RIGIDITY OF STABLE CYLINDERS IN THREE-MANIFOLDS

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ABSTRACT. In this paper we show how the existence of a certain stable cylinder determines (locally) the ambient manifold where it is immersed. This cylinder has to verify a *bifurcation phenomenon*; we make this explicit in the introduction. In particular, the existence of such a stable cylinder implies that the ambient manifold has infinite volume.

1. INTRODUCTION

A stable compact domain Σ on a minimal surface in a Riemannian three-manifold $\mathcal{M}$ is one whose area cannot be decreased up to second order by a variation of the domain leaving the boundary fixed. Stable oriented domains Σ are characterized by the *stability inequality* for normal variations ψN [11]

$$\int_{\Sigma} \psi^2 |A|^2 + \int_{\Sigma} \psi^2 \text{Ric}_{\mathcal{M}}(N, N) \leq \int_{\Sigma} |\nabla \psi|^2$$

for all compactly supported functions $\psi \in H_0^{1,2}(\Sigma)$. Here $|A|^2$ denotes the square of the length of the second fundamental form of Σ , $\text{Ric}_{\mathcal{M}}(N, N)$ is the Ricci curvature of $\mathcal{M}$ in the direction of the normal N to Σ and ∇ is the gradient w.r.t. the induced metric.

One writes the stability inequality in the form

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \text{Area}(\Sigma(t)) = - \int_{\Sigma} \psi L \psi \geq 0,$$

where L is the linearized operator of the mean curvature

$$L = \Delta + |A|^2 + \text{Ric}_{\mathcal{M}}.$$

In terms of L , stability means that $-L$ is nonnegative; i.e., all its eigenvalues are nonnegative. Σ is said to have finite index if $-L$ has only finitely many negative eigenvalues.

From the Gauss equation, one can write the stability operator as $L = \Delta - K + V$, where Δ and K are the Laplacian and Gauss curvature associated to the metric g respectively and $V := 1/2|A|^2 + S$, where S denotes the scalar curvature associated to the metric g .

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The index form of this kind of operators is

$$I(f) = \int_{\Sigma} \left\{ \|\nabla f\|^2 - Vf^2 + Kf^2 \right\}$$

where ∇ and $\|\cdot\|$ are the gradient and norm associated to the metric g . Thus, if Σ is stable, we have

$$\int_{\Sigma} fLf = -I(f) \leq 0,$$

or equivalently

$$(1.1) \quad \int_{\Sigma} f^2(1/2|A|^2 + S) \leq \int_{\Sigma} \left\{ \|\nabla f\|^2 + Kf^2 \right\}.$$

In a seminar paper [5], D. Fischer-Colbrie and R. Schoen proved:

Theorem A. *Let $\mathcal{M}$ be a complete oriented three-manifold of nonnegative scalar curvature. Let Σ be an oriented complete stable minimal surface in $\mathcal{M}$. If Σ is noncompact, conformally equivalent to the cylinder and the absolute total curvature of Σ is finite, then Σ is flat and totally geodesic.*

They also state [5, Remark 2]: *We feel that the assumption of finite total curvature should not be essential in proving that the cylinder is flat and totally geodesic.*

Recently, this question was partially answered in [3] under the assumption that the positive part of the Gaussian curvature is integrable, i.e., $K^+ := \max\{0, K\} \in L^1(\Sigma)$. It was totally answered by M. Reiris [10]; he proved:

Theorem B. *Let $\mathcal{M}$ be a complete oriented three-manifold of nonnegative scalar curvature. Let Σ be an oriented complete stable minimal surface in $\mathcal{M}$ diffeomorphic to the cylinder. Then Σ is flat and totally geodesic.*

Besides, Bray, Brendle and Neves [1] were able to determine the structure of a three-manifold $\mathcal{M}$ under the assumption of the existence of an area minimizing two-sphere. Specifically, they proved:

Theorem C. *Let $\mathcal{M}$ be a compact three-manifold with $\pi_2(\mathcal{M}) \neq 0$. Denote by $\mathcal{F}$ the set of all smooth maps $f : \mathbb{S}^2 \rightarrow \mathcal{M}$ which represent a nontrivial element of $\pi_2(\mathcal{M})$. Set*

$$\mathcal{A}(\mathcal{M}) := \inf \left\{ \text{area}(f(\mathbb{S}^2)) : f \in \mathcal{F} \right\}.$$

Then

$$\mathcal{A}(\mathcal{M}) \inf_{\mathcal{M}} R \leq 8\pi,$$

where R denotes the scalar curvature of $\mathcal{M}$. Moreover, if the equality holds, then the universal cover of $\mathcal{M}$ is isometric to the standard cylinder $\mathbb{S}^2 \times \mathbb{R}$ up to scaling.

In this paper, we will go further. We will see how the existence of a stable cylinder verifying a *bifurcation phenomenon* determines the ambient manifold $\mathcal{M}$. First, let us make clear what we mean by *bifurcation phenomenon*:

Definition 1.1. We say that a complete minimal surface $\Sigma \subset \mathcal{M}$ bifurcates if there exist $\delta > 0$ and a smooth map $u : \Sigma \times (-\delta, \delta) \rightarrow \mathbb{R}$ so that:

- For each $p \in \Sigma$, we have $u(x, 0) = 0$ and $\frac{\partial}{\partial t}|_{t=0}u(p, t) = 1$. Moreover, $u(p, t) \geq 0$ if $t > 0$ and $u(p, t) \leq 0$ if $t < 0$.

- For each $t \in (-\delta, \delta)$, the surface

$$\Sigma_t := \{\exp_p(u(p, t)N(p)) : p \in \Sigma\}$$

is a complete minimal surface. Here, $\exp$ denotes the exponential map in $\mathcal{M}$.

Now, we can state

Theorem 1.1. *Let $\mathcal{M}$ be a complete oriented Riemannian three-manifold with non-negative scalar curvature. Assume there exists $\Sigma \subset \mathcal{M}$ a complete stable minimal surface conformally equivalent to a cylinder that bifurcates. Then Σ is flat, totally geodesic, and S vanishes along Σ . Moreover, there exists an open set $\mathcal{U} \subset \mathcal{M}$ so that $\mathcal{U}$ is locally isometric to $\mathcal{C} \times (-\delta, \delta)$, where $\mathcal{C}$ denotes the standard cylinder $\mathbb{S}^1 \times \mathbb{R}$. Also, if any complete stable cylinder in $\mathcal{M}$ bifurcates for a uniform $\delta > 0$, then $\mathcal{M}$ is locally isometric either to $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{T}^2 \times \mathbb{R}$ (here $\mathbb{T}^2$ is the flat tori).*

We should point out that the condition that Σ bifurcates is necessary. In fact, one can construct the following example: Let $\mathcal{C}(-l, l)$ be the right cylinder of height $2l$ and radius 1 endowed with the flat metric. Close it up with two spherical caps S_i , $i = 1, 2$ (one on the top and another on the bottom). Now, smooth the surface $\mathcal{M}^2 = \mathcal{C}(-l, l) \cup S_1 \cup S_2$ so that it is flat on $\mathcal{C}(-l + \varepsilon, l - \varepsilon)$, for some $\varepsilon > 0$, and has nonnegative Gaussian curvature.

Consider the three-manifold $\mathcal{M}^3 = \mathcal{M}^2 \times \mathbb{R}$. One can see that, if we take a closed geodesic $\gamma(t) \subset \mathcal{C}(-l + \varepsilon, l - \varepsilon)$, $t \in (-l + \varepsilon, l - \varepsilon)$, the surface $\Sigma(t) := \gamma(t) \times \mathbb{R}$ is a complete stable minimal cylinder in $\mathcal{M}$ that bifurcates, but, when we reach $t = l - \varepsilon$, this property might disappear (it could bifurcate as constant mean curvature surfaces at one side, but not minimal).

One interesting consequence of Theorem 1.1 is the following:

Corollary 1.1. *Let $\mathcal{M}$ be a complete oriented Riemannian three-manifold with nonnegative scalar curvature. Assume there exists $\Sigma \subset \mathcal{M}$ a complete stable minimal surface conformally equivalent to a cylinder that bifurcates. Then,*

$$\text{Vol}(\mathcal{M}) = +\infty.$$

Actually, the above conclusion (that is, Corollary 1.1) is also valid when the cylinder bifurcates only at one side.

2. PRELIMINARIES

We denote by $\mathcal{M}$ a complete connected orientable Riemannian three-manifold, with Riemannian metric g . Moreover, throughout this work, we will assume that its scalar curvature is nonnegative, i.e., $S \geq 0$. Also, $\Sigma \subset \mathcal{M}$ will be assumed to be connected and oriented.

We denote by N the unit normal vector field along Σ . Let $p_0 \in \Sigma$ be a point of the surface and let $D(p_0, s)$, for $s > 0$, denote the geodesic disk centered at p_0 of radius s . We assume that $\overline{D(p_0, s)} \cap \partial\Sigma = \emptyset$. Moreover, let r be the radial distance of a point p in $D(p_0, s)$ to p_0 . We write $D(s) = D(p_0, s)$.

We also denote

$$\begin{aligned} l(s) &= \text{Length}(\partial D(s)), \\ a(s) &= \text{Area}(D(s)), \\ K(s) &= \int_{D(s)} K, \\ \chi(s) &= \text{Euler characteristic of } D(s). \end{aligned}$$

Let $\Sigma \subset \mathcal{M}$ be a stable minimal surface diffeomorphic to the cylinder. Then, from Theorem B [10], Σ is flat and totally geodesic. We will give a (more general) proof of this result in the abstract setting of Schrödinger-type operators:

Lemma 2.1. *Let Σ be a complete Riemannian surface. Let $L = \Delta + V - aK$ be a differential operator on Σ acting on compactly supported $f \in H_0^{1,2}(\Sigma)$, where $a > 1/4$ is constant, $V \geq 0$, and Δ and K are the Laplacian and Gauss curvature associated to the metric g respectively.*

Assume that Σ is homeomorphic to the cylinder and $-L$ is nonnegative. Then, $V \equiv 0$ and $K \equiv 0$. Therefore,

$$\text{Ker } L := \{1\};$$

i.e., its kernel is the constant functions. Here, L denotes the Jacobi operator.

Proof. Set $b \geq 1$ and let us consider the radial function

$$f(r) := \begin{cases} (1 - r/s)^b, & r \leq s, \\ 0, & r > s, \end{cases}$$

where r denotes the radial distance from a point $p_0 \in \Sigma$. Then, from [3, Lemma 3.1] (see also [9]), we have

$$\int_{D(s)} (1 - r/s)^{2b} V \leq 2a\pi G(s) + \frac{b(b(1 - 4a) + 2a)}{s^2} \int_0^s (1 - r/s)^{2b-2} l(r),$$

where

$$G(s) := - \int_0^s (f(r)^2)' \chi(r).$$

Therefore, since $a > 1/4$, we can find $b \geq 1$ so that $b(1 - 4a) + 2a \leq 0$. So

$$\int_{D(s)} (1 - r/s)^{2b} V \leq 2a\pi G(s).$$

- **Step 1:** V vanishes identically on Σ .

Suppose there exists a point $p_0 \in \Sigma$ so that $V(p_0) > 0$. From now on, we fix the point p_0 . Then, there exists $\epsilon > 0$ so that $V(q) \geq \delta$ for all $q \in D(\epsilon) = D(p_0, \epsilon)$. Since Σ is topologically a cylinder, there exists $s_0 > 0$ so that for all $s > s_0$ we have $\chi(s) \leq 0$ (see [2, Lemma 1.4]).

Now, from the above considerations, there exists $\beta > 0$ so that

$$0 < \beta \leq 2a\pi G(s).$$

But, following [3], we can see that

$$\begin{aligned} G(s) &= - \int_0^s (f(r)^2)' \chi(r) = - \int_0^{s_0} (f(r)^2)' \chi(r) - \int_{s_0}^s (f(r)^2)' \chi(r) \\ &\leq - \int_0^{s_0} (f(r)^2)' = - (f(s_0)^2 - f(0)^2) = -f(s_0)^2 + 1 \\ &= - (1 - s_0/s)^{2b} + 1, \end{aligned}$$

since $-\int_{s_0}^s (f(r)^2)' \chi(r) \geq 0$. Therefore,

$$G(s) \leq 1 - (1 - s_0/s)^{2b} \rightarrow 0, \text{ as } s \rightarrow +\infty,$$

which is a contradiction. Thus, V vanishes identically along Σ .

- **Step 2:** K vanishes identically on Σ . In particular, Σ is parabolic.

First, note that $L := \Delta - aK$. From [4], there is a smooth positive function u on Σ such that $Lu = 0$. Set $\alpha := 1/a$. Then, from [9] (following ideas of [4]), the conformal metric $\tilde{ds}^2 := u^{2\alpha} ds^2$, where ds^2 is the metric on Σ , is complete and its Gaussian curvature $\tilde{K}$ is nonnegative; i.e., $\tilde{K} \geq 0$.

On the one hand, the respective Gaussian curvatures are related by

$$\alpha \Delta \ln u = K - \tilde{K} u^{2\alpha}.$$

On the other hand, since Σ is topologically a cylinder, the Cohn-Vossen inequality says

$$\int_{\Sigma} \tilde{K} \leq 0;$$

that is, $\tilde{K}$ vanishes identically.

Thus, $K = \alpha \Delta \ln u$. From this last equation, we get

$$aK = \frac{1}{u} \Delta u - \frac{|\nabla u|^2}{u^2};$$

that is,

$$\frac{|\nabla u|^2}{u} = \Delta u - aKu = 0.$$

This last equation implies that u is constant, and since u satisfies $Lu = 0$, we have that K vanishes identically on Σ . In particular, Σ is parabolic (see [6, Lemma 5]).

This implies that the Jacobi operator becomes $L := \Delta$, and so the constant functions are in the kernel. But, since Σ is parabolic, such a kernel has dimension one (see [8]); therefore

$$\text{Ker } L := \{1\}. \quad \square$$

Let $\mathcal{C} := \mathbb{S}^1 \times \mathbb{R}$ be the flat cylinder. Then we can parametrize Σ as the isometric immersion $\psi_0 : \mathcal{C} \rightarrow \mathcal{M}$ where $\Sigma := \psi_0(\mathcal{C})$. Also, let $N_0 : \mathcal{C} \rightarrow N\Sigma$ be the unit normal vector field along Σ .

Assume Σ bifurcates (see Definition 1.1). Then there exist $\delta > 0$ and a smooth map $u : \mathcal{C} \times (-\delta, \delta) \rightarrow \mathbb{R}$ so that the surface $\Sigma_t := \psi_t(\mathcal{C})$, $\psi_t : \mathcal{C} \rightarrow \mathcal{M}$ where

$$\psi_t(p) := \exp_{\psi_0(p)}(u(p, t)N_0(p)), \quad p \in \mathcal{C},$$

is a complete minimal surface.

For each $t \in (-\delta, \delta)$, the lapse function $\rho_t : \Sigma \rightarrow \mathbb{R}$ is defined by

$$\rho_t(p) = g \left(N_t(p), \frac{\partial}{\partial t} \psi_t(p) \right).$$

Clearly, $\rho_0(p) = 1$ for all $p \in \mathcal{C}$. Also, the lapse function satisfies the Jacobi equation

$$(2.1) \quad \Delta_t \rho_t + (\text{Ric}(N_t) + |A_t|^2) \rho_t = 0,$$

since $\psi_t(\mathcal{C})$ is minimal for all $|t| < \delta$.

Lemma 2.2. *There exists $0 < \delta' < \delta$ such that Σ_t is a stable minimal surface for each $t \in (-\delta, \delta)$. Thus, Σ_t is flat, totally geodesic, and S vanishes along Σ_t for each $t \in (-\delta, \delta)$.*

Proof. First, note that the lapse function is not negative for all $|t| < \delta$ and therefore, by (2.1) and the Maximum Principle, either ρ_t vanishes identically or $\rho_t > 0$ for each $|t| < \delta$.

So, since

$$\rho_t \rightarrow \rho_0 \equiv 1 \text{ as } t \rightarrow 0,$$

we can find a uniform constant $0 < \delta' < \delta$ such that $\rho_t > 0$ for all $|t| \leq \delta'$.

Therefore, ρ_t , $|t| \leq \delta'$, is a positive function solving the Jacobi equation. This implies that Σ_t is stable for all $|t| \leq \delta'$ (see [4]).

The last assertion follows from Lemma 2.1 and Σ_t is stable. $\square$

3. PROOF OF THEOREM 1.1

From Definition 1.1 and Lemma 2.2, there exists $\delta > 0$ so that Σ_t is a complete minimal stable surface, which is flat, totally geodesic and $S = 0$ along Σ_t , for each $|t| < \delta$.

Now, we follow ideas of [1]. Since $\text{Ric}(N_t) + |A_t|^2 \equiv 0$ and $H(t) = 0$ for each $|t| < \delta$, from (2.1) and Σ_t being parabolic, we obtain that ρ_t is constant. Thus, since Σ_t is totally geodesic,

$$\begin{aligned} Y : \mathcal{C} \times (-\delta, \delta) &\rightarrow \mathcal{M} \\ (p, t) &\rightarrow Y(p, t) := N_t(p) \end{aligned}$$

is parallel. Also, the flow of N_t is a unit speed geodesic flow (see [7]). Moreover, the map

$$\begin{aligned} \Phi : \Sigma \times (-\delta, \delta) &\rightarrow \mathcal{M} \\ (p, t) &\rightarrow \Phi(p, t) := \exp_{\psi_0(p)}(t N(p)) \end{aligned}$$

is a local isometry onto $\mathcal{U} = \bigcup_{|t| < \delta} \Sigma_t$. Therefore, Φ is a diffeomorphism onto $\mathcal{U}$, which implies that $Y : \mathcal{C} \times (-\delta, \delta) \rightarrow \mathcal{U}$ is a globally defined unit Killing vector field. This implies that $\mathcal{U}$ is locally isometric to $\mathcal{C} \times (-\delta, \delta)$.

Now, assume that any stable minimal complete cylinder bifurcates for a uniform $\delta > 0$. Then, we can start with a complete stable minimal cylinder Σ_0 that bifurcates, and then by the above considerations, Σ_t , for each $|t| < \delta$, is complete, flat, totally geodesic and S vanishes along Σ_t . Moreover, Σ_t is strongly stable for each $|t| < \delta$. Note that Σ_δ is a strongly stable minimal surfaces conformally equivalent to a cylinder. Since it is the limit of strongly stable minimal surfaces Σ_t which are flat and totally geodesic, then Σ_δ is totally geodesic, flat and $S = 0$ along Σ_δ . Therefore, by Definition 1.1 and Lemma 2.2, there exists $\delta > 0$ so that Σ_t , $-\delta < t < 2\delta$,

is flat, totally geodesic and S vanishes along Σ_t . Continuing this argument, Σ_t is flat, totally geodesic and S vanishes along Σ_t for each $t \in \mathbb{I}$, where $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = \mathbb{S}^1$.

As we did above, since $\text{Ric}(N_t) + |A_t|^2 \equiv 0$ and $H(t) = 0$ for each $t \in \mathbb{I}$, from (2.1) and Σ_t being parabolic, we obtain that ρ_t is constant. Thus, since Σ_t is totally geodesic,

$$\begin{aligned} Y : \mathcal{C} \times \mathbb{I} &\rightarrow \mathcal{M} \\ (p, t) &\rightarrow Y(p, t) := N_t(p) \end{aligned}$$

is parallel, where $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = \mathbb{S}^1$. Also, the flow of N_t is a unit speed geodesic flow (see [7]). Moreover, the map

$$\begin{aligned} \Phi : \Sigma \times \mathbb{I} &\rightarrow \mathcal{M} \\ (p, t) &\rightarrow \Phi(p, t) := \exp_{\psi_0(p)}(t N(p)) \end{aligned}$$

is a local isometry, which implies that it is a covering map. Therefore, Φ is a diffeomorphism, which implies that $Y : \mathcal{C} \times \mathbb{I} \rightarrow \mathcal{M}$ is a globally defined unit Killing vector field. This implies that $\mathcal{M}$ is locally isometric either to $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{T}^2 \times \mathbb{R}$ (here $\mathbb{T}^2$ denotes the flat tori).

REFERENCES

- [1] H. Bray, S. Brendle and A. Neves: Rigidity of area-minimizing two-spheres in three-manifolds. *Comm. Anal. Geom.* **18** (2010), no. 4, 821–830. MR2765731
- [2] P. Castillon: An inverse spectral problem on surfaces, *Comment. Math. Helv.* **81** (2006), no. 2, 271–286. MR2225628 (2007b:58042)
- [3] J.M. Espinar and H. Rosenberg: A Colding-Minicozzi stability inequality and its applications. *Trans. A.M.S.* **363** (2011), 2447–2465. MR2763722
- [4] D. Fischer-Colbrie: On complete minimal surfaces with finite Morse index in three manifolds, *Invent. Math.* **82** (1985), 121–132. MR808112 (87b:53090)
- [5] D. Fischer-Colbrie and R. Schoen: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, *Comm. Pure Applied Math.* **33** (1980), 199–211. MR562550 (81i:53044)
- [6] T. Klotz and R. Osserman: Complete surfaces in E^3 with constant mean curvature, *Comment. Math. Helv.* **41** (1966-67), 313–318. MR0211332 (35:2213)
- [7] S. Montiel: Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds, *Indiana Univ. Math. J.* **48** (1999), 711–748. MR1722814 (2001f:53131)
- [8] J. M. Manzano, J. Pérez and M. M. Rodríguez: Parabolic stable surfaces with constant mean curvature. To appear in *Calc. Var. Partial Differential Equations*.
- [9] W. Meeks, J. Pérez and A. Ros: Stable constant mean curvature surfaces, *Handbook of Geometric Analysis*, volume 1 (2008), pages 301–380. International Press, edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. MR2483369 (2009k:53016)
- [10] M. Reiris: Geometric relations of stable minimal surfaces and applications. Preprint.
- [11] R. Schoen and S.T. Yau: Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature, *Comm. Math. Helv.* **39** (1976), 333–341. MR0438388 (55:11302)

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