^{g-1}$. Such a viewpoint has been adopted in [C–H]. The authors prove that, if C is general in its moduli space and its genus is $g \geq 23$, then $\deg(S) \geq 2g - 3$. They also characterise the surfaces for which such a minimum value is attained.

A second viewpoint consists in understanding whether some particular geometric characteristics of surfaces $S \supseteq C_{\text{can}}$ reflect important properties of C . For example, one could try to classify curves C for which $d := \deg(S)$ is very low. Due to the geometric characterisation of the degree, we know that $\deg(S) \geq g - 2$. It is thus natural to ask what happens when $d = g - 2, g - 1, g$, and so on. A first natural step in this direction could consist of understanding the nature of such surfaces of very low degree.

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The classification of surfaces of minimal degree (i.e. with $d = g - 2$) is due to del Pezzo (see [dP1]; see also [Fj3] and [E–H]). Also thanks to such a classification, one obtains the Babbage–Enriques–Noether–Petri Theorem (see [A–C–G–H], p. 131, when $\text{char}(k) = 0$ and [SD] when $\text{char}(k) > 0$).

Theorem A. *Let C be a non-hyperelliptic curve of genus $g \geq 3$. Then $C_{\text{can}} \subseteq \check{\mathbb{P}}_k^{g-1}$ is contained in an integral surface S of degree $g - 2$ if and only if exactly one of the two following conditions holds:*

- i) C is trigonal (i.e. there exists a map $C \rightarrow \mathbb{P}_k^1$ which is finite of degree 3): in this case S is a rational normal scroll.*
- ii) $g = 6$ and C is isomorphic to a plane quintic: in this case S is a Veronese surface.*

Also the classification of surfaces of almost minimal degree (i.e. with $d = g - 1$) is known (see [dP2]; see also [Fj1], [Fj2], [Fj3], [H–W], [Bd–Sc]). The following result has been proven in [B–C–N].

Theorem B. *Let C be a non-hyperelliptic curve of genus $g \geq 5$. Then $C_{\text{can}} \subseteq \check{\mathbb{P}}_k^{g-1}$ is contained in an integral surface S of degree $g - 1$ if and only if one of the three following conditions holds:*

- i) C is bielliptic (i.e. there exists a double cover $C \rightarrow E$ onto an elliptic curve): in this case S is a cone on an elliptic normal curve contained in a hyperplane of $\check{\mathbb{P}}_k^{g-1}$ with vertex not on C_{can} .*
- ii) $g \leq 10$ and C is birationally isomorphic to a plane sextic carrying $10 - g$, possibly infinitely near, double points: in this case S is the anticanonical image of the blow up of $\mathbb{P}_k^2$ at such points.*
- iii) $g = 9$ and C is isomorphic to a smooth complete intersection inside $\mathbb{P}_k^3$ of an integral quadric Q with an integral quartic surface: in this case S is the anticanonical embedding of Q .*

The main result of this paper is the answer to the following question: What can be said if the canonical model of C in $\check{\mathbb{P}}_k^{g-1}$ is contained in an integral surface S of degree g ? We succeed in completely answering this question in Section 4.

Theorem C. *Let C be a non-hyperelliptic curve of genus g . If the minimal degree of a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ containing C_{can} is g , then $7 \leq g \leq 12$ and S is a conic bundle over $\mathbb{P}_k^1$ whose fibres cut out on C a base-point-free g_4^1 . The curve C does not carry any other base-point-free g_4^1 .*

The quickest way to achieve such a result does not pass through any kind of classification of surfaces $S \subseteq \check{\mathbb{P}}_k^{g-1}$ of degree g . It suffices to combine the description of the residual linear system cut out on S by the quadrics through C_{can} with some well-known facts about base-point-free g_4^1 's on a curve C .

We recall such results in Sections 2 and 3, proving Theorem C in Section 4. We also prove therein some corollaries and check that the above result is sharp, at least when k is the complex field. Finally, in Section 5, we show how Theorem C above is related to an invariant of a non-hyperelliptic curve, namely its apolarity, introduced for the first time in [I–R] (see also [R–S]) and investigated in some recent papers (see e.g. [DP–Z1], [DP–Z2] and [B–C–N]).

Notation. We work over an algebraically closed field k . We denote by $\text{char}(k)$ its characteristic.

If V is a vector space, then we denote by $\mathbb{P}(V)$ the corresponding projective space. In particular, we set $\mathbb{P}_k^n := \mathbb{P}(k^{n+1})$. A curve in Section 2 is an integral and projective scheme of dimension 1, without embedded components. From Section 3, a curve is also assumed to be smooth.

Let $X \subseteq \mathbb{P}_k^n$ be a closed subscheme and let $\mathcal{L}$ be a sheaf on X . If $q \in H^0(X, \mathcal{L})$ we will denote by $D_0(q) \subseteq X$ its zero-scheme.

Let $\gamma := (\gamma_0, \dots, \gamma_n) \in \mathbb{N}^{n+1}$ be a multi-index. We set $|\gamma| := \sum_{i=0}^n \gamma_i$, $\gamma! := \prod_{i=0}^n \gamma_i!$, and $t^\gamma := t_0^{\gamma_0} \dots t_n^{\gamma_n} \in k[t_0, \dots, t_n]$. We say that $\gamma \geq 0$ if and only if $\gamma_i \geq 0$ for each $i = 0, \dots, n$. If $\delta := (\delta_0, \dots, \delta_n) \in \mathbb{N}^{n+1}$ is another multi-index, then we write $\gamma \geq \delta$ if and only if $\gamma - \delta \geq 0$.

For other definitions, results and notation we always refer to [Ha].

2. COVERS OF DEGREE 4

We start by recalling some standard facts about covers of degree 4. If C is a curve, then a morphism $\varrho: C \rightarrow \mathbb{P}_k^1$ is a cover of degree 4 if it is finite, flat and $\text{rk}(\varrho_* \mathcal{O}_C) = 4$. In this case there exists a natural exact sequence

$$(2.1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^1} \xrightarrow{\varrho^\#} \varrho_* \mathcal{O}_C \longrightarrow \tilde{\mathcal{E}} \longrightarrow 0,$$

where $\tilde{\mathcal{E}}$ is a locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -sheaf of rank 3 called the *Tschirnhausen module* of ϱ .

Since every locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -sheaf splits, we can fix a decomposition $\tilde{\mathcal{E}} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$. Sequence (2.1) and Serre duality yields $g + 3 = \alpha_1 + \alpha_2 + \alpha_3$.

Theorem 2.2. *Let C be a curve of genus g and $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4. There exists an embedding $i: C \hookrightarrow \mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^1$, a locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -sheaf $\mathcal{F} := \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$ such that $\sum_{j=1}^2 \beta_j = g + 3$ and a morphism $\delta: \pi^* \mathcal{F} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}}(2)$ such that $C = D_0(\delta)$ and $\varrho = \pi|_{D_0(\delta)}$. Moreover,*

$$(2.2.1) \quad \omega_C \cong i^*(\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-2)).$$

Conversely let $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$ and $\mathcal{F} \cong \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$ be such that $\sum_{i=1}^3 \alpha_i = \sum_{j=1}^2 \beta_j$, define $\mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^1$ and let $\delta: \pi^ \mathcal{F} \rightarrow \mathcal{O}_{\mathbb{P}}(2)$. If $\varrho = \pi|_{D_0(\delta)}$ is a cover, then it has degree 4 and the Tschirnhausen module of ϱ is $\tilde{\mathcal{E}}$.*

Proof. See [C-E], Theorem 2.1 and Section 4. $\square$

Definition 2.3. Let C be a curve, $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4 and fix decompositions $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$, $\alpha_1 \leq \alpha_2 \leq \alpha_3$, $\mathcal{F} \cong \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$, $\beta_1 \leq \beta_2$, of the associated sheaves.

The triple $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ is called the *scrollar invariant* of ϱ . The number $\tau := \beta_1 - 4$ is called the *tetragonality* of ϱ .

Both the scrollar invariants and the tetragonality are classically well-known invariants associated to each g_4^1 on a curve.

We have three monomorphisms $\mathcal{O}_{\mathbb{P}_k^1}(\alpha_i) \hookrightarrow \mathcal{E}$ into the three summands, hence three fibrewise independent sections $u \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-\alpha_1)) \cong H^0(\mathbb{P}_k^1, \mathcal{E}(-\alpha_1))$, $v \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-\alpha_2)) \cong H^0(\mathbb{P}_k^1, \mathcal{E}(-\alpha_2))$, and $w \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-\alpha_3)) \cong H^0(\mathbb{P}_k^1, \mathcal{E}(-\alpha_3))$.

Via the isomorphism

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \tilde{\mathcal{F}}) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-\beta_1)) \oplus H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-\beta_2)),$$

we can identify δ with a pair (a, b) where

$$\begin{aligned}
 (2.4) \quad a(u, v, w) &= a_{2\alpha_1-\beta_1}u^2 + a_{\alpha_1+\alpha_2-\beta_1}uv + a_{\alpha_1+\alpha_3-\beta_1}uw \\
 &\quad + a_{2\alpha_2-\beta_1}v^2 + a_{\alpha_2+\alpha_3-\beta_1}vw + a_{2\alpha_3-\beta_1}w^2, \\
 b(u, v, w) &= b_{2\alpha_1-\beta_2}u^2 + b_{\alpha_1+\alpha_2-\beta_2}uv + b_{\alpha_1+\alpha_3-\beta_2}uw \\
 &\quad + b_{2\alpha_2-\beta_2}v^2 + b_{\alpha_2+\alpha_3-\beta_2}vw + b_{2\alpha_3-\beta_2}w^2,
 \end{aligned}$$

$a_h, b_h \in H^0(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}(h))$. In particular, let $A := D_0(a)$, $B := D_0(b)$. Then $C = A \cap B \subseteq \mathbb{P}$. Since both C and $\mathbb{P}$ are smooth, it follows that the surfaces A and B are smooth at the points of C .

Proposition 2.5. *Let C be a curve of genus $g \geq 4$, $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4, $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and τ the corresponding invariants. Then*

$$\frac{g+3}{3} \geq \alpha_1 \geq 1, \quad \min \left\{ \frac{g-5}{2}, 2(\alpha_1-2) \right\} \geq \tau \geq -2.$$

Proof. By definition, $\alpha_1 \leq \alpha_2 \leq \alpha_3$ and $\beta_1 \leq \beta_2$. Thus the inequalities

$$\alpha_1 \leq \frac{g+3}{3}, \quad \tau \leq \frac{g-5}{2}$$

are immediate. Since $h^0(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}) = h^0(C, \mathcal{O}_C) = 1$ and $h^1(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}) = 0$, taking the cohomology of sequence (2.1), one also easily deduces that $h^0(\mathbb{P}_k^1, \mathcal{E}) = 0$, whence $\alpha_1 \geq 1$.

If $2(\alpha_1-2) < \tau$, then in equations (2.4) above we would have $a_{2\alpha_1-\beta_1} = b_{2\alpha_1-\beta_2} = 0$. Hence the unisecant curve $D_0(v, w) \subseteq \mathbb{P}$ would be contained in C , contradicting the irreducibility of C .

If $\tau < -2$, then $\beta_1 \leq 1$ and $\beta_2 \geq g+2$. If $\alpha_1 \leq \alpha_2 \leq \alpha_3 < (g+2)/2$, then $b_{\alpha_i+\alpha_j-\beta_2} = 0$ for each $i, j = 1, 2, 3$ in equations (2.4). Thus $b(u, v, w) = 0$; hence $C = A$, a contradiction. It follows that $\alpha_3 \geq (g+2)/2$. Since $\alpha_1 \geq 1$, we deduce that $\alpha_2 \leq (g+2)/2$. If $\alpha_1 > 1$, then $b(u, v, w) = b'vw + b''w^2$, where b', b'' are forms of non-negative degree.

Thus B splits in $\mathbb{P}$ as the union of the two subbundles $W := D_0(w)$ and $B' := D_0(b'v + b''w)$. In particular, $C = C_1 \cup C_2$, where $C_1 := C \cap W$ and $C_2 := C \cap B'$. Since C is irreducible, it follows that either $C = C_1$ or $C = C_2$. In any case we obtain a contradiction, since C is fibrewise the complete intersection of two conics.

We deduce that $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = (g+2)/2$. If $g \geq 1$, then $b(u, v, w) = b'v^2 + b''vw + b'''w^2$, where b', b'', b''' are constants. Thus B would again be reducible, and an argument analogous to the one used above yields a contradiction. We conclude that the bound $\beta_1 \geq 2$ must hold, whence $\tau \geq -2$. $\square$

3. TETRAGONALITY AND CANONICAL EMBEDDING

For the rest of the paper the word *curve* will always denote a smooth, integral, projective scheme of dimension 1. Since we are interested in the surfaces of the canonical space containing C_{can} , we restrict our attention to the case $g \geq 4$.

Recall that a curve C of genus $g \geq 2$ is called hyperelliptic if there exists a cover $\varphi: C \rightarrow \mathbb{P}_k^1$ of degree 2. Every hyperelliptic curve C carries infinitely many g_4^1 's obtained by composing φ with any double cover $\mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$. In this section we will first study some different characterisations of base-point-free g_4^1 's on a

non-hyperelliptic curve C of genus $g \geq 2$ in terms of its scrollar invariant α and tetragonality τ . We start recalling the following well-known results.

Proposition 3.1. *Let C be a non-hyperelliptic curve of genus $g \geq 4$ and $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4, $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and τ the corresponding invariants. Then the following assertions hold:*

- i) $\alpha_1 \geq 2$ and $\tau \geq -1$.
 - ii) If $\tau < (g-9)/3$, then $\tau = 2(\alpha_1 - 2)$.
 - iii) If $\tau = -1$, then C is either trigonal or isomorphic to a smooth plane quintic.
- If this happens, then $g \leq 6$.

Proof. In order to prove assertion i) it suffices to check that $\tau \geq -1$, due to Proposition 2.5. However, this has been proven in Section 6.3 of [Sch1], taking into account that the number b_2 defined there is exactly $\tau = \beta_1 - 4$.

For the proof of assertion ii) see [Sch2], Corollary 4.2. For the first part of the proof of assertions iii) see Section (6.3) and (6.4) of [Sch1]. The bound on the genus follows from assertion ii), since $\tau = -1$ is odd. $\square$

Now let C be a non-hyperelliptic curve of genus $g \geq 4$. Let $\varrho: C \rightarrow \mathbb{P}_k^1$ be a cover of degree 4 and consider the corresponding invariants $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and τ . Due to Proposition 3.1 i), one deduces that $\alpha_1 \geq 2$ and $\tau \geq -1$. As in the previous section we set $\mathbb{P} := \mathbb{P}(\mathcal{E})$.

The Picard group $\text{Pic}(\mathbb{P})$ of $\mathbb{P}$ is generated by the class ξ of the tautological sheaf $\mathcal{O}_{\mathbb{P}}(1)$ and by the class f of a fibre of π . Moreover, the Chern equation on $\mathbb{P}$ becomes

$$\xi^3 = (g+3)\xi^2 f.$$

The linear system $|\xi - 2f|$ on $\mathbb{P}$ is globally generated (since $\alpha_1 \geq 2$) and it defines a morphism $\phi: \mathbb{P} \rightarrow \Sigma := \text{im}(\phi) \subseteq \check{\mathbb{P}}_k^{g-1}$ onto a rational normal scroll. The restriction of ϕ to C is exactly the canonical embedding (see formula (2.2.1)). If $\alpha_1 > 2$, then ϕ is actually an embedding. Notice that this happens as soon as $\tau \geq 1$, thanks to Proposition 2.5. If $\alpha_1 = 2$, then ϕ is no longer an embedding. In this case (see [Sch1], Section 1.1 and the references therein) Σ is singular and ϕ is a resolution of its singularities.

In any case the planes in $\mathbb{P}$ cutting the g_4^1 on C are mapped on planes in Σ ; in particular, the divisor on the g_4^1 are cut on C_{can} by planes.

The curve C is the complete intersection inside $\mathbb{P}$ of the two conic bundles $A \in |2\xi - (\tau+4)f|$ and $B \in |2\xi - (g-1-\tau)f|$. Thus $C_{can} \subseteq \check{\mathbb{P}}_k^{g-1}$ coincides with the intersection of the two surfaces $A_{can} := \phi(A)$ and $B_{can} := \phi(B)$ whose degrees are

$$\begin{aligned} \deg(A_{can}) &= (2\xi - (\tau+4)f)(\xi - 2f)^2 = 2g - 6 - \tau, \\ \deg(B_{can}) &= (2\xi - (g-1-\tau)f)(\xi - 2f)^2 = g - 1 + \tau. \end{aligned}$$

Since $\tau \leq (g-5)/2$, it follows that $\deg(B_{can}) \leq \deg(A_{can})$. We can summarise the above remarks in the following proposition.

Proposition 3.2. *Let C be a non-hyperelliptic curve of genus $g \geq 4$, $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4 with tetragonality τ . Then the g_4^1 on C corresponding to ϱ is cut out on C_{can} by a family of planes in $\mathbb{P}_k^{g-1}$ whose union Σ is a scroll over $\mathbb{P}_k^1$. Moreover, C_{can} is the complete intersection inside Σ of two conic bundles A_{can} and B_{can} of respective degrees $2g - 6 - \tau$ and $g - 1 + \tau$. If $\tau \geq 1$, then Σ is smooth. $\square$*

One can be even more precise. Indeed, we have the following nice geometric characterisation of the tetragonality of a cover (for a more precise result obtained via different methods, see [Bn–Sa]).

Theorem 3.3. *Let C be a non-hyperelliptic curve of genus $g \geq 4$, $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4 with tetragonality τ and let $\Sigma \subseteq \check{\mathbb{P}}_k^{g-1}$ be as above. Then $2g-6-\tau$ is the minimal degree of an integral surface $S \subseteq \Sigma$, different from B_{can} and containing C_{can} .*

Proof. Let $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and τ be the invariants associated to ϱ . Since C is non-hyperelliptic, we have that $\tau \geq -1$ and $\alpha_1 \geq 2$ (see Proposition 3.1 i)).

If $g = 4$, then $\alpha = (2, 2, 3)$ and $\beta = (3, 4)$, whence $\tau = -1$. In this case, $\Sigma = \check{\mathbb{P}}_k^{g-1}$, and it is well known that C is the complete intersection inside $\check{\mathbb{P}}_k^{g-1}$ of a surface of degree 3 with a quadric which has degree $2 = g - 1 + \tau$.

Now, we assume $g \geq 5$. Let $\lambda\xi - \mu f$ be the class of $\phi^{-1}(S) \subseteq \mathbb{P}$ in $\text{Pic}(\mathbb{P})$: trivially we have that $\lambda \geq 2$. If we assume that

$$\deg(S) = (\lambda\xi - \mu f)(\xi - 2f)^2 = \lambda(g-1) - \mu < 2g - 6 - \tau,$$

then we have

$$(3.3.1) \quad -\mu < (2 - \lambda)(g - 1) - \tau - 4.$$

If $S \neq B_{can}$, then $S \cap B_{can}$ is a curve of degree

$$\deg(S \cap B_{can}) = (2\xi - (g - 1 - \tau)f)(\lambda\xi - \mu f)(\xi - 2f) = \lambda(g - 1) - 2\mu + (\tau + 4)\lambda.$$

Inequality (3.3.1) thus yields

$$\deg(S \cap B_{can}) < 2g - 2 - (\lambda - 2)(g - 5 - \tau).$$

On the one hand $\tau \leq (g - 5)/2$ and $g \geq 5$, whence $g - 5 - \tau \geq 0$ for each $\tau \geq -1$. On the other hand $\lambda - 2 \geq 0$ since $C_{can} \subseteq S$. We conclude that

$$2g - 2 = \deg(C_{can}) \leq \deg(S \cap B_{can}) < 2g - 2 - (\lambda - 2)(g - 5 - \tau) \leq 2g - 2,$$

and this is a contradiction. $\square$

We conclude this section with the following lemma for which we are unable to give a correct reference. Recall that a curve C is called *tetragonal* if and only if it carries a g_4^1 but it does not carry any g_d^1 with $d \leq 3$.

Lemma 3.4. *Let C be a tetragonal curve of genus $g \geq 5$. If $\varrho_1: C \rightarrow \mathbb{P}_k^1$ and $\varrho_2: C \rightarrow \mathbb{P}_k^1$ are distinct covers of degree 4 with tetragonality τ_1 and τ_2 respectively, then $\tau_1 = \tau_2 = 0$. If C is not bielliptic, then $g \leq 9$.*

Proof. For the proof of the equalities $\tau_1 = \tau_2 = 0$, see [Sch1], Section 6.3. We have a well-defined morphism $\psi = \varrho_1 \times \varrho_2: C \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ and $\psi(C)$ is a divisor of bidegree $(4, 4)$ on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. If ψ is birational, then the genus of C is at most 9.

If ψ is not birational, then it is generically finite onto its reduced image $C_0 := \text{im}(\psi)_{red}$. The degree $\deg(\psi)$ of ψ is at most 3, since $\varrho_1 \neq \varrho_2$. It follows that $\deg(\psi) = 2$ necessarily, thus C_0 is a curve of bidegree $(2, 2)$. We deduce that C_0 is either rational or it is a smooth elliptic curve. The first case cannot occur; otherwise C would be hyperelliptic. It follows that it must be bielliptic. $\square$

Let C be tetragonal. In particular, we are implicitly assuming that $g \geq 5$ and that C is neither hyperelliptic nor trigonal. It follows from Proposition 3.1 that the tetragonality τ of the corresponding cover $\varrho: C \rightarrow \mathbb{P}_k^1$ is at least 0. Lemma 3.4 allows us to introduce the following definition.

Definition 3.5. Let C be a tetragonal curve. The tetragonality $\tau(C)$ of C is the common value of the tetragonalitys of all the covers $\varrho: C \rightarrow \mathbb{P}_k^1$ of degree 4.

4. PROOF OF THEOREM C

In this section we prove Theorem C stated in the introduction. We will also check that Theorem C is sharp, at least when $\text{char}(k) = 0$.

Proof of Theorem C. The minimal degree of a surface containing C_{can} is g . Then C is neither trigonal nor tetragonal with $\tau = 0$ (by Theorems A and B); thus $g \geq 7$ and C_{can} is cut out by quadrics. Each quadric through C_{can} cut out on S , residually to C_{can} , a curve Γ such that $\deg(\Gamma) = 2\deg(S) - \deg(C_{can}) = 2$. If $C_{can} \subseteq \text{Sing}(S)$, then $2 = \deg(\Gamma) \geq \deg(C_{can}) = 2g - 2$, a contradiction. Thus $C_{can} \not\subseteq \text{Sing}(S)$.

Let us consider a minimal desingularization $p: \tilde{S} \rightarrow S$ (see [Za] when $\text{char}(k) = 0$ and [Ab] when $\text{char}(k) > 0$). Let $\tilde{C}$ be the strict transform of C_{can} in $\tilde{S}$ and denote by H the total transform of a general hyperplane section of S . The pull-back of the linear system of quadrics through C_{can} has a fixed part F , which is mapped by p to the singular points of S , and a non-empty movable part $|M|$: we have $H \cdot M = \deg(\Gamma) = 2$. We can thus write $2H \sim \tilde{C} + M + F$, whence

$$2H \cdot M = \tilde{C} \cdot M + M^2 + F \cdot M,$$

where $\tilde{C} \cdot M$, M^2 , $F \cdot M$ are all non-negative. Since $2g - 2 = \tilde{C} \cdot H > M \cdot H = 2$, it follows that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(M - \tilde{C})) = 0$. Thus the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(M - \tilde{C}) \longrightarrow \mathcal{O}_{\tilde{S}}(M) \longrightarrow \mathcal{O}_{\tilde{C}}(M) \longrightarrow 0$$

yields

$$(4.1) \quad h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(M)) \geq h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(M)) \geq 2.$$

If either $M^2 \geq 1$ or $F \cdot M \geq 1$, then $\tilde{C} \cdot M \leq 3$. Thanks to inequality (4.1) it follows the existence of a base-point-free g_d^1 on $C \cong \tilde{C}$ with $d \leq 3$, a contradiction, since C is neither hyperelliptic nor trigonal.

We conclude that $\tilde{C} \cdot M = 4$, $M^2 = F \cdot M = 0$. Several facts follow from these equalities. First, a unique element of $|M|$ passes through each point of $\tilde{S}$. Second, C carries a g_4^r with $r \geq 1$; thus $C \cong \tilde{C}$ is tetragonal (recall that C is neither hyperelliptic nor trigonal). Moreover, if $r \geq 2$, the curve C would be hyperelliptic, since $g \geq 4$. Thus $|M|$ cut out a complete g_4^1 , and it follows that inequalities (4.1) are actually equalities, whence $|M| \cong \mathbb{P}_k^1$. Let $\varrho: C \rightarrow \mathbb{P}_k^1$ be the tetragonal map induced by $|M|$ and let $\Sigma \subseteq \mathbb{P}_k^{g-1}$ be the corresponding scroll (see Proposition 3.2).

Assume that the general element of $|M|$ is not reduced. Since $M \cdot H = 2$, for such a general element we necessarily have $M = 2M_{red}$ in $\text{Pic}(\tilde{S})$ (M_{red} is the reduced scheme structure on M) and $M_{red} \cdot H = 1$. Since $M \cdot \tilde{C} = 4$, it follows that $M_{red} \cdot \tilde{C} = 2$. Recall that a unique element of $|M|$ passes through each point of $\tilde{S}$. Thus we would obtain a natural rational map $\tilde{S} \dashrightarrow |M| \cong \mathbb{P}_k^1$. Such a map induces by restriction a well-defined rational map $\tilde{C} \dashrightarrow \mathbb{P}_k^1$, which actually extends

to a morphism, since $\tilde{C}$ is smooth. Moreover the degree of such a morphism is 2 since $M_{red} \cdot \tilde{C} = 2$. We deduce that C should be hyperelliptic, a contradiction. We conclude that the general element in $|M|$ is necessarily reduced.

Let Γ be the image of a general element of $|M|$. It follows that Γ is a reduced curve of degree 2, thus it is either a conic or it is the union of two skew lines. On the one hand, by construction, Γ cuts on C_{can} the general element of the aforementioned g_4^1 . On the other hand, it follows from Proposition 3.2 that each element of the g_4^1 is the complete intersection of a suitable pencil of conics in a plane: such a complete intersection cannot be contained in the union of two skew lines. We conclude that Γ is a conic in a plane cutting the general element of the g_4^1 on C_{can} . Since Σ is exactly the union of such kind of planes, we also have that it contains Γ . It follows that $S \subseteq \Sigma$, thus $g \geq g - 1 + \tau$ (see Theorem 3.3), i.e. $\tau \leq 1$. If the strict equality holds, then C_{can} would be contained in a surface of degree at most $g - 1$, contradicting the minimality of $\deg(S)$. We conclude that $\tau = 1$, which is not even, so that $g \leq 12$ by Proposition 3.1 ii). Since $1 = \tau \leq (g - 5)/2$ (see Proposition 2.5), we also have that $g \geq 7$.

Finally, if C were endowed with two different covers $\varrho_i: C \rightarrow \mathbb{P}_k^1$, $i = 1, 2$, then $\tau_1, \tau_2 = 0$ (Lemma 3.4). Thus C_{can} would be contained in the surface B_{can} , which has degree $g - 1 < g$ by Theorem 3.3. This last fact contradicts the hypothesis. We conclude that C carries exactly one g_4^1 . $\square$

The following corollaries to Theorems A, B, C are immediate.

Corollary 4.2. *Let C be a non-hyperelliptic curve of genus g . If C_{can} is contained in a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ of degree g , then C is either trigonal or tetragonal or isomorphic to a plane sextic.*

Corollary 4.3. *Let C be a non-hyperelliptic, non-trigonal curve of genus $g \geq 13$. If C_{can} is contained in a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ of degree g , then C is bielliptic.*

We conclude this section by proving that Theorem C is sharp when $\text{char}(k) = 0$; i.e. for each $g = 7, \dots, 12$ there always exists a curve C of genus g such that the minimal degree of a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ containing C_{can} is g .

Due to Theorem C all such curves are necessarily endowed with a cover $\varrho: C \rightarrow \mathbb{P}_k^1$ of degree 4. Looking at the proof of Theorem C, we know that we have to ask for those covers such that $\tau = 3$.

The following Bertini-type theorem will be used.

Theorem 4.4. *Let $\mathcal{E} := \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$ and $\mathcal{F} := \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$ be such that $\sum_{i=1}^3 \alpha_i = \sum_{j=1}^2 \beta_j$. Define $\mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^1$ and*

$$\mathcal{U} := \{(a, b) \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \check{\mathcal{F}}) \mid \\ D_0(a, b) \text{ is smooth and } \pi|_{D_0(a, b)} \text{ is a cover of degree 4}\}.$$

Then $\mathcal{U}$ is open. If $\text{char}(k) = 0$ and $S^2 \mathcal{E} \otimes \check{\mathcal{F}}$ is globally generated, then $\mathcal{U} \neq \emptyset$.

Proof. See [C–E], Theorem 4.5, and [Cs], Theorem 2.5. $\square$

Proposition 4.5. *Let $\text{char}(k) = 0$. For each $g = 7, \dots, 12$ there exists a non-hyperelliptic curve C of genus g such that the minimal degree of a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ containing C_{can} is g .*

Proof. For each $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta := (\beta_1, \beta_2)$ we define $\mathcal{E}_\alpha := \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$ and $\mathcal{F}_\beta := \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$.

For $g \neq 11$ we choose α and β satisfying $\sum_{i=1}^3 \alpha_i = \sum_{j=1}^2 \beta_j = g + 3$ in such a way that $S^2 \mathcal{E}_\alpha \otimes \check{\mathcal{F}}_\beta$ is globally generated. Thus, for a general choice of $(a, b) \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \check{\mathcal{F}})$, the subscheme $D_0(a, b)$ is a smooth curve and $\pi|_{D_0(a, b)}$ is a cover of degree 4 (due to Theorem 5.4) of genus g (due to Theorem 2.2).

If $g = 12$ we set $\alpha := (5, 5, 5)$ and $\beta := (5, 10)$. If $g = 10$ we set $\alpha := (4, 4, 5)$ and $\beta := (5, 8)$. If $g = 9$ we set $\alpha := (4, 4, 4)$ and $\beta := (5, 7)$. If $g = 8$ we set $\alpha := (3, 4, 4)$ and $\beta := (5, 6)$. If $g = 7$ we set $\alpha := (3, 3, 4)$ and $\beta := (5, 5)$.

Now we examine separately the case $g = 11$. Notice that in this case we cannot argue as above, so we will construct directly a cover $\varrho: C \rightarrow \mathbb{P}_k^1$ with $\tau = 3$.

Choose a point $p \in \mathbb{P}_k^1 \times \mathbb{P}_k^1$ and denote by $\mathbb{F}$ the blow up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at p . Then $\text{Pic}(\mathbb{F})$ is freely generated by the classes of the total transforms η and h of the two general members of the pencils of lines on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and by the class e of the exceptional divisor.

Consider a smooth and integral divisor $C \in |4\eta + 5h - 2e|$: the curve C has degree 11 and it is endowed with a base-point-free g_4^1 cut out by $|h|$. One checks that the linear system $\Gamma := |2\eta + 3h - e|$ on $\mathbb{F}$ is base-point-free, very ample on an open subset of $\mathbb{F}$ and it maps birationally $\mathbb{F}$ in $\mathbb{P}_k^{g-1}$ as a conic bundle B of degree 11. The restriction $\Gamma|_C$ is the canonical system on C which is thus non-hyperelliptic. Finally B is contained in the 3-dimensional scroll Σ generated by the g_4^1 on C ; thus $\tau \leq 1$. If the strict inequality holds, then $\deg(B)$ should be at least 16, by Theorem 3.3. We conclude that $\tau = 1$. $\square$

5. APOLARITY OF A TETRAGONAL CURVE

We start this section by recalling some facts about the classical Macaulay correspondence. We refer to [I-K] (see also [R-S] and [I-R] for the characteristic 0 case) for other details and the proofs.

We recall that we can consider two graded rings, namely the polynomial ring $T^n := k[x_0, \dots, x_n]$ and the divided power k -algebra R^n in the $n + 1$ variables $y_0, \dots, y_n$.

As explained in [I-K], Appendix A, the k -vector space R^n coincides with $k[y_0, \dots, y_n]$; thus there exists a natural action of GL_{n+1} on R^n . Let $\gamma := (\gamma_0, \dots, \gamma_n) \in \mathbb{N}^{n+1}$: the monomial $y_0^{\gamma_0} \dots y_n^{\gamma_n} \in R^n$ is usually denoted by $y^{[\gamma]}$. The k -algebra structure on R^n is obtained by extending by linearity the monomial multiplication

$$y^{[\gamma]} y^{[\delta]} := \frac{(\gamma + \delta)!}{\gamma! \delta!} y^{[\gamma + \delta]}.$$

The algebra T^n acts on R^n by differentiation. More precisely, if $\gamma := (\gamma_0, \dots, \gamma_n)$ and $\delta := (\delta_0, \dots, \delta_n) \in \mathbb{N}^{n+1}$ we have the natural pairing $T_g^n \times R_d^n \rightarrow R_{d-g}^n$ (we denote by $*_d^n$ the summand of degree d elements in $*^n$) given on monomials by the rule

$$x^\gamma(y^{[\delta]}) := \begin{cases} y^{[\delta - \gamma]} & \text{if } \delta \geq \gamma, \\ 0 & \text{if } \delta \not\geq \gamma. \end{cases}$$

For any linear form $\ell := \sum_{i=0}^n a_i y_i \in R_1^n$ the divided power $\ell^{[d]}$ is defined as

$$\ell^{[d]} := \sum_{\delta \in \mathbb{N}^{n+1}, |\delta|=d} a_0^{\delta_0} \dots a_n^{\delta_n} y_0^{[\delta_0]} \dots y_n^{[\delta_n]}.$$

Such an action defines a perfect pairing $T_d^n \times R_d^n \rightarrow k$ between forms of degree d in R^n and in T^n . In particular, R_1^n and T_1^n are natural dual vector spaces. Therefore the projective spaces with coordinates $y_0, \dots, y_n$ and $x_0, \dots, x_n$ are naturally dual to each other, and we denote them by $\mathbb{P}_k^n$ and $\check{\mathbb{P}}_k^{g-1}$ respectively.

Two homogeneous forms $f \in R^n$ and $q \in T^n$ are called *apolar* if $q(f) = 0$. As explained in [I–K], apolarity allows us to associate an Artinian Gorenstein graded quotient of T^n to a form in R^n as follows. If $f \in R_d^n$, then

$$f^\perp := \{q \in T^n \mid q(f) = 0\}$$

is easily seen to be a homogeneous ideal in T^n and $A^f := T^n/f^\perp$ is an Artinian Gorenstein graded quotient of T^n with socle $0 :_{T^n} T_1^n$ in degree d . Also, the converse holds: if A is an Artinian Gorenstein graded quotient of T^n , say $A := T^n/I$, with socle in degree d , then there exists $f \in R_d^n$ such that $I = f^\perp$. The main result about apolarity, due to Macaulay (see [I–K], Lemma 2.12 and the references therein), is the following theorem.

Theorem 5.1. *The map $f \mapsto A^f$ induces a bijection between $\mathbb{P}(R_d^n)$ and the set of graded Artinian Gorenstein quotient rings of T^n with socle in degree d .*

Now we restrict our attention to non-hyperelliptic curves C of genus $g \geq 3$. Then the homogeneous coordinate ring of C_{can} inside $\check{\mathbb{P}}_k^{g-1}$ satisfies

$$T_C := \bigoplus_{h=0}^{+\infty} H^0(C, \omega_C^h).$$

If we take a general linear subspace $H = D_0(h_1, h_2) \subseteq \check{\mathbb{P}}_k^{g-1}$ of codimension 2, then the algebra $T_C/(h_1, h_2)$ turns out to be naturally an Artinian Gorenstein graded quotient of $T^{g-1}/(h_1, h_2)$ with socle in degree 3 and Hilbert function $(1, g-2, g-2, 1)$. With a proper choice of coordinates $y_0, \dots, y_{g-1}$ in $\mathbb{P}_k^{g-1}$ and of the corresponding dual coordinates $x_0, \dots, x_{g-1}$ in $\check{\mathbb{P}}_k^{g-1}$, we can assume that $H = V(x_{g-2}, x_{g-1})$ so that $T^{g-1}/(h_1, h_2) \cong T^{g-3}$.

Thus we can associate to C and H a polynomial $f \in R_3^{g-3}$ (up to the natural action of GL_{g-3}) in such a way that

$$T_C/(x_{g-2}, x_{g-1}) \cong T^{g-3}/f^\perp.$$

Definition 5.2. With the notation introduced above we say that the polynomial $f \in R^{g-3}$ and the curve C are apolar to each other.

Following the definition given in [I–R] in the case $\mathrm{char}(k) = 0$, apolarity allows us to define a rational map

$$\psi_C: G(g-2, g) \dashrightarrow H_{g-3,3},$$

where $G(g-2, g)$ denotes the Grassmannian of subspaces of codimension 2 of $\check{\mathbb{P}}_k^{g-1}$ and $H_{g-3,3} := R_3^{g-3}/\mathrm{GL}_{g-2}$, without restrictions on $\mathrm{char}(k)$. On the other hand, in $H_{g-3,3}$ is also defined the locus $H_{g-3,3}(h)$ of GL_{g-2} -orbits of cubics (in the sense of a divided power algebra when $\mathrm{char}(k) > 0$) which can be written as a sum of h (divided power) cubes of linear forms. Obviously we must have $h \geq 1$ and

$H_{g-3,3}(h) \subseteq H_{g-3,3}(h+1)$. For simplicity we denote by $\text{ap}(C)$ the *apolarity* of C , i.e. the smallest integer h such that the general point of $\text{im}(\psi_C)$ is in $H_{g-3,3}(h)$.

A natural problem is to inspect curves C with low apolarity. To this purpose a fundamental tool is the following fundamental result, known as the *Apolarity Lemma*.

Lemma 5.3. *Let $\ell_i \in R_1^n$ and let $L_i := D_0(\ell_i) \subseteq \check{\mathbb{P}}_k^{g-1}$, $i = 1, \dots, s$. Then $f = \sum_{i=0}^s \lambda_i \ell_i^{[d]}$ for some $\lambda_i \in k^*$, $i = 0, \dots, s$, if and only if the homogeneous ideal $I_\Gamma \subseteq R^n$ of $\Gamma := \{L_0, \dots, L_s\} \subset \check{\mathbb{P}}_k^{g-1}$ satisfies $I_\Gamma \subseteq f^\perp$.*

Proof. See [B–C–N], Lemma 1.6, and the references therein. $\square$

The condition $I_\Gamma \subseteq f^\perp$ is often summarised in the standard literature by saying that Γ is *apolar* to f .

By combining the above result with Theorems A and B stated in the introduction and Theorems 1.7 and 4.2 of [B–C–N] (see also Theorem 4 of [DP–Z1]), one can prove the following.

Theorem 5.4. *Let C be a non-hyperelliptic curve C of genus $g \geq 5$. The minimal degree of a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ containing C_{can} is $g-2$ (resp. $g-1$) if and only if $\text{ap}(C) = g-2$ (resp. $\text{ap}(C) = g-1$).*

When C is tetragonal we are able to prove the following partial result.

Proposition 5.5. *Let C be a tetragonal curve of genus $g \geq 5$. Then*

$$\text{ap}(C) \leq g-1 + \tau(C) \leq g-1 + \left\lfloor \frac{g-5}{2} \right\rfloor.$$

Proof. Recall that for a tetragonal curve of genus $g \geq 5$ one has $\tau \geq 0$; hence $g-1 + \tau \geq g-1$.

Let $C = A \cap B$ be the corresponding representation in $\mathbb{P}$ with the standard meaning of symbols. Let $H := D_0(h_1, h_2) \subseteq \check{\mathbb{P}}_k^{g-1}$, where h_1 and h_2 are general linear forms. The geometric characterisation of the degree of a surface allows us to assume that $\Gamma := B_{\text{can}} \cap H$ consists of exactly $g-1 + \tau$ pairwise distinct points.

We know that I_Γ is the saturation of $I_{B_{\text{can}}} + (h_1, h_2)$. By construction we have

$$I_{B_{\text{can}}} + (h_1, h_2) \subseteq I_{C_{\text{can}}} + (h_1, h_2).$$

The ideal on the right is saturated, since the same is true for $I_{C_{\text{can}}}$ and C_{can} is arithmetically Gorenstein (see [Mi], Proposition 1.3.4). Thus, by saturating both sides of the above inclusion, the ideal on the right does not change. Modulo (h_1, h_2) , we finally obtain

$$I_{\Gamma|H} \subseteq \frac{I_{C_{\text{can}}} + (h_1, h_2)}{(h_1, h_2)}.$$

We can assume that $h_i = x_{g-3+i}$, $i = 1, 2$, so that $I_{\Gamma|H} \subseteq k[x_0, \dots, x_{g-3}]$ and

$$\frac{T(C)}{(h_1, h_2)} \cong \frac{k[x_0, \dots, x_{g-3}]}{f^\perp}.$$

Thus $I_{\Gamma|H} \subseteq f^\perp$. It then follows from Lemma 5.3 that the general point of $\text{im}(\psi_C)$ is in $H_{g-3,3}(g-1+\tau)$; hence $\text{ap}(C) \leq g-1+\tau$. $\square$

It is thus natural, at this point, to ask the following question.

Question 5.6. Let $g \geq 5$ and let C be a tetragonal curve. Is it true that $\text{ap}(C) = g - 1 + \tau(C)$?

An immediate consequence of Theorem C, Theorem 5.4 and Proposition 5.5 is the following proposition.

Proposition 5.7. *Let C be a non-hyperelliptic curve of genus $g \geq 3$. If the minimal degree of a surface $S \subseteq \mathbb{P}_k^{g-1}$ containing C_{can} is g , then $\text{ap}(C) = g$.*

Proof. On the one hand, due to Theorem C, we have $g \geq 7$, C is tetragonal and $\tau(C) = 1$. Proposition 5.5 thus implies that $\text{ap}(C) \leq g - 1 + \tau(C) = g$. On the other hand, due to Theorem 5.4 we have that $\text{ap}(C) \geq g$. We conclude that $\text{ap}(C) = g$. $\square$

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