ON THE REGULARITY OF THE COMPLEX MONGE-AMPÈRE EQUATIONS

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ABSTRACT. We shall consider the regularity of solutions for the complex Monge-Ampère equations in $\mathbb{C}^n$ or a bounded domain. First we prove interior C^2 estimates of solutions in a bounded domain for the complex Monge-Ampère equations with the assumption of an L^p bound for Δu , $p > n^2$, and of a Lipschitz condition on the right-hand side. Then we shall construct a family of Pogorelov-type examples for the complex Monge-Ampère equations. These examples give generalized entire solutions (as well as viscosity solutions) of the complex Monge-Ampère equation $\det(u_{i\bar{j}}) = 1$ in $\mathbb{C}^n$.

1. INTRODUCTION

Let u be a continuous plurisubharmonic function in $\mathbb{C}^n$ (or a bounded domain $\Omega \subset \mathbb{C}^n$). We shall consider the complex Monge-Ampère equations

$$(1.1) \quad \det \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) = \psi.$$

For simplicity we use the notation

$$u_k = \frac{\partial u}{\partial z_k}, \quad u_{\bar{k}} = \frac{\partial u}{\partial \bar{z}_k}, \quad u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}, \quad \text{etc.}$$

We use $\Delta u = \sum_k u_{k\bar{k}}$ to denote the complex Laplacian operator.

These equations (1.1) have been studied extensively by many mathematicians, for example [1, 2, 3, 9, 12, 14, 15], etc., and many others. In particular, Bedford-Taylor [1] introduced generalized solutions for the complex Monge-Ampère equations; when Ω is a strongly pseudo-convex bounded domain, Bedford-Taylor [1, 2, 3] established the existence, uniqueness, and global Lipschitz regularity of generalized solutions for the Dirichlet problem

$$(1.2) \quad \begin{aligned} \det(u_{i\bar{j}}) &= \psi(z, u, \nabla u), \quad \text{in } \Omega, \\ u &= \varphi, \quad \text{on } \partial\Omega. \end{aligned}$$

Caffarelli-Kohn-Nirenberg-Spruck [9] proved the existence of classical plurisubharmonic solutions of (1.2) under suitable conditions on ψ and φ .

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In the present paper we shall first consider the *interior a priori* estimates of the complex Monge-Ampère equations

$$(1.3) \quad \log \det(u_{i\bar{j}}) = F$$

in a bounded domain $\Omega \subset \mathbb{C}^n$ ($n \geq 2$). The solution $u(z)$, of class $C^3(\Omega)$, is strictly plurisubharmonic, such that

$$(u_{i\bar{j}}) > 0, \forall z \in \Omega.$$

The given function F is of class $C^1(\Omega)$ and satisfies, for some positive constant Λ ,

$$|F| \leq \Lambda, |\nabla F| \leq \Lambda.$$

We can state our interior estimates as follows:

Theorem 1.1. *For any $P_0 > n^2$ and any domain $\Omega' \Subset \Omega$, if $\|\Delta u\|_{L^{P_0}(\Omega)}$ is bounded, then*

$$\sup_{\Omega'} \Delta u \leq C_1 = C_1(\Lambda, \Omega, \Omega', P_0, n, \|\Delta u\|_{L^{P_0}(\Omega)}).$$

Once Δu is bounded, one can apply a complex version of Evans-Krylov theory (see e.g. [16] or [5]) to get a $C^{2,\alpha}$ estimate for some $\alpha \in (0, 1)$. Once $C^{2,\alpha}$ estimate is established, we can consider the linearized equation

$$u^{i\bar{j}}(\nabla u)_{i\bar{j}} = \nabla F.$$

The standard L^p theory applies and for any $p > 1$, we can get that

$$\|u\|_{W^{3,p}(\Omega')} \leq C_2 = C_2(\Lambda, \Omega, \Omega', P_0, p, n, \|u\|_{W^{2,P_0}(\Omega)}).$$

It then follows that for any $\beta \in (0, 1)$,

$$\|u\|_{C^{2,\beta}(\Omega')} \leq C_2 = C_2(\Lambda, \Omega, \Omega', P_0, \beta, n, \|u\|_{W^{2,P_0}(\Omega)}).$$

Remark 1.2. Z. Blocki [4] proved a similar interior regularity theorem by assuming $F \in W^{2,2n}$ and $u \in W^{2,p}$ for $p > 2n(n-1)$. Blocki-Dinew [7] recently proved a local regularity result of Δu under the assumption that $u \in W^{2,p}$, $p > n(n-1)$ and $F \in C^{1,1}$. Note that the regularity of Δu is false if $p < n(n-1)$; see [4, 7] for the example and comments.

For fully nonlinear equations, the regularity of the right-hand side F does not imply necessarily the regularity of weak solutions. Pogorelov constructed a well-known example to the real Monge-Ampère equation with $u(x) = (1+x_1^2)(x_2^2+\cdots+x_n^2)^{1-1/n}$ for $n \geq 3$ in $B_\epsilon(0)$ for ϵ small; $u(x)$ is a weak solution of $\det(D^2u) = f$ in $B_\epsilon(0)$ such that $0 < f \in C^\infty$, while $u \in C^{1,\alpha}$ for $\alpha = 1 - 2/n$, but not any larger α . However, the interior C^2 estimates can be established if one assumes additional hypotheses, such as sufficient regularity of $\partial\Omega$ and $u|_{\partial\Omega}$, or sufficient interior regularity. In [12], B. Guan studied (1.2) for general domains assuming the existence of a sub-solution. In particular he proved, among others, the interior C^2 estimates (depending on ψ up to its second derivatives). When F is assumed to be only Lipschitz and $u|_{\partial\Omega} \equiv 0$, the interior C^2 estimates were studied by F. Schulz [15] by using an integral approach of N.M. Ivochikina [13] to the real Monge-Ampère equations. However, the proof in [15] is not complete (this was first pointed out by Z. Blocki [6]), where the inequality between (6) and (7) in [15] does not follow in the complex case. By using the integral approach, we shall prove the interior C^2 estimates by assuming an L^{P_0} bound of Δu for some $P_0 > n^2$. For real

Monge-Ampère equations, J. Urbas proved $C^{2,\beta}$ regularity of weak solution for any $\beta \in (0, 1)$ (see [17]) assuming $u \in C^{1,\alpha}$ or $u \in W^{2,q}$ for $\alpha > 1 - 2/n$, $q > n(n-1)/2$. Note that the bound on α in Urbas's result is sharp for $n \geq 3$; also for $q < n(n-1)/2$, the interior C^2 estimates fail for $n \geq 3$.

Then we shall construct Pogorelov-type examples for the complex Monge-Ampère equations. Using these examples we construct a generalized solution (Bedford-Taylor [1]), which is also a viscosity solution, of the complex Monge-Ampère equation in $\mathbb{C}^n$ such that

$$(1.4) \quad \det(u_{i\bar{j}}) = 1.$$

We have

Theorem 1.3. *Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and*

$$v(z_1, \dots, z_n) = n^{2/n}(1 + |z_1|^2 + \dots + |z_{n-1}|^2)|z_n|^{2/n}.$$

Then $v(z)$ is a generalized solution and a viscosity solution of (1.4).

Remark 1.4. The examples in Theorem 1.3 were also considered by Z. Blocki [4] when $n = 2$. When $u_{i\bar{j}}$ is assumed to be bounded, Riebeschl-Schulz [14] proved that the $u_{i\bar{j}}$ are all constants if a strictly plurisubharmonic function u solves (1.4).

2. INTERIOR C^2 ESTIMATES

In this section we prove Theorem 1.1. Note that u is of class C^3 and F is of class C^1 ; in the following, $(\Delta u)_{i\bar{j}}$, ΔF , etc., will be understood in the sense of distribution.

Proof. We compute, by taking the derivative of (1.3),

$$u^{i\bar{j}}u_{i\bar{j}k} = F_k.$$

Taking derivatives again, we get

$$(2.1) \quad -u^{i\bar{b}}u^{a\bar{j}}u_{a\bar{b}k}u_{i\bar{j}k} + u^{i\bar{j}}(\Delta u)_{i\bar{j}} = \Delta F.$$

Let ϕ be a test function with support in Ω . We compute the following for some $p \geq 1$:

$$(2.2) \quad \int_{\Omega} \phi^2 u^{i\bar{j}}((\Delta u)^p)_i((\Delta u)^p)_{\bar{j}} \Delta u = p^2 \int_{\Omega} \phi^2 (\Delta u)^{2p-1} u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}}.$$

Integrating by parts, we can get that

$$(2.3) \quad \begin{aligned} & \int_{\Omega} \phi^2 (\Delta u)^{2p-1} u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\ &= -2 \int_{\Omega} \phi \phi_{\bar{j}} (\Delta u)^{2p} u^{i\bar{j}}(\Delta u)_i - (2p-1) \int_{\Omega} \phi^2 (\Delta u)^{2p-1} u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\ & \quad + \int_{\Omega} \phi^2 (\Delta u)^{2p} u^{i\bar{l}} F_{\bar{l}}(\Delta u)_i - \int_{\Omega} \phi^2 (\Delta u)^{2p} u^{i\bar{j}}(\Delta u)_{i\bar{j}}. \end{aligned}$$

It follows from (2.1) and (2.3) that

$$\begin{aligned}
 (2.4) \quad & 2p \int_{\Omega} \phi^2(\Delta u)^{2p-1} u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\
 & = -2 \int_{\Omega} \phi \phi_{\bar{j}}(\Delta u)^{2p} u^{i\bar{j}}(\Delta u)_i + \int_{\Omega} \phi^2(\Delta u)^{2p} u^{i\bar{l}} f_{\bar{l}}(\Delta u)_i \\
 & \quad - \int_{\Omega} \phi^2(\Delta u)^{2p} \Delta F - \int_{\Omega} \phi^2(\Delta u)^{2p} u^{i\bar{b}} u^{a\bar{j}} u_{a\bar{b}\bar{k}} u_{i\bar{j}k}.
 \end{aligned}$$

Recall that ΔF is understood in the distribution sense. Hence we have

$$\begin{aligned}
 - \int_{\Omega} \phi^2(\Delta u)^{2p} \Delta F & = \int_{\Omega} (\phi^2(\Delta u)^{2p})_i F_{\bar{i}} \\
 & = \int_{\Omega} 2\phi \phi_i(\Delta u)^{2p} F_{\bar{i}} + \int_{\Omega} 2p\phi^2(\Delta u)^{2p-1}(\Delta u)_i F_{\bar{i}}.
 \end{aligned}$$

Inserting this into (2.4), it follows that

$$\begin{aligned}
 (2.5) \quad & 2p \int_{\Omega} \phi^2(\Delta u)^{2p-1} u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\
 & = -2 \int_{\Omega} \phi \phi_{\bar{j}}(\Delta u)^{2p} u^{i\bar{j}}(\Delta u)_i + \int_{\Omega} \phi^2(\Delta u)^{2p} u^{i\bar{l}} f_{\bar{l}}(\Delta u)_i + \int_{\Omega} 2\phi \phi_i(\Delta u)^{2p} F_{\bar{i}} \\
 & \quad + \int_{\Omega} 2p\phi^2(\Delta u)^{2p-1}(\Delta u)_i F_{\bar{i}} - \int_{\Omega} \phi^2(\Delta u)^{2p} u^{i\bar{b}} u^{a\bar{j}} u_{a\bar{b}\bar{k}} u_{i\bar{j}k}.
 \end{aligned}$$

To estimate the right-hand side in (2.5), we choose a coordinate such that at one point, $u_{i\bar{j}}$ is diagonalized. Then we compute, at the point,

$$\begin{aligned}
 \left| \phi \phi_{\bar{j}}(\Delta u)^{2p} u^{i\bar{j}}(\Delta u)_i \right| & = \left| \sum_i \frac{1}{u_{i\bar{i}}} \phi \phi_{\bar{i}}(\Delta u)^{2p}(\Delta u)_i \right| \\
 & = \left| \sum_i \frac{1}{u_{i\bar{i}}} \phi \phi_{\bar{i}}(\Delta u)^{2p} \sum_k u_{i\bar{k}k} \right| \\
 & \leq \varepsilon \phi^2(\Delta u)^{2p} \sum_{i,k} \frac{|u_{i\bar{k}k}|^2}{u_{i\bar{i}} u_{k\bar{k}}} + \frac{1}{4\varepsilon} |\nabla \phi|^2(\Delta u)^{2p} \sum_{i,k} \frac{u_{k\bar{k}}}{u_{i\bar{i}}} \\
 & \leq \varepsilon \phi^2(\Delta u)^{2p} u^{i\bar{b}} u^{a\bar{j}} u_{a\bar{b}\bar{k}} u_{i\bar{j}k} + \frac{e^{-\Lambda}}{4\varepsilon} |\nabla \phi|^2(\Delta u)^{2p+n}.
 \end{aligned}$$

Similarly we compute

$$\left| \phi^2(\Delta u)^{2p} u^{i\bar{l}} f_{\bar{l}}(\Delta u)_i \right| \leq \varepsilon \phi^2(\Delta u)^{2p} u^{i\bar{b}} u^{a\bar{j}} u_{a\bar{b}\bar{k}} u_{i\bar{j}k} + \frac{e^{-\Lambda}}{4\varepsilon} \phi^2 |\nabla F|^2(\Delta u)^{2p+n}$$

and

$$\left| 2p\phi^2(\Delta u)^{2p-1}(\Delta u)_i F_{\bar{i}} \right| \leq \varepsilon \phi^2(\Delta u)^{2p} u^{i\bar{b}} u^{a\bar{j}} u_{a\bar{b}\bar{k}} u_{i\bar{j}k} + e^{-\Lambda} \frac{p^2}{\varepsilon} \phi^2(\Delta u)^{2p+n-2} |\nabla F|^2.$$

Inserting the above into (2.5) (for example we can choose $\varepsilon = 1/4$ in the above inequalities), we can get

$$2p \int_{\Omega} \phi^2(\Delta u)^{2p-1} u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \leq C \int_{\Omega} (\phi^2 + |\nabla \phi|^2)(\Delta u)^{2p+n} + Cp^2 \int_{\Omega} \phi^2(\Delta u)^{2p+n-2},$$

where C denotes a universal constant depending only on Λ, n , and it can vary line by line. Inserting the above into (2.2), we can get that

$$(2.6) \quad \int_{\Omega} \phi^2 u^{i\bar{j}} ((\Delta u)^p)_i ((\Delta u)^p)_{\bar{j}} \Delta u \leq Cp^3 \int_{\Omega} (\phi^2 + |\nabla \phi|^2) (\Delta u)^{2p+n}.$$

Note that $u^{i\bar{j}} \Delta u \xi_i \bar{\xi}_j \geq |\xi|^2$ for any vector $\xi = (\xi_1, \dots, \xi_n)$. We can get that

$$(2.7) \quad \int_{\Omega} \phi^2 |\nabla ((\Delta u)^p)|^2 \leq Cp^3 \int_{\Omega} (\phi^2 + |\nabla \phi|^2) (\Delta u)^{2p+n}.$$

It follows that, from (2.7),

$$(2.8) \quad \int_{\Omega} |\nabla (\phi (\Delta u)^p)|^2 \leq Cp^3 \int_{\Omega} (\phi^2 + |\nabla \phi|^2) (\Delta u)^{2p+n},$$

where C depends only on Λ, n . Starting from (2.8), one can use the iteration technique (Moser's iteration) to get an interior L^∞ bound for Δu . By the Sobolev inequality, there exists a constant $c = c(n)$ such that

$$\|\phi (\Delta u)^p\|_{L^{2n/(n-1)}(\Omega)} \leq c \|\nabla (\phi (\Delta u)^p)\|_{L^2(\Omega)}.$$

It follows that

$$(2.9) \quad \|\phi (\Delta u)^p\|_{L^{2n/(n-1)}(\Omega)} \leq Cp^{3/2} \left(\int_{\Omega} (\phi^2 + |\nabla \phi|^2) (\Delta u)^{2p+n} \right)^{1/2}.$$

First we consider $\Omega = B_R(0), \Omega' = B_r(0)$ for $0 < r < R \leq 1$. Now we can choose the cutoff function ϕ and p . Let $r_k = r + (R - r)2^{-k}$, $k \in \mathbb{N}$. Let $p = p_k$ and $\phi = \phi_k$ in (2.9) such that $\phi_k \equiv 1$ in $B_{r_k}(0)$, $\phi_k \equiv 0$ outside $B_{r_{k-1}}(0)$, and $|\nabla \phi_k| \leq 2(r_{k-1} - r_k)^{-1}$. By (2.9), we get

$$(2.10) \quad \|\Delta u\|_{L^{\frac{2np_k}{n-1}}(B_{r_k}(0))} \leq a_k \|\Delta u\|_{L^{2p_k+n}(B_{r_{k-1}}(0))}^{b_k},$$

where

$$a_k = \left(\frac{Cp_k^{3/2}}{r_{k-1} - r_k} \right)^{1/p_k} = \left(\frac{C2^k p_k^{3/2}}{R - r} \right)^{1/p_k}, \quad b_k = \frac{2p_k + n}{2p_k}.$$

Now let $p_0 = a + n(n-1)/2, a > 0$ and $p_k, k \in \mathbb{N}$ satisfy

$$2p_{k+1} + n = \frac{2np_k}{n-1}.$$

It is easy to get that

$$p_k = a \left(\frac{n}{n-1} \right)^k + \frac{n(n-1)}{2}.$$

By (2.10), we can get that

$$(2.11) \quad \|\Delta u\|_{L^{\frac{2np_k}{n-1}}(B_{r_k}(0))} \leq a_k a_{k-1}^{b_k} a_{k-2}^{b_k b_{k-1}} \dots a_1^{b_k \dots b_2} \|\Delta u\|_{L^{\frac{2np_0}{n-1}}(B_{r_0}(0))}^{b_k \dots b_1}.$$

It is easy to show that

$$\lim_{k \rightarrow \infty} \prod_{i=1}^k b_i = p_0/a.$$

Without loss of generality, we assume $a_k \geq 1, \forall k$. It is also clear that

$$\sum_k \log a_k \leq C = C(R, r, a, n, \Lambda),$$

which implies that $a_k a_{k-1}^{b_k} a_{k-2}^{b_k b_{k-1}} \cdots a_1^{b_k \cdots b_2} \leq C = C(R, r, a, n, \Lambda)$. In (2.11), letting $k \rightarrow \infty$, we can get

$$(2.12) \quad \|\Delta u\|_{L^\infty(B_r(0))} \leq C(R, r, a, n, \Lambda) \|\Delta u\|_{L^{P_0}(B_R(0))}^{p_0/a},$$

where $P_0 = 2np_0/(n-1) = n^2 + 2na/(n-1)$. For general domains $\Omega' \Subset \Omega$, we can use finitely many concentric balls b_i, B_i such that $b_i \Subset B_i \Subset \Omega$ and $\overline{\Omega'} \subset \bigcup b_i$. Applying (2.12), one can easily prove that

$$\|\Delta u\|_{L^\infty(\Omega')} \leq C_1 = C_1(\Lambda, \Omega, \Omega', P_0, n, \|\Delta u\|_{L^{P_0}(\Omega)}).$$

If we assume that $\|u\|_{W^{2,p_0}(\Omega)}$ is bounded, then after obtaining interior C^2 estimates as above, one can prove interior $C^{2,\alpha}$ or $W^{3,p}$ estimates for any $\alpha \in (0, 1)$ or $p > 1$ by using Evans-Krylov theory, L^p theory and Schauder theory; for example, see [11] for details. $\square$

3. POGORELOV-TYPE EXAMPLES

In this section we prove Theorem 1.3.

Proof. First we consider $n = 2$. Let

$$(3.1) \quad u^\epsilon(z, w) = 2(1 + |z|^2)(|w|^2 + \epsilon)^{1/2}, \epsilon > 0, (z, w) \in \mathbb{C}^2$$

be a family of smooth functions defined in $\mathbb{C}^2$. A straightforward computation yields

$$\begin{aligned} u_{z\bar{z}}^\epsilon &= 2(|w|^2 + \epsilon)^{1/2}, u_{z\bar{w}}^\epsilon = \bar{z}w(|w|^2 + \epsilon)^{-1/2}, \\ u_{w\bar{w}}^\epsilon &= \frac{1 + |z|^2}{2}(|w|^2 + \epsilon)^{-3/2}(|w|^2 + 2\epsilon). \end{aligned}$$

It is clear that $u^\epsilon(\epsilon > 0)$ is a smooth plurisubharmonic function defined in $\mathbb{C}^2$ and satisfies the complex Monge-Ampère equation

$$(3.2) \quad u_{z\bar{z}}^\epsilon u_{w\bar{w}}^\epsilon - u_{z\bar{w}}^\epsilon u_{w\bar{z}}^\epsilon = F(z, w, \epsilon),$$

where

$$F(z, w, \epsilon) = |w|^2(|w|^2 + \epsilon)^{-1} + 2\epsilon(1 + |z|^2)(|w|^2 + \epsilon)^{-1}.$$

When $\epsilon \rightarrow 0$, u^ϵ converges to a continuous plurisubharmonic function $u^0 = (1 + |z|^2)|w|$. It is clear that $u^0 \in Lip_{loc}(\mathbb{C}^2) \cap W_{loc}^{2,p}(\mathbb{C}^2)$ for any $p \in (0, 2)$. So u^ϵ converges to u^0 in C^α with the $W^{2,p}$ norm on any compact subset of $\mathbb{C}^2$ for any $\alpha \in (0, 1)$ and $p \in (0, 2)$. It is clear that $F(z, w, \epsilon)$ converges to 1 in the L^∞ sense when $\epsilon \rightarrow 0$. In particular, u^0 is a generalized solution (in the sense of [1]) of the complex Monge-Ampère equation in $\mathbb{C}^2$ such that

$$(3.3) \quad u_{z\bar{z}}^0 u_{w\bar{w}}^0 - u_{z\bar{w}}^0 u_{w\bar{z}}^0 = 1.$$

Now we show that u^0 is also a viscosity solution. We refer to [10] for more details about viscosity solutions. For an $n \times n$ matrix X , define

$$G(X) = \begin{cases} 1 - \det(X), & X \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\partial\bar{\partial}u$ denote the complex Hessian of u . To show that u^0 is a viscosity solution of $G(\partial\bar{\partial}u) = 0$, we need to show that for any $v \in C^2$, if $u - v$ has a local minimum at p , then $G(\partial\bar{\partial}v) \geq 0$ at p , and if $u - v$ has a local maximum at p , then $G(\partial\bar{\partial}v) \leq 0$ at p . Since u^0 solves $G(\partial\bar{\partial}u) = 0$ when $|w| \neq 0$ in the classical sense, we just need to check for $|w| = 0$.

Let $z = x_1 + \sqrt{-1}y_1$, $w = x_2 + \sqrt{-1}y_2$. Suppose that $u - v$ has a local minimum at $(a, b, 0, 0)$. Then

$$u^0(x_1, y_1, x_2, y_2) - v(x_1, y_1, x_2, y_2) \geq u(a, b, 0, 0) - v(a, b, 0, 0)$$

for any (x_1, y_1, x_2, y_2) in a small neighborhood of $(a, b, 0, 0)$. So we get

$$v(x_1, y_1, x_2, y_2) - v(a, b, 0, 0) \leq (1 + x_1^2 + y_1^2)\sqrt{x_2^2 + y_2^2}.$$

Taking $x_2 = y_2 = 0$, this gives that

$$v(x_1, y_1, 0, 0) - v(a, b, 0, 0) \leq 0$$

for any (x_1, y_1) in a small neighborhood of (a, b) . This implies that $D_{x_1, y_1}^2 v(a, b, 0, 0) \leq 0$. In particular, $v_{z\bar{z}}(a, b, 0, 0) \leq 0$. Let X be the matrix of the complex Hessian of v at $(a, b, 0, 0)$. If $X \geq 0$, then $X \equiv 0$ and $G(X) = 1 > 0$; if $X \geq 0$ does not hold, then $G(X) = \infty > 0$.

Suppose $u - v$ has a local maximum at $(a, b, 0, 0)$. Then

$$u^0(x_1, y_1, x_2, y_2) - v(x_1, y_1, x_2, y_2) \leq u(a, b, 0, 0) - v(a, b, 0, 0).$$

So we get

$$v(x_1, y_1, x_2, y_2) - v(a, b, 0, 0) \geq (1 + x_1^2 + y_1^2)\sqrt{x_2^2 + y_2^2}$$

for any (x_1, y_1, x_2, y_2) in a small neighborhood of $(a, b, 0, 0)$. Taking $x_1 = a, y_1 = b$, we get that

$$v(a, b, x_2, y_2) - v(a, b, 0, 0) \geq (1 + a^2 + b^2)\sqrt{x_2^2 + y_2^2}.$$

But this contradicts $v \in C^2$. So there is no C^2 function v such that $u - v$ has a local maximum.

Then we shall consider $\mathbb{C}^{n+1}$ ($n \geq 2$). For $(z_1, z_2, \dots, z_n, w) \in \mathbb{C}^{n+1}$, let

$$(3.4) \quad u^\epsilon(z_1, \dots, z_n, w) = (1 + |z_1|^2 + \dots + |z_n|^2)(|w|^2 + \epsilon)^{1/n}.$$

A straightforward computation gives that

$$\begin{aligned} u_{z_i \bar{z}_j}^\epsilon &= (|w|^2 + \epsilon)^{1/(n+1)} \delta_{ij}, u_{z_i \bar{w}}^\epsilon = \frac{\bar{z}_i w}{n+1} (|w|^2 + \epsilon)^{-n/(n+1)}, \\ u_{w \bar{w}}^\epsilon &= \frac{1}{(n+1)^2} (1 + |z_1|^2 + \dots + |z_n|^2) (|w|^2 + \epsilon)^{-(2n+1)/(n+1)} (|w|^2 + (n+1)\epsilon). \end{aligned}$$

It is clear that $u^\epsilon(\epsilon > 0)$ is a smooth plurisubharmonic function in $\mathbb{C}^{n+1}$ which solves $(z_{n+1} = w)$

$$(3.5) \quad \det(u_{i\bar{j}}^\epsilon) = F(z_1, \dots, z_n, w, \epsilon),$$

where

$$F(z_1, \dots, z_n, w, \epsilon) = \frac{|w|^2}{(n+1)^2} (|w|^2 + \epsilon)^{-1} + \frac{\epsilon}{n+1} (1 + |z_1|^2 + \dots + |z_n|^2) (|w|^2 + \epsilon)^{-1}.$$

When $\epsilon \rightarrow 0$, $u^\epsilon \rightarrow u^0 = (1 + |z_1|^2 + \cdots + |z_n|^2)|w|^{2/(n+1)}$. Note that $u^0 \in C_{loc}^\alpha \cap W_{loc}^{2,p}$ for $\alpha = 2/(n+1)$ and $p < 1 + 1/n$. So u^ϵ converges to u^0 in C^α with the $W^{2,p}$ norm for $\alpha = 2/(n+1)$ and $p < 1 + 1/n$ on any compact subset of $\mathbb{C}^{n+1}$, while $F(z_1, \dots, z_n, w, \epsilon)$ converges to 1 in L^∞ . In particular, u^0 is a generalized solution of the complex Monge-Ampère equation in $\mathbb{C}^{n+1}$,

$$(3.6) \quad \det(u_{i\bar{j}}^0) = 1/(n+1)^2.$$

It is similar to check that $u^0(z_1, \dots, z_n, w)$ is a viscosity solution as in $n = 2$. Theorem 1.3 then follows from (3.3) and (3.6). $\square$

Remark 3.1. It is clear that $u = n^{2/n}(\sum_{i=1}^{n-1} |z_i|^2)|z_n|^{2/n}$ is a generalized solution of the degenerated complex Monge-Ampère equation

$$\det(u_{i\bar{j}}) = 0.$$

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