FROM PARKING FUNCTIONS TO GELFAND PAIRS

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ABSTRACT. A pair (G, K) of a group and its subgroup is called a Gelfand pair if the induced trivial representation of K on G is multiplicity free. Let (a_j) be a sequence of positive integers of length n , and let (b_i) be its non-decreasing rearrangement. The sequence (a_i) is called a parking function of length n if $b_i \leq i$ for all $i = 1, \dots, n$. In this paper we study certain Gelfand pairs in relation with parking functions. In particular, we find explicit descriptions of the decomposition of the associated induced trivial representations into irreducibles. We obtain and study a new q -analogue of the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$, $n \geq 1$.

1. INTRODUCTION

For a given natural number n and a sequence of positive integers (a_j) , let (b_i) denote the non-decreasing rearrangement of (a_j) . Then, the sequence (a_i) is called a *parking function* of length n if $b_i \leq i$.

Parking functions, introduced in the papers of Pyke [7], Konheim and Weiss [4] in relation with hashing problems, are related to various topics in algebraic combinatorics: labelled trees, hyperplane arrangements, non-crossing set partitions, diagonal harmonics.... For more, see the solution to problem 5.49 in [9] and the references therein.

A clever argument by Pollak counts the number of parking functions of length n by constructing a one-to-one correspondence between the set of parking functions and the group $\mathbb{Z}_{n+1}^n / \Delta \mathbb{Z}_{n+1}^n$, where $\mathbb{Z}_{n+1} := \mathbb{Z} / (n+1)\mathbb{Z}$ denotes the cyclic group of order $n+1$ and $\Delta \mathbb{Z}_{n+1}^n$ stands for the diagonal in $\mathbb{Z}_{n+1}^n = \mathbb{Z}_{n+1} \times \cdots \times \mathbb{Z}_{n+1}$. This correspondence sends a parking function (a_i) to the element $(a_i - 1 \bmod n+1)$. Consequently, the number of parking functions of length n is $P_n = (n+1)^{n-1}$.

One can go one step further than Pollak and view the set of parking functions of length n as a finite abelian group. This group is naturally endowed with a symmetric group action by permuting the coordinates. The symmetric group action is compatible with the group operation of the group.

On the representation theory side, this line of inquiry leads us to the realm of Gelfand pairs. On the combinatorics side, it motivates us to define a new q -analogue of the Catalan numbers.

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1.1. Outline. Let Γ be a finite Abelian group, and for each $n \geq 1$ denote by $\widetilde{\Gamma}^n$ the quotient of $\Gamma \times \cdots \times \Gamma$ (n copies) by its diagonal subgroup $\Delta\Gamma^n := \{(g, \dots, g) : g \in \Gamma\}$. Consider $\widetilde{\Gamma}^n \rtimes S_n$, the semidirect product of $\widetilde{\Gamma}^n$ with the symmetric group S_n .

Our main results are as follows: The pair $(\widetilde{\Gamma}^n \rtimes S_n, S_n)$ is a Gelfand pair (Lemma 3.2). For finite cyclic groups, $\Gamma = \mathbb{Z}_r$, $r \in \mathbb{N}$ in Theorem 3.5, we find explicit descriptions of the irreducible constituents of the multiplicity-free representation $\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$. Note that when $r = n + 1$, the group $\widetilde{\Gamma}^n$ is the “group of parking functions”, $\mathbb{Z}_{n+1}^n / \Delta\mathbb{Z}_{n+1}^n$. Let $\bigoplus_{\alpha} V_{\alpha}$ be the decomposition of $\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$ into irreducibles, and let $C_n(q) = \sum_{\alpha} q^{\dim V_{\alpha}}$. For $n \in \mathbb{N}$, let

$$D(n) := \{(k_0, \dots, k_n) \in \mathbb{N}^{n+1} : \sum k_i = n, \text{ and } n+1 \text{ divides } \sum_{i=1}^n ik_i\}.$$

The corollaries of our main result imply that

$$C_n(q) = \sum_{\mathbf{k} \in D(n)} q^{\binom{n}{k_0, k_1, \dots, k_n}},$$

where $\binom{n}{k_0, k_1, \dots, k_n}$ is a multinomial coefficient. Hence,

$$\begin{aligned} C_n(1) &= \frac{1}{n+1} \binom{2n}{n}, \text{ and} \\ \left. \frac{dC_n(q)}{dq} \right|_{q=1} &= (n+1)^{n-1}. \end{aligned}$$

Observe that $D(n)$ can be identified with the set of polynomials $p(x) \in \mathbb{Z}[x]$ with non-negative integer coefficients such that $p(1) = n$, and $p'(1)$ is divisible by $n+1$.

Let $S(n)$ denote the set of all integer sequences $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{N}^n$ satisfying $\sum_{i=1}^j b_i \geq j$ for all $j = 1, \dots, n$, and $\sum_{i=1}^n b_i = n$. It is well known that $|S(n)| = C_n(1)$. See (o⁵) in [10]. Define

$$S_n(q) = \sum_{\mathbf{b} \in S(n)} q^{\binom{n}{b_1, \dots, b_n}}.$$

Conjecture 1.1. The polynomials $C_n(q)$ and $S_n(q)$ are identical for all $n \geq 1$.

The conjecture has been verified for $n \leq 10$ using a computer.

We organized our paper as follows. In Section 2 we introduce our notation and background. In Section 3 we prove our main theorem. In Section 4 we introduce and study a q -analogue of the Catalan numbers. We end our article with final remarks and open questions in Section 5.

2. PRELIMINARIES

It is clear from the definition that the symmetric group on n letters S_n acts on the set of parking functions of length n . For $n = 3$, the set of parking functions is divided into 5 orbits:

111

112, 121, 211

113, 131, 311

122, 212, 221

123, 132, 213, 231, 312, 321.

Each line corresponds to an orbit whose first entry is chosen to be non-decreasing.

Note that orbits can be parameterized in terms of non-decreasing parking functions. Recall that the number of non-decreasing integer sequences $(b_1, \dots, b_n)$ such that $1 \leq b_i \leq i$ for all $i = 1, \dots, n$ is the n -th Catalan number,

$$C_n := \frac{1}{n+1} \binom{2n}{n}.$$

2.1. Permutation representation. Given a group G and an action of G on some set X , the space of complex-valued functions $L(X)$ on X becomes a representation of G : Given $g \in G$ and $\psi : X \rightarrow \mathbb{C}$, define the action of G on $L(X)$ as follows:

$$(g \cdot \psi)(x) := \psi(g^{-1} \cdot x).$$

The representation $L(X)$ induced from the action of G on the set X is called a *permutation representation*. For any $x \in G$, by δ_x denote the Dirac mass at x , defined by

$$\delta_x(y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

When G acts on a set X , then X is decomposed into G -orbits. Similarly, the representation $L(X)$ is decomposed into irreducible representations. However, by virtue of linearity, the decomposition of $L(X)$ is much finer than that of the set X .

Denote the set of orbits of X by $\text{Orb}(X)$. Each G -orbit is stable under the G -action; that is, if $x \in O$ for some orbit O and $g \in G$, then $g \cdot x \in O$. Therefore to each orbit we can attach a representation $L(O)$. Since

$$X = \bigsqcup_{O \in \text{Orb}(X)} O,$$

the representation $L(X)$ decomposes as $L(X) = \bigoplus_{O \in \text{Orb}(X)} L(O)$. However, typically one can decompose $L(X)$ further by decomposing the representations $L(O)$ corresponding to the orbits $O \in \text{Orb}(X)$. By $|Y|$ denote the number of elements of a finite set Y . Then

$$\dim L(O) = \text{Number of elements of } O = |O|.$$

2.2. Parking function module. When X is the set of parking functions of length n , we call the representation $L(X)$ the *parking function module* and denote it by $\text{PF}(n)$.

Let us go back to the set of parking functions of length 3. The symmetric group S_3 acts on the set of parking functions of length 3 and divides it into 5 orbits. We may parameterize the orbits of S_3 on $\text{PF}(3)$ by the non-decreasing elements they contain. Then,

$$\text{PF}(3) \cong L(\mathbf{111}) \oplus L(\mathbf{112}) \oplus L(\mathbf{113}) \oplus L(\mathbf{122}) \oplus L(\mathbf{123}).$$

It is well known that S_3 has 3 irreducible representations: the *trivial representation* (of dimension 1) which acts trivially on the one-dimensional complex space, the *sign representation* (of dimension 1) which acts by the sign of the permutation on the one-dimensional complex vector space and a two-dimensional representation.

The symmetric group S_3 acts on 3 letters $\{1, 2, 3\}$, hence on the corresponding function space $L(X)$, called the *standard representation*. The function $\delta_1 + \delta_2 + \delta_3$ is invariant under the S_3 -action and forms a one-dimensional representation (isomorphic to the trivial representation). The vector space V is endowed with the standard inner product $\langle \cdot, \cdot \rangle$, where $\langle \delta_i, \delta_j \rangle$ is 1 only if $i = j$, and 0 otherwise. Let V

be the orthogonal subspace to $\delta_1 + \delta_2 + \delta_3$. Because the inner product is compatible with the S_3 -action, the space V is a two-dimensional representation. In fact, it is irreducible. In the basis $\delta_1, \delta_2, \delta_3$, the subspace V can be expressed as

$$V = \left\{ \sum a_i \delta_i : \sum a_i = 0 \right\}.$$

Of the list above, $L(\mathbf{111})$ is isomorphic to the trivial representation, the three-dimensional representations $L(\mathbf{112}), L(\mathbf{113}), L(\mathbf{122})$ are isomorphic to the standard representation (hence decompose as the sum of a trivial representation and the two-dimensional representation), and $L(\mathbf{123})$ is the regular representation of the group S_3 .

Given a representation V and an irreducible representation V_i of a group G , the number of times the irreducible representation V_i appears in the decomposition of the representation V is called the *multiplicity* of V_i in V .

We summarize what we have seen in a table of multiplicities:

	<i>triv</i>	<i>sign</i>	<i>2-dim</i>
$L(\mathbf{111})$	1		
$L(\mathbf{112})$	1		1
$L(\mathbf{113})$	1		1
$L(\mathbf{122})$	1		1
$L(\mathbf{123})$	1	1	2

Observe that the trivial representation appears in each representation $L(O)$ only *once*. In fact, we will later prove an equivalent form of

Lemma 2.1. *For all $n \geq 1$, the trivial representation of S_n appears in each orbit representation $L(O)$ in $\text{PF}(n)$ exactly once.*

2.3. Intertwining maps and multiplicity-free representations. A representation V of G is called *multiplicity free* if no irreducible representation of G appears more than once in V .

A $\mathbb{C}$ -linear map $T : V \rightarrow W$ between two representations V and W of G is called *intertwining* if

$$T(g \cdot v) = g \cdot T(v)$$

for any $g \in G$ and $v \in V$. If V and W are irreducible, then there are two cases:

- (1) V and W are non-isomorphic, and $T = 0$;
- (2) V and W are isomorphic, and $T = \lambda \mathbf{1}_V$ for some $\lambda \in \mathbb{C}$.

This is called *Schur's Lemma*. It follows that if $V = \bigoplus V_i^{m_i}$, where $m_i \in \mathbb{Z}_{\geq 0}$ is the decomposition of V into irreducibles, then the set of intertwining operators on V , denoted by $\text{End}_G(V)$, is

$$\text{End}_G(V) = \text{End}_G\left(\bigoplus V_i^{m_i}\right) = \bigoplus \text{End}(\mathbb{C}^{m_i}).$$

Therefore, V is multiplicity free if and only if $m_i \leq 1$ for all i if and only if the algebra $\text{End}_G(V)$ is commutative.

2.4. Gelfand pairs. Given a finite group G and a subgroup K , the group G acts on the left coset space G/K . The pair (G, K) is called a *Gelfand pair* if the G -representation $L(G/K)$ is multiplicity free.

Given two subgroups K_1 and K_2 of G , the space of functions $L(K_1 \backslash G / K_2)$ on the double coset space $K_1 \backslash G / K_2$ can be identified with the space of $K_1 \times K_2$ -invariant

functions on G , consisting of those functions $\psi : G \rightarrow \mathbb{C}$ so that $\psi(k_1 g k_2) = \psi(g)$ for all $(k_1, k_2) \in K_1 \times K_2$ and $g \in G$.

The space of intertwining operators $\text{End}_G \mathbb{L}(G/K)$ on $\mathbb{L}(G/K)$ coincides with $\mathbb{L}(K \backslash G/K)$. The algebra $\text{End}_G \mathbb{L}(G/K)$ is called the *Hecke algebra* of the pair (G, K) . It follows from Schur's Lemma that the pair (G, K) is a Gelfand pair if and only if its Hecke algebra is commutative.

Lemma 2.1 is equivalent to the following:

Lemma 2.2. *The pair (G, K) is a Gelfand pair for $G = \text{PF}(n) \ltimes S_n$ and $K = S_n$.*

Equivalently, the pair (G', K') is a Gelfand pair, where $G' = \mathbb{Z}_{n+1}^n \ltimes S_n$ and $K' = \Delta \mathbb{Z}_{n+1}^n \ltimes S_n$. Indeed, the symmetric group S_n fixes the elements on the diagonal $\Delta \mathbb{Z}_{n+1}^n$ pointwise; hence this action is trivial, and $K' \cong \Delta \mathbb{Z}_{n+1}^n \times S_n$. The group G' is an example of a wreath product of finite groups: $G' = \mathbb{Z}_{n+1}^n \ltimes S_n$ is the wreath product of $\mathbb{Z}_{n+1}^n$ by S_n .

2.5. Wreath products. Let Γ be a finite group. The symmetric group S_n acts on Γ^n by permuting the coordinates. Form the semi-direct product $\Gamma^n \rtimes S_n$. The resulting finite group is called the *wreath product* of Γ by S_n and denoted by $\Gamma \wr S_n$. An element of $\Gamma \wr S_n$ is denoted by the symbol $(g_1, \dots, g_n; \sigma)$, where $g_i \in \Gamma$ and $\sigma \in S_n$. The product of two elements $(g_1, \dots, g_n; \sigma)$ and $(h_1, \dots, h_n; \tau)$ is

$$(g_1, \dots, g_n; \sigma)(h_1, \dots, h_n; \tau) = (g_1 h_{\sigma^{-1}(1)}, \dots, g_n h_{\sigma^{-1}(n)}; \sigma\tau).$$

We have previously stated that the pair $(\mathbb{Z}_{n+1} \wr S_n, \Delta \mathbb{Z}_{n+1}^n \ltimes S_n)$ is Gelfand. Inspired by this observation, we can ask for any group Γ :

QUESTION. Is $(\Gamma \wr S_n, \Delta \Gamma^n \times S_n)$ a Gelfand pair?

When $n = 2$, the answer is “yes” for any group Γ . For $n \geq 3$, the results are mixed. Generally, the answer is “no”. We have tested this claim using GAP. Some of the results are:

Group Γ	true	false
S_3	$n \leq 5$	$n = 6$
A_4	$n \leq 3$	$n = 4$
$GL(2, \mathbb{F}_3)$	$n = 2$	$n = 3$
$SL(3, \mathbb{F}_2)$	$n = 2$	$n = 3$

In general, one can view being a Gelfand pair for (G, K) as a measure of distance between the group G and its subgroup K . In some sense, for a Gelfand pair (G, K) , the distance between the subgroup K and the whole group G is not that big. In the examples above, as n gets larger, the group $G = \Gamma \wr S_n$ gets much larger than the subgroup $K = \Gamma \times S_n$. We believe that this is why the answer is typically “no”.

3. GENERALIZED PARKING FUNCTIONS

Let Γ be a finite group. It is known that the pair $(\Gamma \wr S_n, S_n)$ is Gelfand, in the case of a finite cyclic group $\mathbb{Z}_r$ by [6], and in the case of an arbitrary Abelian group Γ by [2].

For the rest of the article, let Γ be a finite Abelian group. By $\Delta \Gamma^n$, denote the diagonal copy of Γ in Γ^n , and by $\widetilde{\Gamma}^n$ the quotient group $\Gamma^n / \Delta \Gamma^n$. Finally, the image of an element $g \in \Gamma^n$ in $\widetilde{\Gamma}^n$ is denoted by $\tilde{g}$.

The symmetric group S_n acts naturally on the product Γ^n , fixes the diagonal $\Delta\Gamma^n$ and hence acts on the quotient group $\Gamma^n/\Delta\Gamma^n$. Denote the resulting S_n -representation by V_n . There is a natural projection

$$(3.1) \quad \begin{aligned} \pi : \Gamma^n \rtimes S_n &\rightarrow \widetilde{\Gamma}^n \rtimes S_n \\ (g, \sigma) &\mapsto (\tilde{g}, \sigma). \end{aligned}$$

Denote the identity element of the symmetric group S_n by e_0 .

Lemma 3.1. *The group $\widetilde{\Gamma}^n \rtimes S_n$ is isomorphic to $(\Gamma^n \rtimes S_n)/(\Delta\Gamma^n \rtimes \{e_0\})$.*

Proof. Let $\pi : \Gamma^n \rtimes S_n \rightarrow \widetilde{\Gamma}^n \rtimes S_n$ be as in (3.1). It is clear that π is surjective. We check that π is a homomorphism. Indeed:

$$\begin{aligned} \pi((g, \sigma)(h, \tau)) &= \pi((g\sigma(h), \sigma\tau)) \\ &= (\widetilde{g\sigma(h)}, \sigma\tau) \\ &= (\tilde{g}, \sigma)(\tilde{h}, \tau). \end{aligned}$$

Suppose now that $\pi(g, \sigma) = (\tilde{0}, e_0)$. Then, $\sigma = e_0$ and $g \in \Delta\Gamma^n$. In other words, the kernel of π is $\Delta\Gamma^n \rtimes \{e_0\}$. In particular, $\Delta\Gamma^n \rtimes \{e_0\}$ is normal in $\Gamma^n \rtimes S_n$. Since Γ^n is Abelian, there is a splitting $\Gamma^n = H \cdot \Delta\Gamma^n$, where $H \cap \Delta\Gamma^n = \{0\}$, and H is isomorphic to $\Gamma^n/\Delta\Gamma^n$. Since

$$(H \rtimes S_n) \cdot (\Delta\Gamma^n \rtimes e_0) = (\Delta\Gamma^n \cdot H) \rtimes S_n = \Gamma^n \rtimes S_n,$$

it follows that $H \rtimes S_n \cong \Gamma^n/\Delta\Gamma^n \rtimes S_n$. In other words, $\Gamma^n \rtimes S_n$ is the semidirect product of $\Delta\Gamma^n \rtimes S_n$ and $\widetilde{\Gamma}^n \rtimes S_n$. □

Lemma 3.2. *$(\widetilde{\Gamma}^n \rtimes S_n, S_n)$ is a Gelfand pair.*

Proof. This follows from [2]. □

Lemma 3.2 implies Lemma 2.1 and Lemma 2.2.

3.1. Γ is a finite cyclic group, $\mathbb{Z}_r$. The irreducible representations of the group $\Gamma \wr S_n = \mathbb{Z}_r \wr S_n$ are described as follows (see [1]): Let $\mathbf{k} = (k_0, \dots, k_{r-1})$ be an r -tuple of non-negative integers, and let $\lambda = \lambda_{\mathbf{k}} = (\lambda_1, \dots, \lambda_{\ell})$ be the partition $(0^{k_0}, 1^{k_1}, \dots, (r-1)^{k_{r-1}})$.¹ Suppose

$$\sum_{i=0}^{r-1} k_i = n.$$

Then, $\ell = \ell(\lambda) = n$.

Let $x_1, \dots, x_n$ be algebraically independent variables, and let

$$M(\mathbf{k}) = \{x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(n)}^{\lambda_n} : \sigma \in S_n, \lambda = \lambda_{\mathbf{k}}\}.$$

We denote by $\mathbb{C}M(\mathbf{k})$ the vector space on $M(\mathbf{k})$. Then, $\Gamma \wr S_n$ acts on $\mathbb{C}M(\mathbf{k})$ by

$$(g, \sigma) \cdot f(x_1, \dots, x_n) := f(g_1^{-1} x_{\sigma(1)}, \dots, g_n^{-1} x_{\sigma(n)}),$$

where $g = (g_1, \dots, g_n) \in \Gamma^n$, $\sigma \in S_n$, and $f \in \mathbb{C}M(\mathbf{k})$. It is shown in [1] that $\mathbb{C}M(\mathbf{k})$ is an irreducible $\Gamma \wr S_n$ -module and all irreducible representations of $\Gamma \wr S_n$ can be obtained this way.

¹In this notation, k_i is the multiplicity of i in the partition λ .

In [6] it is shown that

$$\mathrm{Ind}_{S_n}^{\Gamma \wr S_n}(1) = \bigoplus_{\sum k_i = n} \mathbb{C}M(\mathbf{k}).$$

Remark 3.3. Let $(g, \sigma) \in \Gamma \wr S_n$ and $x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n} \in M(\mathbf{k})$. Then,

$$\begin{aligned} (g, \sigma) \cdot (x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n}) &= (g, \sigma) \cdot (x_1^{\lambda_{\tau^{-1}(1)}} \cdots x_n^{\lambda_{\tau^{-1}(n)}}) \\ &= \frac{1}{\prod_i g_{\sigma(i)}^{\lambda_{\tau^{-1}(i)}}} x_{\sigma(1)}^{\lambda_{\tau^{-1}(1)}} \cdots x_{\sigma(n)}^{\lambda_{\tau^{-1}(n)}} \\ &= \frac{1}{\prod_i g_{\sigma(\tau(i))}^{\lambda_i}} x_{\sigma(\tau(1))}^{\lambda_1} \cdots x_{\sigma(\tau(n))}^{\lambda_n}. \end{aligned}$$

Therefore, the matrix of the representation $\mathbb{C}M(\mathbf{k})$ at $(g, \sigma) \in \Gamma \wr S_n$ is a monomial matrix and its unique non-zero diagonal entry is $1/\prod_i g_{\sigma(i)}^{\lambda_i}$. It follows that the character $\chi_{\mathbb{C}M(\mathbf{k})}$ of $\mathbb{C}M(\mathbf{k})$ evaluated at (g, σ) equals

$$(3.2) \quad \chi_{\mathbb{C}M(\mathbf{k})}(g, \sigma) = \frac{1}{g_{\sigma(1)}^{\lambda_1} \cdots g_{\sigma(n)}^{\lambda_n}}.$$

3.2. Spherical functions. Let (G, K) be a Gelfand pair. Suppose that $\bigoplus_{i=1}^s V_i$ is the decomposition of $\mathrm{Ind}_K^G(1)$ into irreducible subrepresentations. For $1 \leq i \leq s$, let χ_i denote the character of V_i and define the $\mathbb{C}$ -valued function

$$(3.3) \quad \omega_i(x) = \frac{1}{|K|} \sum_{h \in K} \chi_i(x^{-1}h), \text{ for } x \in G.$$

The functions ω_i for $i = 1, \dots, s$ are called the *zonal spherical functions* of the pair (G, K) . They form an orthogonal basis of $\mathrm{End}_G \mathbb{L}(G/K)$.

Let $m_\lambda(x_1, \dots, x_n)$ be the *monomial symmetric function* defined as

$$m_\lambda(x_1, \dots, x_n) = \frac{1}{k_0! k_1! \cdots k_n!} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n}.$$

In [6], it is shown that the zonal spherical function $\omega^{\mathbf{k}}$ corresponding to an irreducible constituent $\mathbb{C}M(\mathbf{k})$ of $\mathrm{Ind}_{S_n}^{\Gamma \wr S_n}(1)$ can be expressed in terms of the monomial symmetric function m_λ ,

$$\omega^{\mathbf{k}}((h, \sigma)) = \frac{m_\lambda(h_1, \dots, h_n)}{m_\lambda(1, \dots, 1)}, \text{ for } (h, \sigma) \in \Gamma \wr S_n,$$

where λ is the partition determined by $\mathbf{k}$.

Remark 3.4. It is easy to see that $m_\lambda(1, \dots, 1) = \binom{n}{k_1, \dots, k_n}$.

3.3. Zonal spherical functions for $(\tilde{\Gamma}n \rtimes S_n, S_n)$. In the rest of the article, we will concentrate on the *generalized parking function module*

$$\mathrm{Ind}_{S_n}^{\tilde{\Gamma}n \rtimes S_n}(1).$$

We start with a general observation.

Fact 3.4 (See Section 1.13, Exercise 6.b in [8]). Let N be a normal subgroup of a finite group G , and let Y be a representation of G/N . Define a function X on G by $X(g) = Y(gN)$. Then, X is an irreducible representation of G if and only if Y is an irreducible representation of G/N . In this case X is said to be “lifted from Y ”.

Since $\widetilde{\Gamma}^n \rtimes S_n$ is isomorphic to $(\Gamma \wr S_n)/(\Delta\Gamma^n \rtimes \{e_0\})$, we can use Fact 3.4: Let $\mathbb{C}M(\mathbf{k})$ be an irreducible representation of $\Gamma \wr S_n$ for some $\mathbf{k} = (k_0, \dots, k_{r-1}) \in \mathbb{Z}_{\geq 0}^r$, where $\sum k_i = n$. The representation $\mathbb{C}M(\mathbf{k})$ descends to a representation Y of $\widetilde{\Gamma}^n$ if and only if the normal subgroup $\Delta\Gamma^n \rtimes \{e_0\}$ acts trivially on the representation $\mathbb{C}M(\mathbf{k})$, in which case, the representation Y is irreducible by the above fact.

Let $(c, e_0) \in \Delta\Gamma^n \rtimes \{e_0\}$ and $x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n} \in M(\mathbf{k})$, where $\xi = e^{\frac{2\pi i}{r}}$ and $c = (\xi^j, \dots, \xi^j)$ for some $j = 0, \dots, r-1$.

We calculate the effect of $(c, e_0) \in \Delta\Gamma^n \rtimes \{e_0\}$ on $x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n}$:

$$(c, e_0) \cdot x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n} = (\xi^{-j \sum \lambda_i}) x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n}.$$

The action of $\Delta\Gamma^n \rtimes \{e_0\}$ on $\mathbb{C}M(\mathbf{k})$ is trivial if and only if the right-hand side equals $x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(n)}^{\lambda_n}$. Given that ξ is a primitive r -th root of unity, this is possible if and only if r divides the sum $\sum \lambda_i = \sum ik_i$.

Thus, $\mathbb{C}M(\mathbf{k})$ is an irreducible representation of $\widetilde{\Gamma}^n \rtimes S_n$ if and only if $\mathbf{k} = (k_0, \dots, k_{r-1})$ satisfies the following two conditions:

- $\sum_i k_i = n$,
- r divides $\sum ik_i$.

Our main result follows from these observations:

Theorem 3.5. *Let $\Gamma = \mathbb{Z}_r$ and $\mathbf{k} = (k_0, \dots, k_{r-1})$ be a sequence of non-negative integers of length r . Then,*

- (1) *The $\widetilde{\Gamma}^n \rtimes S_n$ -representation $\mathbb{C}M(\mathbf{k})$ is irreducible of degree $\binom{n}{k_0, \dots, k_{r-1}}$ if and only if $\sum_i k_i = n$ and r divides $\sum ik_i$.*
- (2) *The pair $(\widetilde{\Gamma}^n \rtimes S_n, S_n)$ is Gelfand. Furthermore, $\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$ decomposes into irreducibles as follows:*

$$(3.5) \quad \text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1) = \bigoplus_{\substack{\sum k_i = n \\ r \mid \sum ik_i}} \mathbb{C}M(\mathbf{k}).$$

Remark 3.6. Let $g \in \Gamma^n$ and let $\tilde{g}$ be its image in $\widetilde{\Gamma}^n$. A simple calculation shows that the double coset $S_n(\tilde{g}, e_0)S_n$ in $\widetilde{\Gamma}^n \rtimes S_n$ is $\{(\sigma(\tilde{g}), \sigma\tau) : \sigma, \tau \in S_n\}$. Therefore $|S_n(\tilde{g}, e_0)S_n|$ is the cardinality of the S_n -orbit $S_n \cdot \tilde{g}$ in $\widetilde{\Gamma}^n$ times $n!$. In addition, the number of double cosets of S_n in $\widetilde{\Gamma}^n \rtimes S_n$ is the number of S_n -orbits in $\widetilde{\Gamma}^n$.

Corollary 3.7. *For any positive integers n and r the following identity holds:*

$$r^{n-1} = \sum \binom{n}{k_0, \dots, k_{r-1}},$$

where the sum runs over all non-negative integer sequences of length r such that $\sum_i k_i = n$ and r divides $\sum_i ik_i$.

4. A q -ANALOG OF THE CATALAN NUMBERS

Set $r = n + 1$ and $\Gamma = \mathbb{Z}_{n+1}$. Recall that as an S_n -set, $\widetilde{\Gamma}^n$ is isomorphic to the set of parking functions. Recall also that

Fact 4.1. The number of S_n -orbits in an S_n -set X is the multiplicity of the trivial representation in the associated representation $L(X)$.

Proposition 4.1. *Let $\Gamma = \mathbb{Z}_{n+1}$. Then, the number of irreducible representations in*

$$\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$$

is the n -th Catalan number, C_n .

Proof. This is immediate from Fact 4.1. □

Let $\bigoplus_{\alpha} V_{\alpha}$ be the decomposition of $\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$ as in (3.5), where $r = n + 1$. Define

$$(4.2) \quad C_n(q) = \sum_{\alpha} q^{\dim V_{\alpha}}.$$

Since the multiplicity of each irreducible representation in $\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$ is 1, the coefficients of the polynomial $C_n(q)$ count the number of occurrences of the irreducible representations of a given dimension in $\text{Ind}_{S_n}^{\widetilde{\Gamma}^n \rtimes S_n}(1)$.

Let $D(n)$ denote the set of all $n + 1$ tuples $\mathbf{k} = (k_0, \dots, k_n) \in \mathbb{N}^{n+1}$ such that $\sum_{i=0}^n k_i = n$ and $n + 1$ divides $\sum_{i=1}^n ik_i$. It follows from Theorem 3.5 that

$$(4.3) \quad C_n(q) = \sum_{\mathbf{k} \in D(n)} q^{\binom{n}{k_0, k_1, \dots, k_n}}.$$

It follows from Proposition 4.1 that

$$(4.4) \quad C_n(1) = C_n = \frac{1}{n+1} \binom{2n}{n}$$

and that

$$(4.5) \quad \left. \frac{dC_n(q)}{dq} \right|_{q=1} = (n+1)^{n-1}.$$

Example 4.6. There are 5 sequences of the form $(k_0, \dots, k_3) \in \mathbb{N}^4$ such that $k_0 + \dots + k_3 = 3$ and $\sum_{i=1}^3 ik_i$ is divisible by 4. Namely,

$$D(3) = \{3000, 1101, 0210, 1020, 0012\}.$$

Then $C_3(q) = q + 3q^3 + q^6$. Similarly, one can compute $C_4(q) = q + 4q^4 + 2q^6 + 6q^{12} + q^{24}$.

Let $E(n)$ be the set of n -element multisets on $\mathbb{Z}_{n+1}$ whose elements sum to 0. It is known that the cardinality $E(n)$ is equal to the n -th Catalan number. See **jjj** on page 264 of [9]. Recently Tewodros Amdeberhan constructed an explicit bijection between the sets $D(n)$ and $E(n)$.

Set $C_n(q) = \alpha_1(n)q^{a_1} + \alpha_2(n)q^{a_2} + \cdots + \alpha_m(n)q^{a_m}$, where $\alpha_i(n) \neq 0$ for $i = 1, \dots, m$ and $1 < a_1 < a_2 < \cdots < a_m$. It is desirable to find a combinatorial interpretation of the coefficients $\alpha_i(n)$, $1 \leq i \leq m$.

4.1. Labeled rooted trees. Let us point out an intriguing relationship between the coefficients of $C_n(q)$ and the Lagrange inversion formula.

Let $t_0 < t_1 < t_2 < \cdots$ be a collection of ordered, commuting variables. Let σ be a plane tree (connected graph with no cycles) on n vertices. Set

$$t^\sigma = \prod_{i \geq 0} t_i^{d_i(\sigma)},$$

where $d_i(\sigma)$ is the number of vertices of σ of degree i . Define

$$s_n = \sum_{\tau} t^\tau,$$

summed over all plane trees with n vertices. For example, $s_4 = t_0^3 t_3 + 3t_0^2 t_1 t_2 + t_0 t_1^3$. Using lexicographic ordering on the monomials of s_n , we denote by v_n the sequence of coefficients of the polynomial s_n . For example, $v_4 = (1, 3, 1)$. It turns out, for $n \leq 6$, that the sequence v_{n+1} agrees with the sequence $(\alpha_1(n), \dots, \alpha_m(n))$ of coefficients of $C_n(q)$. However,

$$(\alpha_i(7))_{i=1}^{14} = (1, 7, 7, 7, 21, 42, 21, 56, 105, 35, 35, 70, 21, 1),$$

and $v_8 = (1, 7, 7, 7, 21, 42, 21, 21, 35, 105, 35, 35, 70, 21, 1)$.

It is well known that the coefficient vectors v_n , $n \geq 1$ can be computed by the Lagrange inversion formula (see page 40 of [9]).

5. FINAL REMARKS AND QUESTIONS

Let $\mathbf{k} = (k_0, \dots, k_n)$ be an element of $D(n)$ and let $\mathbb{C}M(\mathbf{k})$ be the associated irreducible constituent of $\text{Ind}_{S_n}^{\widetilde{F}^n \rtimes S_n}(1)$. Let $\omega_{PF}^{\mathbf{k}}$ be the zonal spherical function.

Given $(h, \sigma) \in (F^n \wr S_n, S_n)$, by $\widetilde{h}$ denote the image of h in $\widetilde{F}^n$. Then,

$$\omega_{PF}^{\mathbf{k}}(\widetilde{h}, \sigma) = \frac{m_\lambda(h_1, \dots, h_n)}{m_\lambda(1, \dots, 1)},$$

where λ is the partition determined by $\mathbf{k} \in D(n)$.

The values of the zonal spherical functions for $\widetilde{F}^n \rtimes S_n$ are not necessarily real. For instance, the zonal spherical function $\omega^{\mathbf{k}} : \widetilde{F}^5 \rtimes S_5 \rightarrow \mathbb{C}$ for $\mathbf{k} = (0, 0, 0, 2, 3)$, depicted in Figure 1, is not real valued.

What about the number of real-valued zonal spherical functions? A few terms of this sequence are 2, 3, 6, 10, $\dots$ (starting at $n \geq 2$). Can we say anything about the injectivity of these functions? How about their Fourier transforms?

Let $\{x_1, y_1, \dots, x_n, y_n\}$ be a set of $2n$ algebraically independent variables on which $\sigma \in S_n$ acts by

$$\sigma \cdot x_i = x_{\sigma^{-1}(i)}, \quad \sigma \cdot y_i = y_{\sigma^{-1}(i)} \quad \text{for } i = 1, \dots, n.$$

FIGURE 1

For non-negative integers $r, s \in \mathbb{N}$, define $p_{r,s} = \sum_{i=1}^n x_i^r y_i^s$. The ring of diagonal co-invariants (of S_n) is the S_n -module

$$R_n = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]/I_+,$$

where I_+ is the ideal generated by the power sums $p_{r,s}$ with $r + s > 0$.

Let $\Gamma = \mathbb{Z}_{n+1}$. It is known that as an S_n -module, R_n is isomorphic to $(\Gamma^n/\Delta\Gamma^n) \otimes \text{sign}$ (see [3]). Notice that R_n is a bi-graded S_n -module. Let $\mathcal{F}(q, t)$ be the corresponding bi-graded character. The multiplicity of the sign character in $\mathcal{F}(q, t)$ is the so-called q, t -Catalan series. What is the relationship between the q, t -Catalan series and our q -analogue? It would be interesting to also investigate similar questions for the associated “twisted Gelfand pair”. (For a definition, see [5], page 398.)

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