SEMIPROJECTIVITY OF UNIVERSAL C^* -ALGEBRAS GENERATED BY ALGEBRAIC ELEMENTS

TATIANA SHULMAN

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ABSTRACT. Let p be a polynomial in one variable whose roots all have multiplicity more than 1. It is shown that the universal C^* -algebra of a relation $p(x) = 0$, $\|x\| \leq 1$, is semiprojective and residually finite-dimensional. Applications to polynomially compact operators are given.

INTRODUCTION

The “general lifting problem” in C^* -algebras is a class of problems which are formulated in the following way: for a quotient C^* -algebra A/I and its element b with some specific properties, find (or prove the existence of) a preimage of b in A with the same properties. Lifting problems are closely related to the theory of stability of relations which, roughly speaking, investigates when “an almost representation of a relation” is “close” to a representation (for strict definitions see [5]).

By a relation we mean a system of equalities of the form $f(x_1, \dots, x_n, x_1^*, \dots, x_n^*) = 0$, where f is a non-commutative polynomial (an element of the corresponding free algebra), which can also include inequalities of the form $\|x_i\| \leq c$ or $\|x_i\| < c$ (they also can be written as polynomial equalities if one extends the list of generators).

Recall that a C^* -algebra D is *projective* if for any C^* -algebra A , its ideal I and every $*$ -homomorphism $\phi : D \rightarrow A/I$, there exists $\tilde{\phi}$ such that the diagram

$$\begin{array}{ccc} & & A \\ & \nearrow \tilde{\phi} & \downarrow \\ D & \xrightarrow{\phi} & A/I \end{array}$$

commutes.

A C^* -algebra D is *semiprojective* if for any C^* -algebra A , any increasing chain of ideals I_n in A and for every $*$ -homomorphism $\phi : D \rightarrow A/I$, where $I = \overline{\bigcup I_n}$,

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there exist n and $*$ -homomorphism $\tilde{\phi} : D \rightarrow A/I_n$ such that the diagram

$$\begin{array}{ccc} & A/I_n & \\ \tilde{\phi} \nearrow & \downarrow & \\ D & \xrightarrow{\phi} & A/I \end{array}$$

commutes.

A C^* -algebra D is *weakly semiprojective* if for any sequence of C^* -algebras A_i and any $*$ -homomorphism $\phi : D \rightarrow \prod A_i / \bigoplus A_i$, there exists a $*$ -homomorphism $\tilde{\phi} : D \rightarrow \prod A_i$ such that the diagram

$$\begin{array}{ccc} & \prod A_i & \\ \tilde{\phi} \nearrow & \downarrow & \\ D & \xrightarrow{\phi} & \prod A_i / \bigoplus A_i \end{array}$$

commutes.

These notions provide an algebraic setting of lifting and stability questions: a relation is liftable (stable, weakly stable) if and only if its universal C^* -algebra is projective (semiprojective, weakly semiprojective), [2], [5]. Of course it is assumed that the universal C^* -algebra of a relation exists. Note that in general a relation need not have a universal C^* -algebra, but if it includes restrictions $\|x_i\| \leq c_i$, for all generators, then the universal C^* -algebra exists (of course if the relation has at least one representation in a Hilbert space).

Thus given a liftable or (weakly) stable relation, there arises a natural question if being combined with restrictions on norms of generators, it is still liftable or (weakly) stable: the positive answer would produce new examples of projective or (weakly) semiprojective C^* -algebras.

It was proved in [8] that each nilpotent is liftable and in [5], [6] the question was asked if a relation

$$x^n = 0, \quad \|x\| \leq 1,$$

is liftable or, in other words, if it defines a projective C^* -algebra. In [13] this question was answered positively and the main tool was to use the technique of M -ideals in Banach spaces.

In [4] it was proved that for each polynomial p in one variable, a relation $p(x) = 0$ is weakly stable (though the proof given in [4] easily implied stability also), but again it remained unknown if the universal C^* -algebra of a relation

$$p(x) = 0, \quad \|x\| \leq 1,$$

is semiprojective. In the present paper, using the approach of [13], we prove this in the case when all the roots of the polynomial have multiplicity more than 1.

For such polynomials we also prove that the corresponding universal C^* -algebras are residually finite-dimensional (have separating families of finite-dimensional representations). Note that this in no way means that these algebras are small. For polynomials of degree more than two they are all non-exact.

One more result (Theorem 11) concerns well-known questions of C. Olsen. Let T be a bounded operator on a Hilbert space and let $p(x)$ be a polynomial in one variable. Olsen's questions are:

1) Does there exist a compact operator K such that $\|p(T + K)\| = \|p(T)\|_e$? (The essential norm of T is denoted by $\|T\|_e$.)

2) Can this K be chosen such that it is common for all polynomials?

The questions are still open, but there are partial results ([1], [9], [10], [11], [3], [14]). In particular Olsen proved ([9]) that under the assumption that $p(T)$ is compact, such K always exists. In this paper we prove that under the same assumption, K can be chosen common for polynomials $p(x)$ and x in the case when all roots of the polynomial have multiplicity more than 1. In other words, we prove (Theorem 11) that for such polynomials, if $p(T)$ is compact, then there exists a compact operator K such that

$$p(T + K) = 0, \quad \|T + K\| = \|T\|_e.$$

1. PRELIMINARY RESULTS AND CONSTRUCTIONS

Below we use the following notation.

By $M(I)$ we denote the multiplier C^* -algebra of a C^* -algebra I .

For a closed ideal I of a C^* -algebra A , we denote by $a \rightarrow \dot{a}$ the standard epimorphism from A to A/I . We say that a is a *lift* of b if $\dot{a} = b$.

Given ideals $I_1 \subseteq I_2 \subseteq \dots \subseteq I \subseteq A$ with $I = \overline{\bigcup I_n}$, we use the notation

$$q_n : A \rightarrow A/I_n$$

and

$$q_{n,\infty} : A/I_n \rightarrow A/I$$

for the induced epimorphisms.

For an arbitrary algebra A , an operator $T : A \rightarrow A$ is called an *elementary* operator if it is of the form $T = \sum_{i=1}^N L_{a_i} R_{b_i}$, where $a_i, b_i \in A^+$. By L_{a_i}, R_{b_i} we denote operators of left and right multiplication by a_i and b_i respectively.

A subspace Y of a Banach space X is called *proximal* if for any $x \in X$ there exists $y \in Y$ such that $\|x - y\| = d(x, Y)$, or, equivalently, if for any $z \in X/Y$ there is a lift x of z such that $\|x\| = \|z\|$.

Below for any C^* -algebra A we denote the set of its selfadjoint elements by A_{sa} .

Theorem 1 ([13]). *Let A be a C^* -algebra, let T be an elementary operator on A . Then, for each ideal I of A , $\overline{TI}$ is proximal in $\overline{TA}$ and $\overline{TI_{sa}}$ is proximal in $\overline{TA_{sa}}$.*

Remark 2. The case of $\overline{TI_{sa}}$ was not considered in [13], but the proof is the same if one considers C^* -algebras as Banach spaces over $\mathbb{R}$.

Theorem 1 implies a corollary which was contained in indirect form in the proof of Theorem 5 of [13]:

Corollary 3. *Let A be a C^* -algebra, I its ideal, and T an elementary operator on A , $b \in A/I$. If, for some lift a_0 of b , an element Ta_0 is also a lift of b , then the set $\overline{\{Ta \mid \dot{a} = b\}}$ contains an element of norm equal to $\|b\|$.*

Proof. Let $E = \{Ta \mid \dot{a} = b\}$. Then

$$E = \{Ta_0 + Ti \mid i \in I\}, \quad \overline{E} = \{Ta_0 + x \mid x \in \overline{TI}\}.$$

All elements of $\overline{E}$ are lifts of b . Indeed let $\dot{a} = b$. Then $a = a_0 + i$, for some $i \in I$. Since T is elementary, it preserves ideals, whence $(Ti)^\cdot = 0$ and $(Ta)^\cdot = (Ta_0)^\cdot = b$. This easily implies that $\overline{E}$ also consists of lifts of b .

Let $b_i, c_i, i = 1, \dots, N$, be elements of A^+ such that $A = \sum_{i=1}^N L_{b_i} R_{c_i}$ and let B be the C^* -algebra generated by $b_i, i = 1, \dots, N$, and I . Let us choose a quasicontral approximate unit u_λ for the ideal I of B . Then, for each $a \in E$, we have $T((1 - u_\lambda)a) \in E$ and

$$\lim \|T((1 - u_\lambda)a)\| = \lim \|(1 - u_\lambda)Ta\| = \|(Ta)^\cdot\| = \|b\|$$

(we used here a well-known fact from [7, Section 3.1.3] that if e_λ is any approximate unit, then

$$\lim_\lambda \|x(1 - e_\lambda)\| = \|\dot{x}\|,$$

for any $x \in A$). Hence $\inf_{y \in \overline{TI}} \|Ta_0 + y\| \leq \|b\|$. Since the norm of any lift of b is not less than $\|b\|$, we get $\inf_{y \in \overline{TI}} \|Ta_0 + y\| = \|b\|$.

Thus $\text{dist}(Ta_0, \overline{TI}) = \inf_{y \in \overline{TI}} \|Ta_0 + y\| = \|b\|$ and, by Theorem 1, there is $y_0 \in \overline{TI}$ such that $\|Ta_0 + y_0\| = \|b\|$. Since $Ta_0 + y_0 \in \overline{E}$, we are done. $\square$

Lemma 4. $\overline{\dot{a}(A/I)\dot{a}} = \overline{aAa}/\overline{aIa}$.

Proof. Define a map $f : aAa \rightarrow \dot{a}(A/I)\dot{a}$ by $f(aba) = \dot{a}b\dot{a}$ and extend it by continuity to $\tilde{f} : \overline{aAa} \rightarrow \overline{\dot{a}(A/I)\dot{a}}$. Since $\tilde{f}|_{\overline{aIa}} = 0$, there is a well-defined map $F : \overline{aAa}/\overline{aIa} \rightarrow \overline{\dot{a}(A/I)\dot{a}}$. This is surjection because its image is closed (as the image of any $*$ -homomorphism between C^* -algebras) and contains $\dot{a}(A/I)\dot{a}$.

Suppose that $x \in \overline{aAa}/\overline{aIa}$ and $F(x) = 0$. Let $X \in \overline{aAa}$ be any preimage of x . Then there are elements $b_n \in A$ such that $X = \lim ab_n a$. Then $\lim \dot{a}b_n \dot{a} = 0$ and there are $i_n \in I$ such that $ab_n a - i_n \rightarrow 0$. Hence $X \in I$. Let $\{u_\lambda\}$ be a quasicontral approximate unit for $I \subseteq A$. Then for any ϵ , there exists n such that $\|X - ab_n a\| \leq \epsilon$ and there exists λ such that $\|X - Xu_\lambda\| \leq \epsilon$, $\|ab_n au_\lambda - ab_n u_\lambda a\| \leq \epsilon$. Then $\|X - ab_n u_\lambda a\| \leq 3\epsilon$, $\text{dist}(X, aIa) \leq 3\epsilon$; whence $X \in \overline{aIa}$. $\square$

A C^* -algebra A is called SAW* ([12]) if for any $a, b \in A_{sa}$, $ab = 0$, there exists $0 \leq z \leq 1$, $z \in A^+$, such that $az = a$, $zb = 0$.

Lemma 5. If A is SAW* and $p \in A$ is a projection, then pAp is SAW*.

Proof. Suppose $a, b \in (pAp)_{sa}$, $ab = 0$. Since A is SAW*, there exists $0 \leq z \leq 1$, $z \in A^+$, such that $az = a$, $zb = 0$. Clearly $0 \leq pzp \leq 1$, $pzp \in (pAp)^+$, $apzp = a$, $pzpb = 0$. $\square$

Below we write $d \ll e$ for selfadjoint elements satisfying $de = ed = d$.

Theorem 6 ([8]). Let A be a C^* -algebra and I its ideal such that the quotient A/I is SAW*. Let $b \in A/I$ be a nilpotent of order n , $n \geq 2$. Then there are elements

$$(1) \quad 0 \leq d_{n-1} \ll e_{n-1} \ll d_{n-2} \ll e_{n-2} \ll \dots \ll e_2 \ll d_1 \ll e_1 \leq e_0 = 1$$

and their lifts

$$(2) \quad 0 \leq D_{n-1} \ll E_{n-1} \ll D_{n-2} \ll E_{n-2} \ll \dots \ll E_2 \ll D_1 \ll E_1 \leq E_0 = 1$$

such that

- (i) $b = \sum_{j=1}^{n-1} (e_{j-1} - e_j)bd_j$,
- (ii) for any lift B of b , $\sum_{j=1}^{n-1} (E_{j-1} - E_j)BD_j$ is a nilpotent lift of b of order n .

Now let an element b of a C^* -algebra be algebraic with the minimal polynomial $p(t) = (t - \lambda_1)^{k_1} \dots (t - \lambda_n)^{k_n}$. In [4] a family of n pairwise orthogonal projections was associated with b in the following way. Let e_i be the spectral idempotents for b and let $s = \sum e_i^* e_i$. One can prove that $e_i(b - \lambda_i)^{k_i} e_i = 0$ for each i , s is invertible and

$$(3) \quad p_i = s^{1/2} e_i s^{-1/2}$$

are projections with sum 1. Let $c = s^{1/2} b s^{-1/2}$. Then

$$(4) \quad p_i(c - \lambda_i)^{k_i} p_i = 0,$$

$$(5) \quad \sum_{i=1}^n p_i c p_i = c.$$

Theorem 7 ([4]). *The following are equivalent:*

- (i) $b \in A/I$ has a lift $\tilde{b} \in A$ with $p(\tilde{b}) = 0$;
- (ii) the family $\{p_i\}$ can be lifted to a family of projections in A with sum 1.

Since in what follows we use ideas from [4], we now sketch arguments from [4] of how (ii) implies (i). Let a family of projections $\{Q_i\}$ with $\sum Q_i = 1$ be a lift of $\{p_i\}$. Since $p_i(c - \lambda_i)p_i$ is a nilpotent of order k_i in a C^* -algebra $p_i A / I p_i = Q_i A Q_i / Q_i I Q_i$, by [8] it has a nilpotent lift $t_i \in Q_i A Q_i$ of order k_i . Now one can check that $\sum t_i + \lambda_i Q_i$ is a lift of c and $p(\sum t_i + \lambda_i Q_i) = 0$. As a lift of b we can take $S^{-1/2}(\sum t_i + \lambda_i Q_i)S^{1/2}$, where S is any invertible positive lift of s (to find such S we can take any selfadjoint lift of s and then use functional calculus).

2. MAIN RESULTS

Lemma 8. *Let $p(t) = (t - \lambda_1)^{k_1} \dots (t - \lambda_n)^{k_n}$ be a polynomial such that $k_i \geq 2$, $i = 1, \dots, n$. Let A be a C^* -algebra and I its ideal such that A/I is SAW*. Suppose that $b \in A/I$, $p(b) = 0$, $\|b\| = 1$, and a family of projections $\{p_i\}$ associated with b by (3) can be lifted to a family of projections in A with sum 1. Then there is a lift $\tilde{b} \in A$ of b such that $p(\tilde{b}) = 0$, $\|\tilde{b}\| = 1$.*

Proof. Let s, c be as above and let projections $\{Q_i\}$ in A be lifts of $\{p_i\}$ with sum 1. By (4), $p_i(c - \lambda_i)p_i$ is a nilpotent of order k_i in the C^* -algebra $p_i(A/I)p_i$. By Lemma 4, $p_i(A/I)p_i = Q_i A Q_i / Q_i I Q_i$. It follows from Lemma 5 and Theorem 6 that there are $e_j^{(i)}, d_j^{(i)} \in p_i(A/I)p_i$ and their lifts $E_j^{(i)}, D_j^{(i)} \in Q_i A Q_i$ satisfying (1) and (2) such that

$$p_i(c - \lambda_i)p_i = \sum_{j=1}^{k_i-1} (e_{j-1}^{(i)} - e_j^{(i)})(c - \lambda_i)d_j^{(i)}$$

and, for any lift C of c ,

$$\sum_{j=1}^{k_i-1} (E_{j-1}^{(i)} - E_j^{(i)})(C - \lambda_i)D_j^{(i)}$$

is a nilpotent lift of $p_i(c - \lambda_i)p_i$ of order k_i . Let $q : A \rightarrow A/I$ be the canonical surjection. We have

$$\begin{aligned} q \left(\sum_{i=1}^n \sum_{j=1}^{k_i-1} \left((E_{j-1}^{(i)} - E_j^{(i)})(C - \lambda_i)D_j^{(i)} + \lambda_i Q_i \right) \right) \\ = \sum_{i=1}^n \sum_{j=1}^{k_i-1} \left((e_{j-1}^{(i)} - e_j^{(i)})(c - \lambda_i)d_j^{(i)} + \lambda_i p_i \right) \\ = \sum_{i=1}^n p_i(c - \lambda_i)p_i + \lambda_i p_i = \sum_{i=1}^n p_i c p_i = c. \end{aligned}$$

Denote $\sum_{j=1}^{k_i-1} (E_{j-1}^{(i)} - E_j^{(i)})(C - \lambda_i)D_j^{(i)}$ by X_i . Since $X_i^{k_i} = 0$, $X_i Q_j = Q_j X_i = 0$ if $i \neq j$, and $\sum Q_i = 1$, we have

$$\begin{aligned} p \left(\sum_{i=1}^n \sum_{j=1}^{k_i-1} \left((E_{j-1}^{(i)} - E_j^{(i)})(C - \lambda_i)D_j^{(i)} + \lambda_i Q_i \right) \right) \\ = \prod_{l=1}^n \left(\sum_{i=1}^n (X_i + \lambda_i Q_i) - \lambda_l \right)^{k_l} = \prod_{l=1}^n \left(\sum_{i=1}^n (X_i + (\lambda_i - \lambda_l) Q_i) \right)^{k_l} \\ = \prod_{l=1}^n \left(X_l + \sum_{i \neq l} (X_i + \lambda_i - \lambda_l) Q_i \right)^{k_l} = \prod_{l=1}^n \left(\sum_{i \neq l} (X_i + \lambda_i - \lambda_l) Q_i \right)^{k_l} = 0. \end{aligned}$$

It follows from (2) that $\sum_{j=1}^{k_i-1} (E_{j-1}^{(i)} - E_j^{(i)})D_j^{(i)} = Q_i$, whence

$$\sum_{i=1}^n \sum_{j=1}^{k_i-1} \left((E_{j-1}^{(i)} - E_j^{(i)})(C - \lambda_i)D_j^{(i)} + \lambda_i Q_i \right) = \sum_{i=1}^n \sum_{j=1}^{k_i-1} (E_{j-1}^{(i)} - E_j^{(i)})C D_j^{(i)}.$$

Let $S \in A$ be any invertible positive lift of s . Since $b = s^{-1/2} c s^{1/2}$, the set

$$\mathcal{E} = \left\{ \sum_{i=1}^n \sum_{j=1}^{k_i-1} S^{-1/2} (E_{j-1}^{(i)} - E_j^{(i)}) S^{1/2} X S^{-1/2} D_j^{(i)} S^{1/2} \mid X \in A, q(X) = b \right\}$$

consists of lifts of b and we have $p(X) = 0$, for any $X \in \mathcal{E}$. Then $\overline{\mathcal{E}}$ also consists of lifts of b and $p(X) = 0$, for any $X \in \overline{\mathcal{E}}$.

Define an elementary operator $T : A \rightarrow A$ by

$$T(X) = \sum_{i=1}^n \sum_{j=1}^{k_i-1} S^{-1/2} (E_{j-1}^{(i)} - E_j^{(i)}) S^{1/2} X S^{-1/2} D_j^{(i)} S^{1/2}.$$

Then $\mathcal{E} = \{TX \mid X \in A, q(X) = b\}$. By Corollary 3, in $\overline{\mathcal{E}}$ there is an element of norm 1. $\square$

Theorem 9. Let $p(t) = (t - \lambda_1)^{k_1} \dots (t - \lambda_n)^{k_n}$ be a polynomial such that $k_i \geq 2$, $i = 1, \dots, n$. The universal C^* -algebra of a relation $p(b) = 0, \|b\| \leq 1$, is semiprojective.

Proof. By Theorem 14.1.7 of [5], it suffices to prove that if $b \in M(I)/I$, where $I = \bigcup I_n$, $I_1 \subseteq I_2 \subseteq \dots$, $p(b) = 0$, $\|b\| \leq 1$, then there exist N and $\tilde{b} \in M(I)/I_N$ such that $q_{n,\infty}(\tilde{b}) = b$, $p(\tilde{b}) = 0$, $\|\tilde{b}\| \leq 1$. Here by $q_{n,\infty}$ we denote the canonical surjection from $M(I)/I_N$ onto $M(I)/I$. Clearly we can assume $\|b\| = 1$ and look for $\tilde{b}$ with $\|\tilde{b}\| = 1$.

By [2], there exists N such that $\{p_i\}$ can be lifted to a family $\{Q_i\}$ of projections in $M(I)/I_N$ with sum 1. Denote $M(I)/I$ by A and I/I_N by J . Then $A/J \cong M(I)/I$. By Kasparov's Technical Theorem, any corona-algebra is SAW^* , so the assertion follows now from Lemma 8. $\square$

Recall that a C^* -algebra is residually finite-dimensional (RFD) if the intersection of kernels of its finite-dimensional representations is zero.

Theorem 10. *Let $p(t) = (t - \lambda_1)^{k_1} \dots (t - \lambda_n)^{k_n}$ be a polynomial such that $k_i \geq 2$, $i = 1, \dots, n$. The universal C^* -algebra of a relation $p(b) = 0$, $\|b\| \leq 1$, is residually finite-dimensional.*

Proof. Let H be $\ell^2(\mathbb{N})$. We will identify M_n with $B(\ell^2\{1, \dots, n\}) \subseteq B(H)$. Let $\mathcal{A} \subseteq \prod M_n$ be a C^* -algebra of all sequences $\{x_n\}$ such that their $*$ -strong limit x_n exists and let $\mathcal{I}$ be an ideal of all sequences $\{x_n\}$ such that their $*$ -strong limit x_n is zero. Then a map $i : \mathcal{A}/\mathcal{I} \rightarrow B(H)$ sending each sequence $\{x_n\}$ to its $*$ -strong limit is an isomorphism. The algebra $B(H)$ is SAW^* . Indeed if $T, S \in B(H)$, $TS = 0$, then for Z equal to the projection on $(\text{Ran } S)^\perp$ one has $TZ = T$, $ZS = 0$. Thus $\mathcal{A}/\mathcal{I}$ is SAW^* . Let $p_1, \dots, p_n$ be a family of projections in $\mathcal{A}/\mathcal{I}$ with sum 1. Let H_j be a range of projection $i(p_j)$ and let $\{e_i^{(j)}\}_{i \in \mathbb{N}}$ be an orthonormal basis in H_j . Then $e_1^{(1)}, \dots, e_1^{(n)}, e_2^{(1)}, \dots, e_2^{(n)}, \dots$ is an orthonormal basis in H . Let R_k be a projection on the first k basis vectors. Then sequences $\{R_k P_j R_k\}_{k \in \mathbb{N}} \in \mathcal{A}$ are projections with sum 1 and are lifts of P_j . By Lemma 8 each $*$ -homomorphism from the universal C^* -algebra of a relation $p(b) = 0$, $\|b\| \leq 1$, to $\mathcal{A}/\mathcal{I}$ lifts to a $*$ -homomorphism to $\mathcal{A}$. Let π be the universal representation of the universal C^* -algebra of a relation $p(b) = 0$, $\|b\| \leq 1$. Then any lift of $i^{-1} \circ \pi$ has to be injective and hence gives a separating family of finite-dimensional representations. $\square$

Theorem 11. *Let $T \in B(H)$ and let $p(t) = (t - \lambda_1)^{k_1} \dots (t - \lambda_n)^{k_n}$ be a polynomial such that $k_i \geq 2$, $i = 1, \dots, n$. Suppose that $p(T)$ is compact. Then there is a compact operator K such that $p(T + K) = 0$, $\|T + K\| = \|T\|_e$.*

Proof. Let b be the image of T in the Calkin algebra $B(H)/K(H)$, $C = \|T\|_e$. Then $p(b) = 0$, $\|b\| = C$. By Kasparov's Technical Theorem, Calkin algebra is SAW^* , and the assertion follows now from Lemma 8 and the well-known fact that orthogonal projections (with sum 1) in Calkin algebra can be lifted to orthogonal projections (with sum 1) in $B(H)$. $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN, DENMARK

E-mail address: shulman@math.ku.dk, tatiana_shulman@yahoo.com