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k-HARMONIC MAPS INTO A RIEMANNIAN MANIFOLD WITH CONSTANT SECTIONAL CURVATURE

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ABSTRACT. J. Eells and L. Lemaire introduced k-harmonic maps, and Shaobo Wang showed the first variational formula. When $k = 2$, it is called biharmonic maps (2-harmonic maps). There have been extensive studies in the area. In this paper, we consider the relationship between biharmonic maps and k-harmonic maps, and we show the non-existence theorem of 3-harmonic maps. We also give the definition of k-harmonic submanifolds of Euclidean spaces and study k-harmonic curves in Euclidean spaces. Furthermore, we give a conjecture for k-harmonic submanifolds of Euclidean spaces.

INTRODUCTION

The theory of harmonic maps has been applied to various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional $E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g$, for smooth maps $\phi : M \to N$.

On the other hand, in 1981, J. Eells and L. Lemaire [4] proposed the problem to consider the *k-harmonic maps*: they are critical maps of the functional

$$E_k(\phi) = \int_M e_k(\phi) v_g \quad (k = 1, 2, \cdots),$$

where $e_k(\phi) = \frac{1}{2}\|(d+d^*)^k\phi\|^2$ for smooth maps $\phi : M \to N$. G.Y. Jiang [6] studied the first and second variational formulas of the bi-energy E_2, and critical maps of E_2 are called *biharmonic maps* (*2-harmonic maps*). There have been extensive studies on biharmonic maps.

In 1989, Shaobo Wang [10] studied the first variational formula of the k-energy E_k, whose critical maps are called k-harmonic maps. Harmonic maps are always k-harmonic maps by definition. However, the author [7] showed biharmonic is not always k-harmonic ($k \geq 3$). More generally, s-harmonic is not always k-harmonic ($s < k$). Furthermore, the author [7] showed the second variational formula of the k-energy.

In this paper, we study k-harmonic maps into a Riemannian manifold with constant sectional curvature K.

In §1, we introduce notation and fundamental formulas of the tension field.

In §2, we recall k-harmonic maps.

In §3, we give the relationship between biharmonic maps and k-harmonic maps.

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In §4, we study 3-harmonic maps into a non-positive sectional curvature and obtain a non-existence theorem.

Finally, in §5, we define k-harmonic submanifolds of Euclidean spaces. Also, we show that a k-harmonic curve is a straight line. Furthermore, we give a conjecture for k-harmonic submanifolds in Euclidean spaces.

1. Preliminaries

Let (M, g) be an m dimensional Riemannian manifold, (N, h) an n dimensional one, and $\phi : M \to N$ a smooth map. We use the following notation.

The second fundamental form $B(\phi)$ of ϕ is a covariant differentiation $\widetilde{\nabla} d\phi$ of the 1-form $d\phi$, which is a section of $\odot^2 T^*M \otimes \phi^{-1}TN$. For every $X, Y \in \Gamma(TM)$, let

$$\begin{aligned} B(X,Y) &= (\widetilde{\nabla} d\phi)(X,Y) = (\widetilde{\nabla}_X d\phi)(Y) \\ &= \overline{\nabla}_X d\phi(Y) - d\phi(\nabla_X Y) = \nabla^N_{d\phi(X)} d\phi(Y) - d\phi(\nabla_X Y). \end{aligned} \tag{1}$$

Here, $\nabla, \nabla^N, \overline{\nabla}, \widetilde{\nabla}$ are the induced connections on the bundles TM, TN, $\phi^{-1}TN$ and $T^*M \otimes \phi^{-1}TN$, respectively.

If M is compact, we consider critical maps of the energy functional

$$E(\phi) = \int_M e(\phi) v_g, \tag{2}$$

where $e(\phi) = \frac{1}{2}\|d\phi\|^2 = \sum_{i=1}^m \frac{1}{2}\langle d\phi(e_i), d\phi(e_i)\rangle$, which is called the *energy density* of ϕ, the inner product $\langle \cdot, \cdot \rangle$ is a Riemannian metric h, and $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g). The *tension field* $\tau(\phi)$ of ϕ is defined by

$$\tau(\phi) = \sum_{i=1}^m (\widetilde{\nabla} d\phi)(e_i, e_i) = \sum_{i=1}^m (\widetilde{\nabla}_{e_i} d\phi)(e_i). \tag{3}$$

Then, ϕ is a *harmonic map* if $\tau(\phi) = 0$.

The curvature tensor field $R^N(\cdot, \cdot)$ of the Riemannian metric on the bundle TN is defined as follows:

$$R^N(X,Y) = \nabla^N_X \nabla^N_Y - \nabla^N_Y \nabla^N_X - \nabla^N_{[X,Y]} \quad (X, Y \in \Gamma(TN)). \tag{4}$$

$\overline{\triangle} = \overline{\nabla}^* \overline{\nabla} = -\sum_{k=1}^m (\overline{\nabla}_{e_k} \overline{\nabla}_{e_k} - \overline{\nabla}_{\nabla_{e_k} e_k})$ is the *rough Laplacian*.

Also, Jiang [6] showed that $\phi : (M, g) \to (N, h)$ is a biharmonic (2-harmonic) if and only if

$$\overline{\triangle}\tau(\phi) - \sum_{i=1}^m R^N(\tau(\phi), d\phi(e_i)) d\phi(e_i) = 0.$$

Throughout the paper, we omit the sign $\sum$ without mentioning this omission.

2. k-harmonic maps

J. Eells and L. Lemaire [4] proposed the notation of k-harmonic maps. The Euler-Lagrange equations for the k-harmonic maps were shown by Shaobo Wang [10]. In this section, we recall k-harmonic maps.

We consider a smooth variation $\{\phi_t\}_{t\in I_\epsilon}(I_\epsilon = (-\epsilon, \epsilon))$ of ϕ with parameters t; i.e. we consider the smooth map F given by

$$F : I_\epsilon \times M \to N, F(t,p) = \phi_t(p),$$

where $F(0,p) = \phi_0(p) = \phi(p)$, for all $p \in M$.

The corresponding variational vector field V is given by

$$V(p) = \frac{d}{dt}|_{t=0}\phi_{t,0} \in T_{\phi(p)}N,$$

where V is a section of $\phi^{-1}TN$, i.e. $V \in \Gamma(\phi^{-1}TN)$.

We also denote by $\nabla, \overline{\nabla}$ and $\widetilde{\overline{\nabla}}$ the induced Riemannian connection on $T(I_\epsilon\times M)$, $F^{-1}TN$ and $T^*(I_\epsilon \times M) \otimes F^{-1}TN$, respectively.

Definition 2.1 ([4])**.** For $k = 1, 2, \cdots$ the *k-energy functional* is defined by

$$E_k(\phi) = \frac{1}{2}\int_M \|(d + d^*)^k\phi\|^2 v_g, \quad \phi \in C^\infty(M,N).$$

Then, ϕ is *k-harmonic* if it is a critical point of E_k; i.e., for all smooth variations $\{\phi_t\}$ of ϕ with $\phi_0 = \phi$,

$$\left.\frac{d}{dt}\right|_{t=0} E_k(\phi_t) = 0.$$

We say that a k-harmonic map is *proper* if it is not harmonic.

Lemma 2.2 ([10])**.**

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\triangle}^{s-1}\tau(F)|_{t=0} = &-\overline{\triangle}^{s}V + \overline{\triangle}^{s-1}R^N(V, d\phi(e_j))d\phi(e_j)\\ &+\sum_{l=1}^{s-1}\overline{\triangle}^{l-1}\{-\overline{\nabla}_{e_j}R^N(V, d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad - R^N(V, d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\}.\end{aligned}$$

Proof. For all $\omega \in \Gamma(\phi^{-1}TN)$,

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\triangle}\omega &= -\{\overline{\nabla}_{\frac{\partial}{\partial t}}(\overline{\nabla}_{e_j}\overline{\nabla}_{e_j} - \overline{\nabla}_{\nabla_{e_j}e_j})\omega\}\\ &= -\{\overline{\nabla}_{e_j}\overline{\nabla}_{\frac{\partial}{\partial t}}(\overline{\nabla}_{e_j}\omega) + R^N(dF(\frac{\partial}{\partial t}), dF(e_j))\overline{\nabla}_{e_j}\omega\\ &\quad - \overline{\nabla}_{\nabla_{e_j}e_j}\overline{\nabla}_{\frac{\partial}{\partial t}}\omega - R^N(dF(\frac{\partial}{\partial t}), dF(\nabla_{e_j}e_j))\omega\}\\ &= -\{\overline{\nabla}_{e_j}(\overline{\nabla}_{e_j}\overline{\nabla}_{\frac{\partial}{\partial t}}\omega + R^N(dF(\frac{\partial}{\partial t}), dF(e_j))\omega)\\ &\quad + R^N(dF(\frac{\partial}{\partial t}), dF(e_j))\overline{\nabla}_{e_j}\omega\\ &\quad - \overline{\nabla}_{\nabla_{e_j}e_j}\overline{\nabla}_{\frac{\partial}{\partial t}}\omega - R^N(dF(\frac{\partial}{\partial t}), dF(\nabla_{e_j}e_j))\omega\}.\end{aligned}$$

Repeating this and using

$$\overline{\nabla}_{\frac{\partial}{\partial t}}\tau(F)|_{t=0} = -\overline{\triangle}V + R^N(V, d\phi(e_j))d\phi(e_j),$$

we have the lemma. $\square$

Lemma 2.3 ([10]).

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(F)|_{t=0} =& -\overline{\nabla}_{e_i}\overline{\triangle}^{s}V+\overline{\nabla}_{e_i}\overline{\triangle}^{s-1}R^N(V,d\phi(e_j))d\phi(e_j)\\ &+\sum_{l=1}^{s-1}\overline{\nabla}_{e_i}\overline{\triangle}^{l-1}\{-\overline{\nabla}_{e_j}R^N(V,d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad -R^N(V,d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad +R^N(V,d\phi(\nabla_{e_j}e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\}\\ &+R^N(V,d\phi(e_i))\overline{\triangle}^{s-1}\tau(\phi).\end{aligned}$$

Proof.

$$\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(F)=\overline{\nabla}_{e_i}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\triangle}^{s-1}\tau(F)+R^N(dF(\frac{\partial}{\partial t}),dF(e_i))\overline{\triangle}^{s-1}\tau(F).$$

Using Lemma 2.2, we have the lemma. □

Lemma 2.4 ([10]).

$$\begin{aligned}&\int_M\langle\overline{\nabla}_{e_j}R^N(V,d\phi(e_j))V_1-R^N(V,d\phi(\nabla_{e_j}e_j))V_1,V_2\rangle v_g\\ &\qquad=-\int_M\langle R^N(V,d\phi(e_j))V_1,\overline{\nabla}_{e_j}V_2\rangle v_g,\end{aligned}$$

$V_1,V_2\in\Gamma(\phi^{-1}TN)$.

Proof.

$$\begin{aligned}\mathrm{div}(\langle R^N(V,d\phi(e_i))V_1,V_2\rangle e_i)=&\langle\nabla_{e_j}\langle R^N(V,d\phi(e_i))V_1,V_2\rangle e_i,e_j\rangle\\ =&\langle\langle\overline{\nabla}_{e_j}R^N(V,d\phi(e_i))V_1,V_2\rangle e_i\\ &+\langle R^N(V,d\phi(e_i))V_1,\overline{\nabla}_{e_j}V_2\rangle e_i\\ &+\langle R^N(V,d\phi(e_i))V_1,V_2\rangle\nabla_{e_j}e_i,e_j\rangle.\end{aligned}$$

By Green's theorem, we have

$$\begin{aligned}0=&\int_M\mathrm{div}\langle R^N(V,d\phi(e_i))V_1,V_2\rangle e_i v_g\\ =&\int_M\langle\overline{\nabla}_{e_j}R^N(V,d\phi(e_i)V_1,V_2)\rangle\delta_{ij}\\ &+\langle R^N(V,d\phi(e_i))V_1,\overline{\nabla}_{e_j}V_2\rangle\delta_{ij}\\ &+\langle R^N(V,d\phi(e_i))V_1,V_2\rangle\langle\nabla_{e_j}e_i,e_j\rangle v_g.\end{aligned}$$

Here,

$$\begin{aligned}\langle R^N(V,d\phi(e_i))V_1,V_2\rangle\langle\nabla_{e_j}e_i,e_j\rangle=&\langle R^N(V,d\phi(\langle\nabla_{e_j}e_i,e_j\rangle e_i))V_1,V_2\rangle\\ =&-\langle R^N(V,d\phi(\nabla_{e_i}e_i))V_1,V_2\rangle.\end{aligned}$$

Therefore, we have the lemma. □

Theorem 2.5 ([10]). *Let $k = 2s$ $(s = 1, 2, \cdots)$,*

$$\frac{d}{dt}\bigg|_{t=0} E_{2s}(\phi_t) = -\int_M \langle \tau_{2s}(\phi), V\rangle,$$

where

$$\begin{aligned}\tau_{2s}(\phi) =&\overline{\triangle}^{2s-1}\tau(\phi) - R^N(\overline{\triangle}^{2s-2}\tau(\phi), d\phi(e_j))e\phi(e_j)\\ &-\sum_{l=1}^{s-1}\{R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-2}\tau(\phi), \overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j)\\ &\qquad - R^N(\overline{\triangle}^{s+l-2}\tau(\phi), \overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j)\},\end{aligned}$$

where $\overline{\triangle}^{-1} = 0$.

Proof.

$$\begin{aligned}E_{2s}(\phi) &= \int_M \langle\underbrace{(d^*d)\cdots(d^*d)}_{s}\phi, \underbrace{(d^*d)\cdots(d^*d)}_{s}\phi\rangle v_g\\ &= \int_M \langle\overline{\triangle}^{s-1}\tau(\phi), \overline{\triangle}^{s-1}\tau(\phi)\rangle v_g.\end{aligned}$$

By using Lemma 2.2 and Lemma 2.4, we calculate $\frac{d}{dt}E_{2s}(\phi_t)$:

$$\begin{aligned}&\frac{d}{dt}E_{2s}(\phi_t)|_{t=0}\\ &=\int_M \langle\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\triangle}^{s-1}\tau(F), \overline{\triangle}^{s-1}\tau(F)\rangle v_g|_{t=0}\\ &=\int_M \langle -\overline{\triangle}^{s}V + \overline{\triangle}^{s-1}R^N(V, d\phi(e_j))d\phi(e_j)\\ &\quad+\sum_{l=1}^{s-1}\overline{\triangle}^{l-1}\{-\overline{\nabla}_{e_j}R^N(V, d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad - R^N(V, d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\overline{\triangle}^{s-l-1}\tau(\phi), \overline{\triangle}^{s-1}\tau(\phi)\}\rangle\\ &=\int_M \langle V, -\overline{\triangle}^{2s-1}\tau(\phi)\rangle v_g\\ &\quad+\int_M \langle V, R^N(\overline{\triangle}^{2s-2}\tau(\phi), d\phi(e_j))d\phi(e_j)\rangle v_g\\ &\quad+\sum_{l=1}^{s-1}\int_M \langle -\overline{\nabla}_{e_j}R^N(V, d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad - R^N(V, d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi)\\ &\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\overline{\triangle}^{s-l-1}\tau(\phi), \overline{\triangle}^{s+l-2}\tau(\phi)\rangle v_g\end{aligned} \tag{5}$$

$$
\begin{aligned}
&= \int_M \langle V, -\overline{\triangle}^{2s-1}\tau(\phi)\rangle v_g \\
&\quad + \int_M \langle V, R^N(\overline{\triangle}^{2s-2}\tau(\phi), d\phi(e_j))d\phi(e_j)\rangle v_g \\
&\quad + \sum_{l=1}^{s-1}\{\int_M \langle R^N(V, d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi), \overline{\nabla}_{e_j}\overline{\triangle}^{s+l-2}\tau(\phi)\rangle v_g \\
&\quad + \int_M \langle -R^N(V, d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi), \overline{\triangle}^{s+l-2}\tau(\phi)\rangle v_g\} \\
&= \int_M \langle V, -\overline{\triangle}^{2s-1}\tau(\phi)\rangle v_g \\
&\quad + \int_M \langle V, R^N(\overline{\triangle}^{2s-2}\tau(\phi), d\phi(e_j))d\phi(e_j)\rangle v_g \\
&\quad + \sum_{l=1}^{s-1}\{\int_M \langle R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-2}\tau(\phi), \overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j), V\rangle v_g \\
&\quad - \int_M \langle R^N(\overline{\triangle}^{s+l-2}\tau(\phi), \overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j), V\rangle v_g\} \\
&= \int_M \langle V, -\overline{\triangle}^{2s-1}\tau(\phi) + R^N(\overline{\triangle}^{2s-2}\tau(\phi), d\phi(e_j))d\phi(e_j) \\
&\quad + \sum_{l=1}^{s-1}\{R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-2}\tau(\phi), \overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j) \\
&\quad - R^N(\overline{\triangle}^{s+l-2}\tau(\phi), \overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j)\}\rangle v_g.
\end{aligned}
$$

Therefore, we have the theorem. □

Theorem 2.6 ([10]). *Let* $k = 2s+1 \quad (s = 0, 1, 2, \cdots)$,

$$
\frac{d}{dt}\bigg|_{t=0} E_{2s+1}(\phi_t) = -\int_M \langle \tau_{2s+1}(\phi), V\rangle,
$$

where

$$
\begin{aligned}
\tau_{2s+1}(\phi) =& \overline{\triangle}^{2s}\tau(\phi) - R^N(\overline{\triangle}^{2s-1}\tau(\phi), d\phi(e_j))d\phi(e_j) \\
& - \sum_{l=1}^{s-1}\{R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-1}\tau(\phi), \overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j) \\
& \qquad - R^N(\overline{\triangle}^{s+l-1}\tau(\phi), \overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j)\} \\
& - R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s-1}\tau(\phi), \overline{\triangle}^{s-1}\tau(\phi))d\phi(e_j),
\end{aligned}
$$

where $\overline{\triangle}^{-1} = 0$.

Proof. When $s = 0$, it is a well-known harmonic map. Hence we consider the case of $s = 1, 2, \cdots$:

$$
\begin{aligned}
E_{2s+1}(\phi) &= \int_M \langle d\underbrace{(d^*d)\cdots(d^*d)}_{s}\phi, d\underbrace{(d^*d)\cdots(d^*d)}_{s}\phi\rangle v_g \\
&= \int_M \langle \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(\phi), \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(\phi)\rangle v_g.
\end{aligned}
$$

By using Lemma 2.3 and Lemma 2.4, we calculate $\frac{d}{dt}E_{2s+1}(\phi_t)$:

$$\begin{aligned}
&\frac{d}{dt}E_{2s+1}(\phi_t)|_{t=0}\\
&=\int_M \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(F), \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(F)\rangle v_g|_{t=0}\\
&=\int_M \langle -\overline{\nabla}_{e_i}\overline{\triangle}^{s}V + \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}R^N(V, d\phi(e_j))d\phi(e_j)\\
&\quad+\sum_{l=1}^{s-1}\overline{\nabla}_{e_i}\overline{\triangle}^{l-1}\{-\overline{\nabla}_{e_j}R^N(V, d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\
&\qquad\qquad - R^N(V, d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi)\\
&\qquad\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\
&\qquad\qquad + R^N(V, d\phi(e_i))\overline{\triangle}^{s-1}\tau(\phi)\}, \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(\phi)\rangle v_g.
\end{aligned}$$

Here, using

$$\int_M \langle \overline{\nabla}_{e_i}\omega_1, \overline{\nabla}_{e_i}\omega_2\rangle v_g = \int_M \langle \overline{\triangle}\omega_1, \omega_2\rangle v_g,$$

where $\omega_1, \omega_2 \in \Gamma(\phi^{-1}TN)$, we have

$$\begin{aligned}
&\frac{d}{dt}E_{2s+1}(\phi_t)|_{t=0}\\
&=\int_M \langle V, -\overline{\triangle}^{2s}\tau(\phi)\rangle v_g\\
&\quad+\int_M \langle R^N(V, d\phi(e_j))d\phi(e_j), \overline{\triangle}^{2s-1}\tau(\phi)\rangle v_g\\
&\quad+\sum_{l=1}^{s-1}\int_M \langle -\overline{\nabla}_{e_j}R^N(V, d\phi(e_j))\overline{\triangle}^{s-l-1}\tau(\phi)\\
&\qquad\qquad - R^N(V, d\phi(e_j))\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi)\\
&\qquad\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\overline{\triangle}^{s-l-1}\tau(\phi), \overline{\triangle}^{s+l-1}\tau(\phi)\rangle v_g\\
&\quad+\int_M \langle R^N(V, d\phi(e_i))\overline{\triangle}^{s-1}\tau(\phi), \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(\phi)\rangle v_g
\end{aligned} \tag{6}$$

$$
\begin{aligned}
=\int_M \langle V, -\overline{\triangle}^{2s}\tau(\phi) + R^N(\overline{\triangle}^{2s-1}\tau(\phi), d\phi(e_j))d\phi(e_j) \\
+\sum_{l=1}^{s-1}\{R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-1}\tau(\phi), \overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j) \\
- R^N(\overline{\triangle}^{s+l-1}\tau(\phi), \overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))d\phi(e_j)\} \\
+ R^N(\overline{\nabla}_{e_j}\overline{\triangle}^{s-1}\tau(\phi), \overline{\triangle}^{s-1}\tau(\phi))d\phi(e_j)\rangle v_g.
\end{aligned}
$$

Therefore, we have the theorem. □

By Theorems 2.5 and 2.6, we have the following [10].

Corollary 2.7. *A harmonic map is always a k-harmonic map $(k = 1, 2, \cdots)$.*

3. The relationship between biharmonic and k-harmonic

In [7], the author showed that s-harmonic is not always k-harmonic $(s < k)$. Especially, biharmonic is not always k-harmonic $(k \geq 3)$. Therefore, we study the relationship between biharmonic and k-harmonic $(2 < k)$. We obtain some results.

Proposition 3.1. *Let $\phi : (M, g) \to (N, h)$ be an isometric immersion into a Riemannian manifold with constant sectional curvature K. Then ϕ is biharmonic if and only if*

$$\overline{\triangle}\tau(\phi) = Km\tau(\phi).$$

Proof. ϕ is biharmonic if and only if

$$
\begin{aligned}
0 =& \overline{\triangle}\tau(\phi) - R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) \\
=& \overline{\triangle}\tau(\phi) - K\{\langle d\phi(e_i), d\phi(e_i)\rangle\tau(\phi) - \langle d\phi(e_i), \tau(\phi)\rangle d\phi(e_i)\} \\
=& \overline{\triangle}\tau(\phi) - Km\tau(\phi).
\end{aligned}
$$

Thus, we have the proposition. □

Lemma 3.2. *Let $\phi : (M, g) \to (N, h)$ be an isometric immersion into a Riemannian manifold with constant sectional curvature K. If ϕ is biharmonic,*

$$\langle d\phi(e_i), \overline{\triangle}^l\tau(\phi)\rangle = 0 \quad (l = 0, 1, \cdots). \tag{7}$$

Proof. By using Proposition 3.1, we have

$$
\begin{aligned}
\langle d\phi(e_i), \overline{\triangle}^l\tau(\phi)\rangle &= mK\langle d\phi(e_i), \overline{\triangle}^{l-1}\tau(\phi)\rangle \\
&\cdots \\
&= (mK)^l\langle d\phi(e_i), \tau(\phi)\rangle \\
&= 0.
\end{aligned}
$$

□

Lemma 3.3. *Let $\phi : (M, g) \to (N, h)$ be an isometric immersion into a Riemannian manifold with constant sectional curvature K. If ϕ is biharmonic,*

$$\langle d\phi(e_i), \overline{\nabla}_{e_i}\overline{\triangle}^l\tau(\phi)\rangle = -(mK)^l||\tau(\phi)||^2. \tag{8}$$

Proof. By using Proposition 3.1, we have

$$\begin{aligned}\langle d\phi(e_i), \overline{\nabla}_{e_i}\overline{\triangle}^l\tau(\phi)\rangle =& mK\langle d\phi(e_i), \overline{\nabla}_{e_i}\overline{\triangle}^{l-1}\tau(\phi)\rangle \\ &\cdots \\ =&(mK)^l\langle d\phi(e_i), \overline{\nabla}_{e_i}\tau(\phi)\rangle \\ =& -(mK)^l||\tau(\phi)||^2,\end{aligned}$$

where, in the last equation, we only notice that

$$0 = e_i\langle d\phi(e_i), \tau(\phi)\rangle = \langle \overline{\nabla}_{e_i} d\phi(e_i), \tau(\phi)\rangle + \langle d\phi(e_i), \overline{\nabla}_{e_i}\tau(\phi)\rangle. \qquad \square$$

Using these lemmas, we show the following two theorems.

Theorem 3.4. *Let $\phi : (M, g) \to (N, h)$ be a biharmonic isometric immersion into a Riemannian manifold with constant sectional curvature K ($\neq 0$). If ϕ is $2s$-harmonic ($s \geq 2$), ϕ is harmonic.*

Proof. By Theorem 2.5, ϕ is $2s$-harmonic if and only if

$$\begin{aligned}&\overline{\triangle}^{2s-1}\tau(\phi) - K\{m\overline{\triangle}^{2s-2}\tau(\phi) - \langle d\phi(e_j), \overline{\triangle}^{2s-2}\tau(\phi)\rangle d\phi(e_j)\} \\ &- \sum_{l=1}^{s-1}\{K(\langle\overline{\triangle}^{s-l-1}\tau(\phi), d\phi(e_j)\rangle\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-2}\tau(\phi) \\ &\qquad - \langle d\phi(e_j), \overline{\nabla}_{e_j}\overline{\triangle}^{s+l-2}\tau(\phi)\rangle\overline{\triangle}^{s-l-1}\tau(\phi) \\ &\qquad - \langle\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi), d\phi(e_j)\rangle\overline{\triangle}^{s+l-2}\tau(\phi) \\ &\qquad + \langle d\phi(e_j), \overline{\triangle}^{s+l-2}\tau(\phi)\rangle\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))\} = 0.\end{aligned}$$

By Proposition 3.1 and Lemmas 3.2 and 3.3, we have

$$\begin{aligned}0 =& (mK)^{2s-1}\tau(\phi) - (mK)^{2s-1}\tau(\phi) \\ &- \sum_{l=1}^{s-1}\{K((mK)^{2s-3}||\tau(\phi)||^2\tau(\phi) + (mK)^{2s-3}||\tau(\phi)||^2\tau(\phi))\} \\ =& -2(s-1)K(mK)^{2s-3}||\tau(\phi)||^2\tau(\phi).\end{aligned}$$

Thus, we have the theorem. $\square$

Theorem 3.5. *Let $\phi : (M, g) \to (N, h)$ be a biharmonic isometric immersion into a Riemannian manifold with constant sectional curvature K ($\neq 0$). If ϕ is $(2s+1)$-harmonic ($s \geq 1$), ϕ is harmonic.*

Proof. By Theorem 2.6, ϕ is $(2s+1)$-harmonic if and only if

$$\begin{aligned}
&\overline{\triangle}^{2s}\tau(\phi) - K\{m\overline{\triangle}^{2s-1}\tau(\phi) - \langle d\phi(e_j), \overline{\triangle}^{2s-1}\tau(\phi)\rangle d\phi(e_j)\} \\
&- \sum_{l=1}^{s-1}\{K(\langle\overline{\triangle}^{s-l-1}\tau(\phi), d\phi(e_j)\rangle\overline{\nabla}_{e_j}\overline{\triangle}^{s+l-1}\tau(\phi) \\
&\qquad - \langle d\phi(e_j), \overline{\nabla}_{e_j}\overline{\triangle}^{s+l-1}\tau(\phi)\rangle\overline{\triangle}^{s-l-1}\tau(\phi) \\
&\qquad - \langle\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi), d\phi(e_j)\rangle\overline{\triangle}^{s+l-1}\tau(\phi) \\
&\qquad + \langle d\phi(e_j), \overline{\triangle}^{s+l-1}\tau(\phi)\rangle\overline{\nabla}_{e_j}\overline{\triangle}^{s-l-1}\tau(\phi))\} \\
&- K\{\langle\overline{\triangle}^{s-1}\tau(\phi), d\phi(e_i)\rangle\overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(\phi) \\
&\qquad - \langle d\phi(e_i), \overline{\nabla}_{e_i}\overline{\triangle}^{s-1}\tau(\phi)\rangle\overline{\triangle}^{s-1}\tau(\phi)\} = 0.
\end{aligned}$$

By Proposition 3.1 and Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
0 &= (mK)^{2s}\tau(\phi) - (mK)^{2s}\tau(\phi) \\
&\quad - \sum_{l=1}^{s-1}\{K\left((mK)^{2s-2}||\tau(\phi)||^2\tau(\phi) + (mK)^{2s-2}||\tau(\phi)||^2\tau(\phi)\right)\} \\
&\quad - K(mK)^{2s-2}||\tau(\phi)||^2\tau(\phi) \\
&= -(2s-1)K(mK)^{2s-2}||\tau(\phi)||^2\tau(\phi).
\end{aligned}$$

Thus, we have the theorem. □

4. 3-harmonic maps into non-positive curvature

In this section we show the non-existence theorem of 3-harmonic maps. G. Y. Jiang showed the following.

Theorem 4.1 ([6])**.** *Assume that M is compact and N is a non-positive curvature, i.e., a Riemannian curvature of N, $K \leq 0$. Then every biharmonic map $\phi : M \to N$ is harmonic.*

We consider this theorem for 3-harmonic maps. First, we recall the following theorem.

Theorem 4.2 ([7])**.** *Let $l = 1, 2, \cdots$. If $\overline{\triangle}^{l}\tau(\phi) = 0$ or $\overline{\nabla}_{e_i}\overline{\triangle}^{(l-1)}\tau(\phi) = 0$ $(i = 1, 2, \cdots, m)$, then $\phi : M \to N$ from a compact Riemannian manifold into a Riemannian manifold is a harmonic map.*

Using this theorem, we obtain the next result.

Proposition 4.3. *Let $\phi : (M, g) \to (N, h)$ be an isometric immersion from a compact Riemannian manifold into a Riemannian manifold with non-positive constant sectional curvature $K \leq 0$. Then 3-harmonic is harmonic.*

Proof. Indeed, by computing the Laplacian of the 4-energy density $e_4(\phi)$, we have

$$\begin{aligned}
\triangle e_4(\phi) =& ||\overline{\nabla}_{e_i}\overline{\triangle}\tau(\phi)||^2 - \langle\overline{\triangle}^2\tau(\phi), \overline{\triangle}\tau(\phi)\rangle \\
=& ||\overline{\nabla}_{e_i}\overline{\triangle}\tau(\phi)||^2 \\
& - \langle R^N(\overline{\triangle}\tau(\phi), d\phi(e_i))d\phi(e_i), \overline{\triangle}\tau(\phi)\rangle \\
& - \langle R^N(\overline{\nabla}_{e_i}\tau(\phi), \tau(\phi))d\phi(e_i), \overline{\triangle}\tau(\phi)\rangle,
\end{aligned} \tag{9}$$

due to the fact that ϕ is 3-harmonic. Here, we consider the right hand side of (9):

$$\begin{aligned}\langle R^N(\overline{\nabla}_{e_i}\tau(\phi),\tau(\phi))d\phi(e_i),\overline{\triangle}\tau(\phi)\rangle &= \langle K\{\langle\tau(\phi),d\phi(e_i)\rangle\overline{\nabla}_{e_i}\tau(\phi)\\ &\quad -\langle d\phi(e_i),\overline{\nabla}_{e_i}\tau(\phi)\rangle\tau(\phi)\},\overline{\triangle}\tau(\phi)\rangle\\ &= K\{||\tau(\phi)||^2\langle\tau(\phi),\overline{\triangle}\tau(\phi)\rangle\}.\end{aligned}$$

Using Green's theorem, we have

$$\begin{aligned}0=\int_M \triangle e_4(\phi) = \int_M & ||\overline{\nabla}_{e_i}\overline{\triangle}\tau(\phi)||^2\\ & -\langle R^N(\overline{\triangle}\tau(\phi),d\phi(e_i))d\phi(e_i),\overline{\triangle}\tau(\phi)\rangle\\ & -K||\tau(\phi)||^2||\overline{\nabla}_{e_j}\tau(\phi)||^2 v_g.\end{aligned}\tag{10}$$

Then both terms of (10) are non-negative, so we have

$$\begin{aligned}0=\triangle e_4(\phi) = & ||\overline{\nabla}_{e_i}\overline{\triangle}\tau(\phi)||^2\\ & -\langle R^N(\overline{\triangle}\tau(\phi),d\phi(e_i))d\phi(e_i),\overline{\triangle}\tau(\phi)\rangle\\ & -K||\tau(\phi)||^2||\overline{\nabla}_{e_j}\tau(\phi)||^2.\end{aligned}\tag{11}$$

Especially, we have

$$\overline{\nabla}_{e_i}\overline{\triangle}\tau(\phi)=0.$$

Using Theorem 4.2, we obtain the proposition. □

5. k-harmonic curves into a Euclidean space

In this section, we consider k-harmonic curves into a Euclidean space $\mathbb{E}^n$ and we give a conjecture. B. Y. Chen [1] defined biharmonic submanifolds of Euclidean spaces.

Definition 5.1 ([1])**.** Let $x: M\to\mathbb{E}^n$ be an isometric immersion into a Euclidean space. $x: M\to\mathbb{E}^n$ is called a biharmonic submanifold if

$$\triangle^2 x = 0,\ \text{that is,}\ \triangle H = 0,\tag{12}$$

where $H=-\frac{1}{m}\triangle x$ is the mean curvature vector of the isometric immersion x and $\triangle$ is the Laplacian of M.

B. Y. Chen and S. Ishikawa [2] proved that any biharmonic surface in $\mathbb{E}^3$ is minimal. Chen [1] also gave a conjecture.

Conjecture 5.2 ([1])**.** *The only biharmonic submanifolds in Euclidean spaces are the minimal ones.*

There are several results for this conjecture ([5], [3] and [8], etc.). However, the conjecture is still open. I. Dimitric [3] considered a curve case ($m=1$) and obtained the following theorem.

Theorem 5.3 ([3])**.** *Let $x: C\to\mathbb{E}^n$ be a smooth curve parametrized by arc length, with the mean curvature vector H satisfying $\triangle H=0$. Then the curve is a straight line, i.e., totally geodesic in $\mathbb{E}^n$.*

We generalize this theorem. First, we define k-harmonic submanifolds in Euclidean spaces.

Definition 5.4. Let $x : M \to \mathbb{E}^n$ be an isometric immersion into a Euclidean space. $x : M \to \mathbb{E}^n$ is called a k-harmonic submanifold if

$$\triangle^k x = 0, \text{ that is, } \triangle^{k-1} H = 0 \ (k = 1, 2, \cdots), \tag{13}$$

where $H = -\frac{1}{m} \triangle x$ is the mean curvature vector of the isometric immersion x and $\triangle$ is the Laplacian of M.

We also consider a curve case ($m = 1$) and obtain the following theorem.

Theorem 5.5. *Let $x : C \to \mathbb{E}^n$ be a smooth curve parametrized by arc length, with the mean curvature vector H satisfying $\triangle^{k-1} H = 0$ ($k = 1, 2, \cdots$). Then the curve is a straight line, i.e., totally geodesic in $\mathbb{E}^n$.*

Proof. We have $0 = \triangle^{k-1} H = - \triangle^k x = (-1)^{k+1} \frac{d^{2k}}{ds^{2k}} x, \ k = 1, 2, \cdots$. Hence x has to be a $(2k-1)$-th power polynomial in s,

$$x = \frac{1}{2k-1} a_{2k-1} s^{2k-1} + \frac{1}{2k-2} a_{2k-2} s^{2k-2} + \cdots + a_1 s + a_0,$$

where a_i ($i = 0, 1, \cdots, 2k-1$) are constant vectors. Since s is the natural parameter we have

$$\begin{aligned} 1 =& \langle \frac{dx}{ds}, \frac{dx}{ds} \rangle = \langle \sum_{i=1}^{2k-1} a_i s^{i-1}, \sum_{i=1}^{2k-1} a_i s^{i-1} \rangle \\ =& |a_{2k-1}|^2 s^{4k-4} + 2\langle a_{2k-1}, a_{2k-2} \rangle s^{4k-5} + \{2\langle a_{2k-1}, a_{2k-3} \rangle + |a_{2k-2}|^2\} s^{4k-6} \\ & \cdots \\ & + \{2\langle a_1, a_3 \rangle + |a_2|^2\} s^2 + 2\langle a_1, a_2 \rangle s + |a_1|^2. \end{aligned}$$

On the right hand side we have a polynomial in s, so we must have

$$a_{2k-1} = a_{2k-2} = a_{2k-3} = a_{2k-4} = \cdots = a_3 = a_2 = 0, \ |a_1|^2 = 1.$$

In other words, $x(s) = a_1 s + a_0$ with $|a_1|^2 = 1$, and therefore the curve is a straight line. □

Conjecture 5.6. *The only k-harmonic submanifolds in Euclidean spaces are the minimal ones.*

Especially, when $k = 2$, it is the B. Y. Chen conjecture.

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