RANDOM LENGTH-SPECTRUM RIGIDITY FOR FREE GROUPS

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ABSTRACT. We say that a subset $S \subseteq F_N$ is *spectrally rigid* if whenever $T_1, T_2 \in \text{cv}_N$ are points of the (unprojectivized) outer space such that $\|g\|_{T_1} = \|g\|_{T_2}$ for every $g \in S$, then $T_1 = T_2$ in cv_N . It is well known that F_N itself is spectrally rigid; it also follows from the result of Smillie and Vogtmann that there does not exist a finite spectrally rigid subset of F_N . We prove that if A is a free basis of F_N (where $N \geq 2$), then almost every trajectory of a non-backtracking simple random walk on F_N with respect to A is a spectrally rigid subset of F_N .

1. INTRODUCTION

The phenomenon of “marked length spectrum rigidity” plays an important role in the study of negative curvature and its generalizations. Let M be a closed connected manifold and $G = \pi_1(M)$. A Riemannian metric ρ on M of strictly negative (but not necessarily constant) curvature defines a length function $\|\cdot\|_\rho : G \rightarrow \mathbb{R}$, where for $g \in G$, $g \neq 1$, $\|g\|_\rho$ is the length of the unique closed geodesic with respect to ρ representing the free homotopy class $[g]$. One also sets $\|1_G\|_\rho = 0$. Note that for $g \in G$ the value $\|g\|_\rho$ is equal to the translation length of g as an isometry of the universal cover $(\widetilde{M}, \widetilde{\rho})$. Note also that $\|g\|_\rho = \|g^{-1}\|_\rho = \|h^{-1}gh\|_\rho$, so that the function $\|\cdot\|_\rho$ is constant on conjugacy classes in G . The function $\|\cdot\|_\rho$ is usually called the *marked length spectrum* of ρ and the image of this function (which is a subset of $\mathbb{R}_{\geq 0}$) is called the *length spectrum* of ρ . The *marked length spectrum rigidity conjecture* states that knowing the function $\|\cdot\|_\rho : G \rightarrow \mathbb{R}$ determines the isometry type of (M, ρ) . There are many variations and generalizations of this conjecture, including representation-theoretic and nonpositive curvature versions. The conjecture has been proved in a number of important cases (e.g. [9, 10, 11, 15, 17, 29]; see a survey of Spatzier [32] for more details), but remains open in general.

For closed hyperbolic surfaces the situation is particularly nice and well-understood. Thus it is known that if S_g is a closed oriented surface of genus $g \geq 2$, then there exist elements $h_1, \dots, h_{6g-5} \in G = \pi_1(S_g)$ such that whenever ρ_1, ρ_2 are two points in the Teichmüller space $\mathcal{T}(S_g)$ (i.e. marked hyperbolic metrics on S_g) such that $\|h_i\|_{\rho_1} = \|h_i\|_{\rho_2}$ for $i = 1, \dots, 6g-5$, then $\rho_1 = \rho_2$ in $\mathcal{T}(S_g)$. Thus already knowing that $\|\cdot\|_{\rho_1}$ and $\|\cdot\|_{\rho_2}$ agree on the finite set $\{h_1, \dots, h_{6g-5}\}$ implies that $\rho_1 = \rho_2$ in $\mathcal{T}(S_g)$. Therefore we may call the subset $\{h_1, \dots, h_{6g-5}\} \subseteq G$

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“spectrally rigid”. The notion of a spectrally rigid set makes sense in any context where full marked length spectrum rigidity is already known. We think that looking for “small” spectrally rigid sets in such cases is an interesting general problem representing the next level in the study of length spectrum rigidity.

If G is a group acting by isometries on an $\mathbb{R}$ -tree T , then, under some mild assumptions (that are satisfied, in particular, in the outer space context discussed below), it is well known that the translation length function $||\cdot||_T : G \rightarrow \mathbb{R}$ determines the action of G on T up to a G -equivariant isometry (see [30]). However, it turns out that, unlike for the case of hyperbolic surfaces, this type of rigidity falls apart even for very nice actions if we restrict ourselves to finite subsets of G . For a free group F_N (where $N \geq 2$) the *Culler-Vogtmann outer space* cv_N is an analog of the Teichmüller space of a hyperbolic surface. The space cv_N consists of minimal free discrete isometric actions of F_N on $\mathbb{R}$ -trees, considered up to F_N -equivariant isometry. Every element $T \in cv_N$ arises as the universal cover of a finite graph Γ , whose fundamental group is identified with F_N via a particular isomorphism, where the edges of Γ are given positive real lengths and their lifts to T are given the same lengths. There is an important subset, $CV_N \subseteq cv_N$, consisting of all $T \in cv_N$ such that the quotient metric graph T/F_N has volume 1. The space CV_N is the *projectivized Culler-Vogtmann outer space*, which in fact was first introduced in [12]. Both cv_N and CV_N play an important role in the study of $Out(F_N)$. We say that a subset $R \subseteq F_N$ is *spectrally rigid* if whenever $T_1, T_2 \in cv_N$ are such that $||g||_{T_1} = ||g||_{T_2}$ for every $g \in R$, then $T_1 = T_2$ in cv_N . As noted above, $R = F_N$ is spectrally rigid and in fact it is enough to make R consist of representatives of all conjugacy classes in F_N . A surprising result of Smillie and Vogtmann [31] shows that there does not exist a finite spectrally rigid subset of F_N , where $N \geq 3$. They prove, in particular, that for any finite subset $R \subseteq F_N$ there exists a one-parameter family $(T_t)_{t \in [0,1]}$ of distinct points of CV_N such that for every $t \in [0,1]$ the length functions $||\cdot||_{T_0}$ and $||\cdot||_{T_t}$ agree on R . A similar statement follows from the recent work of Duchin, Leininger, and Rafi for flat metrics on surfaces of finite type [13]. Moreover, the paper [13] gives a complete characterization of when a set of simple closed curves on a finite type surface is spectrally rigid with respect to the space of flat metrics on that surface.

As noted above, we believe that for any context where full length spectrum rigidity is known it becomes interesting to look for “small” (in some reasonable sense) spectrally rigid subsets of the group G in question and to try to characterize which subsets of G are spectrally rigid. Apart from the above-mentioned paper of Duchin, Leininger, and Rafi, we are not aware of any other results in this direction. In the present paper, as well as in our joint work with Carette [5], we study infinite but “sparse” subsets of F_N that are spectrally rigid with respect to the collection of length functions on F_N coming from cv_N .

Let $N \geq 2$ and let A be a free basis of F_N . Consider a simple nonbacktracking random walk on F_N corresponding to A , starting at $1 \in F_N$. Thus a trajectory of this walk is a semi-infinite freely reduced word $\xi = x_1 x_2 \dots x_n \dots$, where $x_i \in A^{\pm 1}$. For $n \geq 1$ denote $\xi(n) = x_1 x_2 \dots x_n \in F_N$, the initial segment of ξ of length n .

We prove the following:

Theorem 1.1. *Let $N \geq 2$ and let A be a free basis of F_N . For a.e. trajectory ξ of the nonbacktracking simple random walk on F_N with respect to A the following holds:*

For every $n_0 \geq 1$ the subset

$$\{\xi(n) : n \geq n_0\} \subseteq F_N$$

is spectrally rigid.

It is known (cf. [18]) that for a.e. trajectory ξ of the walk above we have

$$\lim_{n \rightarrow \infty} \frac{\|\xi(n)\|_A}{n} = 1.$$

Here for $g \in F_N$, $\|g\|_A$ is the cyclically reduced length of g over A . Thus $\|\xi(n)\|_A = n + o(n)$, which shows that the set $\{\xi(n) : n \geq n_0\}$ in Theorem 1.1 represents a “sparse” collection of conjugacy classes in F_N .

The proof of Theorem 1.1 relies on the machinery of *geodesic currents* on free groups (see Section 2 for the definition and basic properties) and particularly the notion of a “geometric intersection form” between trees and currents, developed in the work of the author and Martin Lustig [20, 23, 24]. The presence of $\mathbb{R}_{\geq 0}$ -linear structure on the space of currents turns out to be particularly useful. Let us point out the new ideas introduced in this paper. In the earlier work [20, 23] we have constructed a “length pairing” or a “geometric intersection form”

$$\langle \cdot, \cdot \rangle : \text{cv}_N \times \text{Curr}(F_N) \rightarrow \mathbb{R}_{\geq 0},$$

where $\text{Curr}(F_N)$ is the space of geodesic currents on F_N (see Section 2 below for details). This geometric intersection form turns out to have a number of nice properties, one of which is being $\mathbb{R}_{\geq 0}$ -linear in the second argument. The elements of $\text{Curr}(F_N)$, that is, geodesic currents, are traditionally defined as *positive* Borel measures on $\partial^2 F_N = \partial F_N \times \partial F_N - \text{diag}$ satisfying some additional properties, namely being F_N -invariant, flip-invariant and finite on compact sets. It turns out that for the proof of Theorem 1.1 we need to extend the geometric intersection form to the space of *signed* geodesic currents, which satisfy all the axioms of the standard currents, except that as measures they are no longer required to be positive and are allowed to take arbitrary (but finite) real values on compact subsets. The extended intersection form now takes values in $\mathbb{R}$ and enjoys $\mathbb{R}$ -linearity in the second argument, which is the space $\mathcal{SCurr}(F_N)$ of all signed currents on F_N . Crucially, the $\mathbb{R}$ -linear span of the standard rational currents is dense in $\mathcal{SCurr}(F_N)$ (see Proposition 4.5 below). It is well known and fairly easy to see [18] that for a random trajectory ξ as in Theorem 1.1 we have

$$\lim_{n \rightarrow \infty} \frac{\eta_{\xi(n)}}{n} = \nu_A,$$

where $\eta_{\xi(n)}$ is the rational current corresponding to $\xi(n)$ and where ν_A is the *uniform current* on F_N . Thus, projectively, the set $\{\xi(n) : n \geq n_0\}$ (viewed as a subset of $\text{Curr}(F_N)$) in Theorem 1.1 is rather small: after rescaling, this set represents a sequence converging to a single point of $\text{Curr}(F_N)$. A key step in the proof of Theorem 1.1 is Proposition 5.1, which says that the $\mathbb{R}$ -linear span of the set $\{\eta_{\xi(n)} : n \geq n_0\}$ is dense in $\mathcal{SCurr}(F_N)$. Informally, Proposition 5.1 says that, although the sequence $\frac{\eta_{\xi(n)}}{n}$ converges to a single point in $\mathcal{SCurr}(F_N)$ (namely ν_A), it converges to that point “from all possible directions”. Once Proposition 5.1 is established, it is not hard to derive Theorem 1.1 using the basic properties of the geometric intersection form.

Remark 1.2. In an ongoing joint work with Mathieu Carrière [5] we prove that the set of primitive elements in a free group F_N (where $N \geq 2$) is spectrally rigid, and that, moreover, for $N \geq 3$, every $\text{Aut}(F_N)$ -orbit of a nontrivial element of F_N is spectrally rigid as well. The proofs there require substantially different methods, as it is not hard to show (by using Whitehead graph properties of primitive elements) that, unlike for the spectrally rigid sets constructed in the present paper, the linear span of the set of rational currents of primitive elements is not dense in the space of signed geodesic currents. The result of [5] raises a natural question of when for a subgroup $H \leq \text{Aut}(F_N)$ and an element $f \in F_N$ the orbit $\text{Aut}(F_N)f$ is spectrally rigid.

It turns out that for the case where H is cyclic generated by an atoroidal fully irreducible automorphism, an H -orbit is never spectrally rigid. As noted above, Smillie and Vogtmann proved [31] that for any finite set $R \subseteq F_N$ (where $N \geq 3$), the set R is not spectrally rigid. In fact, the proof of Theorem 1 in [31] goes through verbatim and shows that any (possibly infinite) subset $R \subseteq F_N$ (again where $N \geq 3$) is not spectrally rigid provided it satisfies the following “weak aperiodicity” property: there is a free basis A of F_N , an element $a \in A$ and an integer $M \geq 1$ such that whenever a^k is a subword of a cyclically reduced word over A representing an element conjugate to some $g \in R$, then $|k| \leq M$. Let $N \geq 3$ and let $\varphi \in \text{Aut}(F_N)$ be atoroidal and fully irreducible (also known as “iwip”, which stands for irreducible with irreducible powers). Proposition 1.18 in [3], together with basic train-track properties, then implies that for any nontrivial $g \in F_N$ the set $\{\varphi^n(g) : n \in \mathbb{Z}\}$ satisfies the “weak aperiodicity” property. Hence the set $\{\varphi^n(g) : n \in \mathbb{Z}\}$ is not spectrally rigid in F_N .

2. OUTER SPACE AND THE SPACE OF GEODESIC CURRENTS

We give here only a brief overview of basic facts related to outer space and the space of geodesic currents. We refer the reader to [28, 19, 20] for more detailed background information about geodesic currents. Other useful references regarding currents and their applications are [22, 26, 23, 24, 14, 4, 16, 6, 25, 8].

2.1. Outer space. Let $N \geq 2$. The *unprojectivized outer space* cv_N consists of all minimal free and discrete isometric actions on F_N on $\mathbb{R}$ -trees (where two such actions are considered equal if there exists an F_N -equivariant isometry between the corresponding trees). There are several different topologies on cv_N that are known to coincide, in particular the equivariant Gromov-Hausdorff convergence topology and the so-called *axis* or *length function* topology. Every $T \in \text{cv}_N$ is uniquely determined by its *translation length function* $\|\cdot\|_T : F_N \rightarrow \mathbb{R}$, where $\|g\|_T$ is the translation length of g on T . Two trees $T_1, T_2 \in \text{cv}_N$ are close if the functions $\|\cdot\|_{T_1}$ and $\|\cdot\|_{T_2}$ are close pointwise on a large ball in F_N . The closure $\overline{\text{cv}}_N$ of cv_N in either of these two topologies is well-understood and known to consist precisely of all the so-called *very small* minimal isometric actions of F_N on $\mathbb{R}$ -trees; see [2] and [7].

A *simplicial chart* or *marking* on F_N is an isomorphism $\alpha : F_N \rightarrow \pi_1(\Gamma)$ where Γ is a finite connected graph without degree-one and degree-two vertices. We think of Γ as an oriented graph, where every edge e comes equipped with an inverse edge $e^{-1} \neq e$ such that $(e^{-1})^{-1} = e$ and such that the initial vertex of e is the terminal vertex of $\bar{e}$ and vice versa. An *orientation* on Γ is a partition $E\Gamma = E^+\Gamma \sqcup E^-\Gamma$ such that for every $e \in E\Gamma$ exactly one of $e, \bar{e}$ belongs to $E^+\Gamma$. A *metric graph structure*

on Γ is a function $\mathcal{L} : E\Gamma \rightarrow \mathbb{R}_{>0}$ such that $\mathcal{L}(e) = \mathcal{L}(e^{-1})$ for every $e \in E\Gamma$. A *marked metric graph* structure on F_N is a pair $(\alpha, \mathcal{L})$, where $\alpha : F_N \rightarrow \pi_1(\Gamma)$ is a marking and where $\mathcal{L}$ is a metric graph structure on Γ . Every marked metric graph structure $(\alpha, \mathcal{L})$ determines a point of cv_N obtained by taking the universal covering (which is a topological tree) $\tilde{\Gamma}$ of Γ and lifting the metric structure $\mathcal{L}$ to $\tilde{\Gamma}$. Moreover, every point of cv_N arises in this way. For a fixed chart α , varying the metric structure $\mathcal{L}$ on Γ determines an open cone inside cv_N .

2.2. Geodesic currents. Let $\partial^2 F_N := \{(x, y) | x, y \in \partial F_N, x \neq y\}$. The action of F_N by translations on its hyperbolic boundary ∂F_N defines a natural diagonal action of F_N on $\partial^2 F_N$. A *geodesic current* on F_N is a positive Borel measure on $\partial^2 F_N$, finite on compact subsets, which is F_N -invariant and is also invariant under the “flip” map $\partial^2 F_N \rightarrow \partial^2 F_N, (x, y) \mapsto (y, x)$. The space $\text{Curr}(F_N)$ of all geodesic currents on F_N has a natural $\mathbb{R}_{\geq 0}$ -linear structure and is equipped with the weak*-topology of pointwise convergence on continuous functions. Every point $T \in \text{cv}_N$ defines a *simplicial chart* on $\text{Curr}(F_N)$ which allows one to think about geodesic currents as systems of nonnegative weights satisfying certain Kirchhoff-type equations; see [20] for details. We briefly recall this description here. Let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be a simplicial chart on F_N . We view $\tilde{\Gamma}$ as a simplicial tree, and thus an element of cv_N , by giving every edge length 1. Then F_N is F_N -equivariantly quasi-isometric to $\tilde{\Gamma}$, which determines a homeomorphism $\partial F_N \rightarrow \partial \tilde{\Gamma}$, which we will view as an identification between them. For a nondegenerate geodesic segment $\gamma = [p, q]$ in $\tilde{\Gamma}$ the *two-sided cylinder* $\text{Cyl}_{\Gamma}(\gamma) \subseteq \partial^2 F_N$ consists of all $(x, y) \in \partial^2 F_N$ such that the geodesic from x to y in $\tilde{\Gamma}$ passes through $\gamma = [p, q]$. Given a nontrivial reduced (i.e. without backtracking) edge-path v in Γ and a current $\mu \in \text{Curr}(F_N)$, the “weight” $\langle v, \mu \rangle_{\alpha}$ is defined as $\mu(\text{Cyl}_{\Gamma}(\gamma))$, where γ is any lift of v to $\tilde{\Gamma}$ (the fact that the measure μ is F_N -invariant implies that a particular choice of a lift of v does not matter). A current μ is uniquely determined by the family of weights $\langle v, \mu \rangle_{\alpha}$ where v varies over all nontrivial reduced finite edge-paths in Γ . The weak*-topology on $\text{Curr}(F_N)$ corresponds to pointwise convergence of the weights for every such v .

If A is a free basis of F_N , then the wedge of circles corresponding to the elements of A determines a simplicial chart on F_N ; the universal cover of this wedge of circles is the Cayley graph T_A of F_N with respect to A . In this case the weights are indexed by nontrivial freely reduced words v over A , and we denote the corresponding weights by $\langle v, \mu \rangle_A$.

For every $g \in F_N, g \neq 1$ there is an associated *counting current* $\eta_g \in \text{Curr}(F_N)$. If $\alpha : F_N \rightarrow \pi_1(\Gamma)$ is a simplicial chart on F_N , the conjugacy class $[g]$ of g can be realized by a reduced “cyclic path” w , that is, a graph immersion from a clockwise oriented simplicial circle to Γ that represents $[\alpha(g)]$. We can think of edges of w as being labelled by edges of Γ . Then for every nontrivial freely reduced edge-path v in Γ the weight $\langle v, \eta_g \rangle_{\alpha}$ is equal to the total number of occurrences of $v^{\pm 1}$ in w , where an *occurrence* of v in w is a vertex on w such that we can read v in w clockwise without going off the circle. We refer the reader to [20] for a detailed exposition on the topic. By construction the counting current η_g depends only on the conjugacy class $[g]$ of g and it also satisfies $\eta_g = \eta_{g^{-1}}$. One can check [20] that for $\varphi \in \text{Out}(F_N)$ and $g \in F_N, g \neq 1$ we have $\varphi \eta_g = \eta_{\varphi(g)}$. Scalar multiples $c \eta_g \in \text{Curr}(F_N)$, where $c \geq 0, g \in F_N, g \neq 1$ are called *rational currents*. A

key fact about $\text{Curr}(F_N)$ states that the set of all rational currents is dense in $\text{Curr}(F_N)$ [19, 20].

If A is a free basis of F_N and Γ_A is the wedge of circles corresponding to elements of A , a reduced cyclic path w in Γ_A can be viewed as a reduced *cyclic word* over A , that is, a cyclically reduced word over A written on a simplicial circle in the clockwise direction, without specifying the base-point of the circle.

The spaces cv_N and $\text{Curr}(F_N)$ come equipped with natural actions of $\text{Out}(F_N)$. However, the details of how these actions are defined are not relevant for this paper and we omit them.

2.3. Intersection form. In [23] Kapovich and Lustig constructed a natural geometric *intersection form* which pairs trees and currents:

Proposition 2.1 ([23]). *Let $N \geq 2$. There exists a unique continuous map $\langle \cdot, \cdot \rangle : \overline{\text{cv}}_N \times \text{Curr}(F_N) \rightarrow \mathbb{R}_{\geq 0}$ with the following properties:*

- (1) *We have $\langle T, c_1\mu_1 + c_2\mu_2 \rangle = c_1\langle T, \mu_1 \rangle + c_2\langle T, \mu_2 \rangle$ for any $T \in \overline{\text{cv}}_N$, $\mu_1, \mu_2 \in \text{Curr}(F_N)$, $c_1, c_2 \geq 0$.*
- (2) *We have $\langle cT, \mu \rangle = c\langle T, \mu \rangle$ for any $T \in \overline{\text{cv}}_N$, $\mu \in \text{Curr}(F_N)$ and $c \geq 0$.*
- (3) *We have $\langle T\varphi, \mu \rangle = \langle T, \varphi\mu \rangle$ for any $T \in \overline{\text{cv}}_N$, $\mu \in \text{Curr}(F_N)$ and $\varphi \in \text{Out}(F_N)$.*
- (4) *We have $\langle T, \eta_g \rangle = \|g\|_T$ for any $T \in \overline{\text{cv}}_N$ and $g \in F_N, g \neq 1$.*
- (5) *For every $\mu \in \text{Curr}(F_N)$ and for every tree $T \in \text{cv}_N$ given by a simplicial chart $\alpha : F_N \rightarrow \pi_1(\Gamma)$ with a metric structure $\mathcal{L}$ on Γ we have*

$$\langle T, \mu \rangle = \sum_{e \in E^+\Gamma} \langle e, \mu \rangle_\alpha \mathcal{L}(e).$$

Recall that $\overline{\text{cv}}_N$ is the closure of cv_N in the length function topology or, equivalently, in the equivariant Gromov-Hausdorff convergence topology. The case of $\text{cv}_N \times \text{Curr}(F_N)$ was dealt with in an earlier papers [27, 20].

3. UNIFORM MEASURE ON THE BOUNDARY

Convention 3.1. From now and until the end of the paper, unless specified otherwise, we fix a free basis $A = \{a_1, \dots, a_N\}$ of F_N , where $N \geq 2$.

Recall that T_A is the Cayley graph of F_N with respect to A . For a word v (not necessarily freely reduced) we denote by $|v|$ the length of v , denote by $|v|_A$ the freely reduced length of v and denote by $\|v\|_A$ the cyclically reduced length of v . Thus $\|v\|_A = \|v\|_{T_A}$. We will think of elements of F_N as freely reduced words over A , so that when writing $v \in F_N$, we will implicitly assume that v is a freely reduced word.

The boundary ∂F_N consists of geodesic rays originating at the vertex 1 in T_A . Every such ray ξ can also be viewed as a semi-infinite freely reduced word $\xi = x_1x_2\dots x_n\dots$ over A , where $x_i \in A^{\pm 1}$. For $n \geq 1$ we denote $\xi(n) = x_1\dots x_n \in F_N$, the initial segment of ξ of length n . If v is a nontrivial freely reduced word over A , denote by $\text{Cyl}_A(v)$ the set of all $\xi \in \partial F_N$ such that v is an initial segment of ξ . The set $\text{Cyl}_A(v) \subseteq \partial F_N$ is the *one-sided cylinder* defined by v . The set of all one-sided cylinders forms a basis of open (also closed) sets for the topology on ∂F_N . Any finite Borel measure on ∂F_N is determined by its values on all the cylinders $\text{Cyl}_A(v)$, $v \in F_N, v \neq 1$.

For $M \geq 1$ denote by $S_A(M)$ the set of all freely reduced words over A in F_N of length M , so that $S_A(M)$ is the sphere of radius M in the Cayley graph of F_N corresponding to the generating set A . Then $d(M) := \#(S_A(M)) = 2N(2N-1)^{M-1}$.

Definition 3.2 (Uniform measure). The *uniform measure* m_A corresponding to A is a Borel probability measure on ∂F_N given by the condition

$$m_A(\text{Cyl}_A(v)) = \frac{1}{d(|v|_A)}, \quad v \in F_N, v \neq 1.$$

It is not hard to see that the above condition indeed defines a Borel probability measure on ∂F_N . Moreover, this measure is exactly the exit measure of the simple nonbacktracking random walk on $F_N = F(A)$ starting at $1 \in F_N$. We refer the reader to [18] for more details. Note also that for m_A -a.e. $\xi \in \partial F$ the semi-infinite freely reduced word $\xi = x_1 x_2 \dots x_n \dots$ contains every finite freely reduced word $v \in F(A)$ as a subword infinitely many times. This fact is a consequence of the law of large numbers and one can say something more precise: if $\langle v, \xi(n) \rangle$ is the number of times that v occurs as a subword of $\xi(n) = x_1 \dots x_n$, then

$$\lim_{n \rightarrow \infty} \frac{\langle v, \xi(n) \rangle}{n} = \frac{1}{\#S_A(M)}.$$

This observation is responsible for the fact (see Remark 5.3 below) that for m_A -a.e. ξ we have

$$\lim_{n \rightarrow \infty} \frac{\eta_{\xi(n)}}{n},$$

where ν_A is the *uniform current* on F_N corresponding to A .

4. FINITE-DIMENSIONAL APPROXIMATIONS OF $\text{Curr}(F_N)$

4.1. Inverse limit description of $\text{Curr}(F_N)$. We briefly recall here an inverse limit description of $\text{Curr}(F_N)$ by finite-dimensional approximations; see [19, 20] for more details.

Let $M \geq 1$. Consider the Euclidean space $\mathbb{R}^{d(M)}$ with coordinates indexed by elements $v \in S_A(M)$. We write an element $p \in \mathbb{R}^{d(M)}$ as $p = (p_v)_{v \in S_A(M)}$, where p_v is the v -th coordinate of p . Now let $Z_M \subseteq \mathbb{R}^{d(M)}$ be the set of all $p \in \mathbb{R}^{d(M)}$ such that p satisfies the following conditions:

$$\begin{aligned} p_v &\geq 0 \text{ for every } v \in S_A(M), \\ p_v &= p_{v^{-1}} \text{ for every } v \in S_A(M), \\ \sum_{a \in A^{\pm 1}: |ua|_A=M} p_{ua} &= \sum_{b \in A^{\pm 1}: |bu|_A=M} p_{bu} \quad \text{for every } u \in S_A(M-1). \end{aligned}$$

If $M = 1$, the last condition above is absent when Z_1 is defined.

Thus $Z_M \subseteq \mathbb{R}^{d(M)}$ is a closed polyhedral cone. For $M \geq 2$ define the projection map $\pi_M : Z_M \rightarrow Z_{M-1}$ by the formula

$$(\pi_M(p))_u = \sum_{a \in A^{\pm 1}: |ua|_A=M} p_{ua},$$

where $p \in Z_M$ and $u \in S_A(M-1)$. Then (see [19, 20]) the map $\pi_M : Z_M \rightarrow Z_{M-1}$ is onto. Moreover, the inverse limit of the sequence of maps

$$(\dagger) \quad \xrightarrow{\pi_{M+1}^M} Z_M \xrightarrow{\pi_M} Z_{M-1} \xrightarrow{\pi_{M-1}^{M-2}} \dots \xrightarrow{\pi_2^1} Z_1$$

is naturally homeomorphic to $\text{Curr}(F_N)$. It is useful to see this identification explicitly. For a current $\mu \in \text{Curr}(F_N)$ and $M \geq 1$ let $\tau_M(\mu) = p \in Z_M$, where $p_v = \langle v, \mu \rangle_A$ for every $v \in S_A(M)$. Then $\pi_M \circ \tau_M = \tau_{M-1}$ for every $M \geq 2$. The sequence $(\tau_M(\mu))_{M \geq 1}$ is an element of the inverse limit of $(\dagger)$ corresponding to $\mu \in \text{Curr}(F_N)$. Note that the maps π_M and τ_M are $\mathbb{R}_{\geq 0}$ -linear.

For a cyclic word w over A denote $\tau_M(w) := \tau_M(\eta_w) \in Z_M$.

The following lemma is an immediate consequence of the definitions and of the fact that for a cyclic word w the weight $\langle v, \eta_w \rangle_A$ is equal to the number of occurrences of $v^{\pm 1}$ in w .

Lemma 4.1. *Let $M \geq 2$ and let $u \in S_A(M)$. Then for any freely reduced word w over A of length $\geq 2M$ such that the words w and wu are cyclically reduced as written, the difference $\tau_M(wu) - \tau_M(w)$ depends only on u and the initial and terminal segments of w of length M .*

Corollary 4.2. *Let $M \geq 2$ and let $u \in S_A(M)$ be cyclically reduced. Suppose w is a freely reduced word of length $\geq 2M$ such that u and w have the same initial segment of length M and such that u and w have the same terminal segment of length M . Then:*

- (1) *We have $\tau_M(wu) - \tau_M(w) = \tau_M(u)$.*
- (2) *For every $2 \leq M' \leq M$ we have $\tau_{M'}(wu) - \tau_{M'}(w) = \tau_{M'}(u)$.*

Proof. By Lemma 4.1 we may assume that $w = u^2$. Then $\tau_M(wu) = \tau_M(u^3) = 3\tau_M(u)$, $\tau_M(w) = \tau_M(u^2) = 2\tau_M(u)$ and hence $\tau_M(wu) - \tau_M(w) = \tau_M(u)$. This establishes part (1) of the corollary. Part (2) immediately follows from part (1). $\square$

4.2. Signed currents and extended intersection form. For technical reasons in this paper we need to work with currents that are not necessarily positive as measures. Let $\mathcal{SCurr}(F_N)$ be the space of all F_N -invariant and flip invariant locally finite (finite on compact sets) Borel measures on $\partial_2 F_N$ that are not necessarily positive. We call elements of $\mathcal{SCurr}(F_N)$ *signed geodesic currents*. Thus $\mathcal{SCurr}(F_N)$ is a vector space over $\mathbb{R}$. As before, we equip $\mathcal{SCurr}(F_N)$ with the weak-* topology. For a simplicial chart $\alpha : F_N \rightarrow \pi_1(\Gamma)$ and a measure $\mu \in \mathcal{SCurr}(F_N)$ the weights $\langle v, \mu \rangle_\alpha$ are defined in the same way as for elements of $\text{Curr}(F_N)$. Also, similar to the $\text{Curr}(F_N)$ situation, we have that $\lim_{n \rightarrow \infty} \mu_n = \mu$, where $\mu_n, \mu \in \mathcal{SCurr}(F_N)$ if and only if for every finite nondegenerate reduced path v in Γ we have $\lim_{n \rightarrow \infty} \langle v, \mu_n \rangle_\alpha = \langle v, \mu \rangle_\alpha$. The set $\text{Curr}(F_N)$ is a closed $\mathbb{R}_{\geq 0}$ -linear subspace of $\mathcal{SCurr}(F_N)$. We can extend the intersection form $\langle \cdot, \cdot \rangle : \text{cv}_N \times \text{Curr}(F_N) \rightarrow \mathbb{R}_{\geq 0}$ to $\langle \cdot, \cdot \rangle : \text{cv}_N \times \mathcal{SCurr}(F_N) \rightarrow \mathbb{R}$ by the formula

$$\langle T, \mu \rangle = \sum_{e \in E^+ \Gamma} \langle e, \mu \rangle_\alpha \mathcal{L}(e),$$

where $\mu \in \mathcal{SCurr}(F_N)$ and $T \in \text{cv}_N$ are given by a simplicial chart $\alpha : F_N \rightarrow \pi_1(\Gamma)$ with a metric structure $\mathcal{L}$ on Γ . Then exactly the same argument as in [20] yields:

Proposition 4.3. *The map $\langle \cdot, \cdot \rangle : \text{cv}_N \times \mathcal{SCurr}(F_N) \rightarrow \mathbb{R}$ is continuous, $\mathbb{R}_{\geq 0}$ -homogeneous in the first argument, $\mathbb{R}$ -linear in the second argument, and $\text{Out}(F_N)$ -invariant (with respect to the natural left actions of $\text{Out}(F_N)$ on cv_N and $\mathcal{SCurr}(F_N)$).*

It is obvious from the definition and from Proposition 2.1 that this extended intersection form agrees with the old one on $\text{cv}_N \times \text{Curr}(F_N)$. Moreover, we also get a sequence of finite-dimensional approximations of $\mathcal{SCurr}(F_N)$ similar to that

of $\text{Curr}(F_N)$. Namely, we define $\tilde{Z}_M$ exactly as Z_M but omitting the condition $p_v \geq 0, v \in S_A(M)$. Then $\tilde{Z}_M \subseteq \mathbb{R}^{d(M)}$ is an $\mathbb{R}$ -linear subspace containing Z_M , the map $\pi_M : Z_M \rightarrow Z_{M-1}$ naturally extends to an $\mathbb{R}$ -linear map $\tilde{\pi}_M : \tilde{Z}_M \rightarrow \tilde{Z}_{M-1}$ (given by the same formula as π_M) and the map $\tau_M : \text{Curr}(F_N) \rightarrow Z_M$ naturally extends to an $\mathbb{R}$ -linear continuous map $\tilde{\tau}_M : \mathcal{SCurr}(F_N) \rightarrow \tilde{Z}_M$ given by the same formula as τ_M . Also, we still have the property $\tilde{\pi}_M \circ \tilde{\tau}_M = \tilde{\tau}_{M-1}$ for every $M \geq 2$. Note that knowing all the weights $\langle v, \mu \rangle_A$, where $v \in S_A(M)$, uniquely determines all the “lower-level” weights $\langle u, \mu \rangle_A$, where $1 \leq |u|_A \leq M$, by iterating linear formulas corresponding to the last equation in the definition of Z_M .

One can show that $\mathcal{SCurr}(F_N)$ is equal to the inverse limit of the sequence $\tilde{\pi}_M$, although this fact is not needed in this paper.

Notation 4.4. If X is an $\mathbb{R}$ -vector space and $Y \subseteq X$, we denote by $\text{Span}(Y)$ the linear span of Y in X , that is, the set of all finite $\mathbb{R}$ -linear combinations of elements of Y .

Proposition 4.5. *Let $N \geq 2$. Then the following hold:*

- (1) *For every $\mu \in \mathcal{SCurr}(F_N)$ there exist $\mu_1, \mu_2 \in \text{Curr}(F_N)$ such that $\mu = \mu_1 - \mu_2$.*
- (2) *The span $\text{Span}(\{\eta_g : g \in F_N - \{1\}\})$ is dense in $\mathcal{SCurr}(F_N)$.*

Proof. Note that part (1) directly implies part (2) since it is known that the set of rational currents is dense in $\text{Curr}(F_N)$.

To see that part (1) holds, recall that by the Hahn-Jordan Decomposition Theorem, for any $\mu \in \mathcal{SCurr}(F_N)$ there exists a decomposition $\partial^2 F_N = E_1 \sqcup E_2$, where E_1, E_2 are Borel sets and where for every Borel subset $S \subseteq E_1$ we have $\mu(S) \geq 0$ and for every Borel subset $S \subseteq E_2$ we have $\mu(S) \leq 0$. Moreover, in view of F_N -invariance and flip-invariance of μ , the uniqueness part of the Hahn-Jordan Theorem easily implies that the sets E_1, E_2 may be assumed to be F_N -invariant and flip-invariant.

Define measures μ_1, μ_2 on $\partial^2 F_N$ by $\mu_1(S) := \mu(S \cap E_1)$ and $\mu_2(S) := -\mu(S \cap E_2)$, where $S \subseteq \partial^2 F_N$. Then $\mu = \mu_1 - \mu_2$ and μ_1, μ_2 are positive Borel measures on $\partial^2 F$ that are finite on compact subsets. Moreover, since E_1, E_2 are F_N -invariant and flip-invariant, the measures μ_1, μ_2 are also F_N -invariant and flip-invariant, so that $\mu_1, \mu_2 \in \text{Curr}(F_N)$. $\square$

5. LINEAR SPANS AND RANDOM RAYS

Proposition 5.1. *There exists a subset of ∂F_N of m_A -measure 1 such that every point ξ from that subset has the following property:*

For every $n_0 \geq 1$ we have

$$\mathcal{SCurr}(F_N) = \overline{\text{Span}(\{\eta_{\xi(n)} : n \geq n_0\})}.$$

Proof. Let $\Omega \subseteq \partial F_N$ be the set of all $\xi \in \partial F_N$ such that every nontrivial freely reduced word over A occurs as a subword of ξ infinitely many times. Then, as is well known, $m_A(\Omega) = 1$.

Let $\xi \in \Omega$ be arbitrary.

Let $n_0 \geq 2$ and let w be an arbitrary nontrivial cyclically reduced word over A . Let $M \geq 2$ be an arbitrary integer. Let v be the initial segment of ξ of length M . We can always choose freely reduced words s, t of length M such that the word $vs(w^n t)^m$ is freely and cyclically reduced as written for all $m, n \geq 1$.

Let $n \geq 1$ be arbitrary. Since $\xi \in \Omega$, the word $vs(w^nt)^2$ occurs as a subword of ξ infinitely many times. Choose such an occurrence that starts at position $\geq n_0 + 1$ in ξ . Thus there exists an initial segment of ξ of the form $vf_nvs(w^nt)^2$ where $|vf_n| \geq n_0 + 1$. Then by Corollary 4.2 we have $\tau_M(vf_nvs(w^nt)^2) - \tau_M(vf_nvs w^nt) = \tau_M(vsw^nt)$. Hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} (\tau_M(vf_nvs(w^nt)^2) - \tau_M(vf_nvs w^nt)) = \tau_M(w).$$

Then (in view of the last equation in the definition of Z_M and $\tilde{Z}_M$) we can find $n = n_M \geq 1$ such that for every freely reduced word u over A with $0 < |u| \leq M$ we have

$$\left| \left\langle u, \frac{1}{n} (\eta_{vf_nvs(w^nt)^2} - \eta_{vf_nvs w^nt}) \right\rangle - \langle u, \eta_w \rangle \right| \leq 2^{-M}.$$

Put $\mu_M = \frac{1}{n} (\eta_{vf_nvs(w^nt)^2} - \eta_{vf_nvs w^nt})$. Therefore, by the definition of topology on $\mathcal{SCurr}(F_N)$, we have $\lim_{M \rightarrow \infty} \mu_M = \eta_w$ in $\mathcal{SCurr}(F_N)$. Hence $\eta_w \in \overline{\text{Span}(\{\eta_{\xi(n)} : n \geq n_0\})}$. Proposition 4.5 now implies that

$$\mathcal{SCurr}(F_N) = \overline{\text{Span}(\{\eta_{\xi(n)} : n \geq n_0\})},$$

as required. $\square$

We can now prove the main result, Theorem 1.1 from the Introduction:

Theorem 5.2. *There exists a subset of ∂F_N of m_A -measure 1 such that every point ξ from that subset has the following property:*

For every $n_0 \geq 1$ the set

$$\{\xi(n) : n \geq n_0\} \subseteq F_N$$

is spectrally rigid.

Proof. Let $\Omega \subseteq \partial F_N$ be the set with $m_A(\Omega) = 1$ provided by Proposition 5.1. Let $\xi \in \Omega$ be arbitrary and let $n_0 \geq 1$. Suppose $T_1, T_2 \in \text{cv}_N$ are such that for every $n \geq n_0$ we have $\|\xi(n)\|_{T_1} = \|\xi(n)\|_{T_2}$. Hence for every $n \geq n_0$,

$$\langle T_1, \eta_{\xi(n)} \rangle = \langle T_2, \eta_{\xi(n)} \rangle.$$

By linearity and continuity of the intersection form on $\text{cv}_N \times \mathcal{SCurr}(F_N)$ it follows that

$$\langle T_1, \mu \rangle = \langle T_2, \mu \rangle$$

for every $\mu \in \overline{\text{Span}(\{\eta_{\xi(n)} : n \geq n_0\})}$. Hence, by Proposition 5.1, for every nontrivial $w \in F_N$ we have

$$\|w\|_{T_1} = \langle T_1, \eta_w \rangle = \langle T_2, \eta_w \rangle = \|w\|_{T_2}.$$

Thus the length functions $\|\cdot\|_{T_1}$ and $\|\cdot\|_{T_2}$ are equal on F_N and hence $T_1 = T_2$ in cv_N . This shows that the set $\{\xi(n) : n \geq n_0\}$ is spectrally rigid in F_N , as claimed. $\square$

Remark 5.3. As noted in [18], for an m_A -a.e. point $\xi \in \partial F_N$ we have $\lim_{n \rightarrow \infty} \frac{1}{n} \eta_{\xi(n)} = \nu_A$. Here $\nu_A \in \text{Curr}(F_N)$ is the *uniform current* corresponding to A , which is given by the weights $\langle v, \nu_A \rangle = \frac{1}{d(v|_A)}$ for every nontrivial freely reduced word v over A . The proof of Theorem 5.2 is based on exploiting the fact that the sequence $\frac{1}{n} \eta_{\xi(n)}$ converges to ν_A “from all possible directions”.

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