LORENTZ NORM INEQUALITIES FOR THE HARDY OPERATOR INVOLVING SUPREMA

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ABSTRACT. The weighted Lorentz norm inequalities for the Hardy operator involving suprema are characterized.

INTRODUCTION

Let $b \in (0, \infty]$. Denote by $\mathfrak{M}^+$ the class of all nonnegative Lebesgue measurable functions on $(0, b)$. If $w, f \in \mathfrak{M}^+$, the distribution function f_* of f with respect to w is given by

$$f_*(x) = \int_{\{|f|>x\}} w, \quad x \in [0, \infty).$$

For $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathfrak{M}^+$, the Lorentz space $L^{p,q}(w)$ consists of all functions f satisfying $\|f\|_{L^{p,q}(w)} < \infty$, where

$$\|f\|_{L^{p,q}(w)} = \begin{cases} \left[q \int_0^\infty t^{q-1} f_*(t)^{\frac{q}{p}} dt \right]^{\frac{1}{q}} & \text{for } q \in (0, \infty), \\ \sup_{t \in (0, \infty)} t f_*(t)^{\frac{1}{p}} & \text{for } q = \infty. \end{cases}$$

In case $p = q$ the Lorentz space $L^{p,q}(w)$ coincides with the Lebesgue space $L^q(w)$ with the norm $\|f\|_{L^q(w)} = \|fw^{\frac{1}{q}}\|_q$.

The main use of the Lorentz space is the interpolation theory, where the standard reference is [2], as well as many other applications. However, the study of the mapping properties of operators of classical analysis in the Lorentz spaces had also attracted much attention. In particular, it is concerned with the weighted Hardy type operators, for which boundedness in the Lorentz spaces was intensively studied by many authors. We mention here just a few of the contributors [1, 3, 4, 5, 9, 11, 12].

On the other hand, some operators involving suprema had been recently encountered in various research projects; see [6, 7] for the weighted Hardy operators, and [10] with the supplement for the Riemann-Liouville fractional integrals. These operators may be found in many branches of analysis and applications to PDE and function spaces theory (see [6] for additional references).

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In detail the paper [6] deals with the characterization of the weighted Lebesgue norm inequalities for the operator

$$(0.1) \quad (Tf)(x) = \sup_{x \leq \tau < b} u(\tau) \int_0^\tau f(y) \sigma(y) dy,$$

where u is a continuous function on $(0, b)$, $\sigma \in \mathfrak{M}^+$.

We characterize weighted Lorentz norm inequalities

$$(0.2) \quad \|Tf\|_{L^{p,q}(w)} \leq C \|f\|_{L^{r,s}(v)} \quad (f \in \mathfrak{M}^+)$$

for the Hardy operator involving suprema (0.1).

Throughout this paper, $A \lesssim B$ and $B \gtrsim A$ mean that $A \leq cB$, where the constant c depends only on p, q, α and may be different in different places. If both $A \lesssim B$ and $A \gtrsim B$, then we write $A \approx B$.

1. MAIN RESULTS

First we will give a couple of auxiliary results.

Lemma 1.1. *Let g be a continuous function on $(0, b)$ such that*

$$f(x) := \sup_{\tau \in [x, b)} g(\tau) < \infty$$

for all $x \in (0, b)$. Then f is a continuous on $(0, b)$.

In paper [11] it was shown (see [11, equality (2.6)]) that

$$\|h\|_{L^{p,q}(w)} = \left(\frac{q}{p}\right)^{\frac{1}{q}} \|h\|_{L^q(\tilde{w})},$$

where $\tilde{w}(x) = w(x) \left(\int_x^\infty w\right)^{\frac{q}{p}-1}$, for an arbitrary nonnegative nondecreasing function h . The following lemma is an analogue of this result for a nonnegative nonincreasing function.

Lemma 1.2. *Let h be a nonincreasing and nonnegative function on $(0, b)$, $w \in \mathfrak{M}^+$, $p, q \in (0, \infty)$, and $\rho(x) = w(x) \left(\int_0^x w\right)^{\frac{q}{p}-1}$. Then*

$$\|h\|_{L^{p,q}(w)} = \left(\frac{q}{p}\right)^{\frac{1}{q}} \|h\|_{L^q(\rho)}.$$

The main results are contained in the following two theorems.

Theorem 1.3. *Let $r, q \in (1, \infty)$, $s \in [1, \infty]$, $p \in (0, \infty)$, $\max\{r, s\} \leq q$. Let u be a continuous function on $(0, b)$, $w, v, \sigma \in \mathfrak{M}^+$, $\tilde{u}(x) = \sup_{x \leq \tau < b} u(\tau)$. The inequality (0.2) holds iff $A < \infty$, where*

$$A := \sup_{0 < x < b} \left[\tilde{u}(x) \left[\int_0^x w \right]^{\frac{1}{p}} + \left[\int_x^b \tilde{u}(t)^q \left[\int_0^t w \right]^{\frac{q}{p}-1} w(t) dt \right]^{\frac{1}{q}} \right] \left\| \frac{\sigma \chi_{(0,x]}}{v} \right\|_{L^{r',s'}(v)}.$$

Proof. Sufficiency. By [6, Lemma 2.2 (i)], we have

$$(Tf)(x) = \sup_{x \leq \tau < b} \tilde{u}(\tau) \int_0^\tau f(y) \sigma(y) dy.$$

Let $\{b_n\}_1^\infty \subset (0, b)$ be a sequence such that $b_n \uparrow b$ as $n \rightarrow \infty$. Fix an arbitrary function $f \in \mathfrak{M}^+$ and put $f_n := \min\{n, f\}\chi_{(0, b_n]}$, $\sigma_n := \min\{n, \sigma\}$, $\tilde{u}_n := \min\{n, \tilde{u}\}$. Then

$$(T_n f_n)(x) = \sup_{x \leq \tau < b} \tilde{u}_n(\tau) \int_0^\tau f_n(y) \sigma_n(y) dy < \infty$$

for all $x \in (0, b)$.

Fix an arbitrary $n \in \mathbb{N}$. We will prove the following relation:

$$(1.1) \quad \|T_n f_n\|_{L^{p,q}(w)} \lesssim A \|f\|_{L^{r,s}(v)}.$$

If $(T_n f_n)(x) = 0$ for every $x \in (0, b)$, then (1.1) holds. So we assume that there exists a point $x_0 \in (0, b)$ such that $(T_n f_n)(x_0) > 0$. Put $E_k := \{x \in (0, b) \mid (T_n f_n)(x) > 2^k\}$, $k \in \mathbb{Z}$, and

$$K_1 := \inf\{k \in \mathbb{Z} \mid (0, b) \setminus E_k \neq \emptyset\} - 1, \quad K_2 := \max\{k \in \mathbb{Z} \mid E_k \neq \emptyset\} + 1.$$

We have

$$(1.2) \quad \|T_n f_n\|_{L^{p,q}(w)}^q = \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} q t^{q-1} (T_n f_n)_*(t)^{\frac{q}{p}} dt \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} \left(\int_{E_k} w \right)^{\frac{q}{p}} \\ = \sum_{k < K_2} 2^{kq} \left(\int_{E_k} w \right)^{\frac{q}{p}} \approx 2^{K_1 q} \left(\int_0^b w \right)^{\frac{q}{p}} + \sum_{K_1 < k < K_2} 2^{kq} \left(\int_0^{x_k} w \right)^{\frac{q}{p}},$$

where $x_k := \sup E_k$, $K_1 < k < K_2$. Note that $(T_n f_n)(x_k) = 2^k$ for $K_1 < k < K_2$ by the continuity (by Lemma 1.1) of the function $T_n f_n$. Since $(T_n f_n)(x_{k-1}) = 2^{k-1} < (T_n f_n)(x_k)$, we have that

$$(1.3) \quad (T_n f_n)(x_k) = \sup_{x_k \leq \tau \leq x_{k-1}} \tilde{u}_n(\tau) \int_0^\tau f_n \sigma_n > u_n(x_{k-1}) \int_0^{x_{k-1}} f_n \sigma_n.$$

The first term of (1.2) is estimated in the following way:

$$(1.4) \quad 2^{K_1 q} \left(\int_0^b w \right)^{\frac{q}{p}} = \sup_{x > b_n} 2^{K_1 q} \left(\int_0^x w \right)^{\frac{q}{p}} \lesssim \sup_{x > b_n} (T_n f_n)(x)^q \left(\int_0^x w \right)^{\frac{q}{p}} \\ = \sup_{x > b_n} \tilde{u}_n(x)^q \left(\int_0^x w \right)^{\frac{q}{p}} \left[\int_0^x f_n \sigma_n \right]^q \\ \leq \sup_{x > b_n} \tilde{u}_n(x)^q \left(\int_0^x w \right)^{\frac{q}{p}} \|f_n\|_{L^{r,s}(v)}^q \left\| \frac{\sigma_n \chi_{(0,x]}}{v} \right\|_{L^{r',s'}(v)}^q \leq A^q \|f\|_{L^{r,s}(v)}^q.$$

For the estimate of the second term of (1.2) we put $x_{K_1} = b$, $x_{K_2} = 0$ and

$$W_k := \begin{cases} \int_{x_{k+1}}^{x_k} w(t) \left(\int_0^t w \right)^{\frac{q}{p}-1} dt, & q < p, \\ \left(\int_{x_{k+1}}^{x_k} w \right)^{\frac{q}{p}}, & q \geq p. \end{cases}$$

Observe that

$$(1.5) \quad W_k \lesssim \min \left\{ \int_{x_{k+1}}^{x_k} w(t) \left(\int_0^t w \right)^{\frac{q}{p}-1} dt, \left(\int_{x_{k+1}}^{x_k} w \right)^{\frac{q}{p}} \right\}.$$

In the case of $q < p$ we obtain

$$\begin{aligned} \sum_{K_1 < k < K_2} 2^{kq} \left(\int_0^{x_k} w \right)^{\frac{q}{p}} &= \frac{q}{p} \sum_{K_1 < k < K_2} 2^{kq} \int_0^{x_k} w(t) \left(\int_0^t w \right)^{\frac{q}{p}-1} dt \\ &= \frac{q}{p} \sum_{K_1 < k < K_2} 2^{kq} \sum_{k \leq m < K_2} \int_{x_{m+1}}^{x_m} w(t) \left(\int_0^t w \right)^{\frac{q}{p}-1} dt \\ &= \frac{q}{p} \sum_{K_1 < m < K_2} W_m \sum_{K_1 < k \leq m} 2^{kq} \approx \sum_{K_1 < m < K_2} 2^{mq} W_m. \end{aligned}$$

If $q \geq p$, then by [8, Proposition 2.1], we estimate

$$\begin{aligned} \sum_{K_1 < k < K_2} 2^{kq} \left(\int_0^{x_k} w \right)^{\frac{q}{p}} &= \sum_{K_1 < k < K_2} 2^{kq} \left(\sum_{k \leq m < K_2} \int_{x_{m+1}}^{x_m} w \right)^{\frac{q}{p}} \\ &\approx \sum_{K_1 < k < K_2} 2^{kq} \left(\int_{x_{k+1}}^{x_k} w \right)^{\frac{q}{p}} = \sum_{K_1 < k < K_2} 2^{kq} W_k. \end{aligned}$$

In both cases, we have

$$\begin{aligned} \sum_{K_1 < k < K_2} 2^{kq} \left(\int_0^{x_k} w \right)^{\frac{q}{p}} &\approx \sum_{K_1 < k < K_2} (T_n f_n)(x_k)^q W_k \\ &= \sum_{K_1 < k < K_2} \left(\sup_{x_k \leq \tau \leq x_{k-1}} \tilde{u}_n(\tau) \int_0^\tau f_n \sigma_n \right)^q W_k \\ &= \sum_{K_1 < k < K_2} \left(\tilde{u}_n(z_k) \int_0^{z_k} f_n \sigma_n \right)^q W_k = S_1 + S_2, \end{aligned}$$

where $z_k \in [x_k, x_{k-1})$, $K_1 < k < K_2$, such that (see (1.3))

$$\begin{aligned} (T_n f_n)(x_k) &= \sup_{x_k \leq \tau \leq x_{k-1}} \tilde{u}_n(\tau) \int_0^\tau f_n \sigma_n = \tilde{u}_n(z_k) \int_0^{z_k} f_n \sigma_n, \\ S_1 &= \sum_{K_1 < k < K_2} \left(\tilde{u}_n(z_k) \int_{z_{k+2}}^{z_k} f_n \sigma_n \right)^q W_k, \quad z_{K_2} = z_{K_2+1} = 0, \\ S_2 &= \sum_{K_1 < k < K_2-2} \left(\tilde{u}_n(z_k) \int_0^{z_{k+2}} f_n \sigma_n \right)^q W_k. \end{aligned}$$

Now, by (1.5), we find

$$\begin{aligned} S_1 &\lesssim \sum_{K_1 < k < K_2} \tilde{u}_n(z_k)^q \|f_n \chi_{[z_{k+2}, z_k]}\|_{L^{r,s}(v)}^q \left\| \frac{\sigma_n \chi_{[z_{k+2}, z_k]}}{v} \right\|_{L^{r',s'}(v)}^q \left(\int_0^{z_k} w \right)^{\frac{q}{p}} \\ &\leq A^q \sum_{K_1 < k < K_2} \|f_n \chi_{[z_{k+2}, z_k]}\|_{L^{r,s}(v)}^q \lesssim A^q \|f\|_{L^{r,s}(v)}^q. \end{aligned}$$

Using the relation (1.5) and the criteria of boundedness of the Hardy operator between Lorentz spaces [5, Theorem 3] we get

$$\begin{aligned} S_2 &\lesssim \sum_{K_1 < k < K_2 - 2} \int_{x_{k+1}}^{x_k} w(t) \left(\int_0^t w \right)^{\frac{q}{p}-1} \left(\tilde{u}(t) \int_0^t f_n \sigma \right)^q dt \\ &\lesssim \sup_{0 < x < b} \left[\int_x^b \tilde{u}(t)^q \left[\int_0^t w \right]^{\frac{q}{p}-1} w(t) dt \right] \left\| \frac{\sigma \chi_{(0,x]}}{v} \right\|_{L^{r',s'}(v)}^q \|f\|_{L^{r,s}(v)}^q \leq A^q \|f\|_{L^{r,s}(v)}^q. \end{aligned}$$

Thus, (1.1) is proved.

Now we prove that for arbitrary $t \in (0, b)$,

$$(1.6) \quad \{x \in (0, b) \mid (Tf)(x) > t\} = \bigcup_{n \in \mathbb{N}} \{x \in (0, b) \mid (T_n f_n)(x) > t\}.$$

Since $(Tf)(x) \geq (T_n f_n)(x)$, the left part of (1.6) contains the right part of (1.6).

For the proof of the inverse embedding we fix a point $x \in (0, b)$ such that $(Tf)(x) > t$. By the definition of $(Tf)(x)$ there exists a point $z \in [x, b)$ such that $\tilde{u}(z) \int_0^z f \sigma > t$. Since

$$\tilde{u}_n(z) \int_0^z f_n \sigma_n \uparrow \tilde{u}(z) \int_0^z f \sigma,$$

there exists $n_0 \in \mathbb{N}$ such that $\tilde{u}_{n_0}(z) \int_0^z f_{n_0} \sigma_{n_0} > t$. Hence,

$$(T_{n_0} f_{n_0})(x) \geq (T_{n_0} f_{n_0})(z) \geq \tilde{u}_{n_0}(z) \int_0^z f_{n_0} \sigma_{n_0} > t,$$

and the equality (1.6) is proved.

Using (1.1) and (1.6), the Monotone Convergence Theorem implies (0.2) with the constant $C \lesssim A$.

Necessity. For arbitrary $x \in (0, b)$ we have

$$(Tf)(x) \geq \int_0^x f \sigma \cdot \sup_{x \leq \tau < b} u(\tau) = \tilde{u}(x) \int_0^x f \sigma.$$

Put $\lambda := (Tf)(x)$. Then $(0, x) \subset \{y \in (0, b) \mid (Tf)(y) > t\}$ for all $t \in (0, \lambda)$, and

$$\begin{aligned} C \|f\|_{L^{r,s}(v)} &\geq \|Tf\|_{L^{p,q}(w)} \geq \left(\int_0^\lambda q t^{q-1} (Tf)_*(t)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\ &\geq \lambda \left(\int_0^x w \right)^{\frac{1}{p}} \geq \left(\int_0^x w \right)^{\frac{1}{p}} \tilde{u}(x) \int_0^x f \cdot \frac{\sigma}{v} \cdot v. \end{aligned}$$

Hence,

$$C \geq \left(\int_0^x w \right)^{\frac{1}{p}} \tilde{u}(x) \sup_{f \in \mathfrak{M}^+, f \neq 0} \frac{\int_0^x f \cdot \frac{\sigma}{v} \cdot v}{\|f\|_{L^{r,s}(v)}} \gtrsim \left(\int_0^x w \right)^{\frac{1}{p}} \tilde{u}(x) \left\| \frac{\sigma \chi_{(0,x]}}{v} \right\|_{L^{r',s'}(v)}.$$

By Lemma 1.2 we have

$$\begin{aligned} \|Tf\|_{L^{p,q}(w)} &= \left(\frac{q}{p}\right)^{\frac{1}{q}} \|Tf\|_{L^q(\rho)} \geq \left[\int_0^b \frac{q}{p} \left[\int_0^y w \right]^{\frac{q}{p}-1} \left[\tilde{u}(y) \int_0^y f \sigma \right]^q w(y) dy \right]^{\frac{1}{q}} \\ &\geq \left(\int_x^b \frac{q}{p} \tilde{u}(y)^q \left[\int_0^y w \right]^{\frac{q}{p}-1} w(y) dy \right)^{\frac{1}{q}} \int_0^x f \sigma, \end{aligned}$$

whence the estimate

$$C \geq \left(\frac{q}{p}\right)^{\frac{1}{q}} \left(\int_x^b \tilde{u}(y)^q \left[\int_0^y w \right]^{\frac{q}{p}-1} w(y) dy \right)^{\frac{1}{q}} \left\| \frac{\sigma \chi_{(0,x]}}{v} \right\|_{L^{r',s'}(v)}$$

follows. $\square$

Theorem 1.4. *Let $r \in (1, \infty)$, $s \in [1, \infty]$, $p \in (0, \infty)$, $q = \infty$. Let u be a continuous function on $(0, b)$, $w, v, \sigma \in \mathfrak{M}^+$. The inequality (0.2) holds iff $B < \infty$, where*

$$B := \sup_{0 < x < b} \tilde{u}(x) \left[\int_0^x w \right]^{\frac{1}{p}} \left\| \frac{\sigma \chi_{(0,x]}}{v} \right\|_{L^{r',s'}(v)},$$

$$\tilde{u}(x) = \sup_{x \leq \tau < b} u(\tau).$$

Proof. Sufficiency. Using the same ideas and notation as in the proof of the sufficient part of Theorem 1.3, we obtain (in case $q = \infty$) the relation

$$(1.7) \quad \|T_n f_n\|_{L^{p,\infty}(w)} \lesssim \max \left\{ 2^{K_1} \left(\int_0^b w \right)^{\frac{1}{p}} ; \sup_{K_1 < k < K_2} 2^k \left(\int_0^{x_k} w \right)^{\frac{1}{p}} \right\},$$

instead of (1.2). In the same way as (1.4) we prove the bound

$$2^{K_1} \left(\int_0^b w \right)^{\frac{1}{p}} \lesssim B \|f\|_{L^{r,s}(v)}.$$

For estimation of the second term of the right part of (1.7), we write

$$\begin{aligned} \sup_{K_1 < k < K_2} 2^k \left(\int_0^{x_k} w \right)^{\frac{1}{p}} &\approx \sup_{K_1 < k < K_2} (T_n f_n)(x_k) \left(\int_0^{x_k} w \right)^{\frac{1}{p}} \\ &= \sup_{K_1 < k < K_2} \tilde{u}_n(z_k) \int_0^{z_k} f_n \sigma_n \left(\int_0^{x_k} w \right)^{\frac{1}{p}} = S_3, \end{aligned}$$

where $z_k \in [x_k, x_{k-1})$, $K_1 < k < K_2$, such that

$$(T_n f_n)(x_k) = \sup_{x_k \leq \tau \leq x_{k-1}} \tilde{u}_n(\tau) \int_0^\tau f_n \sigma_n = \tilde{u}_n(z_k) \int_0^{z_k} f_n \sigma_n.$$

Now, we find

$$\begin{aligned} S_3 &\lesssim \sup_{K_1 < k < K_2} \tilde{u}_n(z_k) \|f_n \chi_{(0,z_k]}\|_{L^{r,s}(v)} \left\| \frac{\sigma_n \chi_{(0,z_k]}}{v} \right\|_{L^{r',s'}(v)} \left(\int_0^{x_k} w \right)^{\frac{1}{p}} \\ &\leq B \|f\|_{L^{r,s}(v)}. \end{aligned}$$

Thus, (1.1) with $q = \infty$ is proved and the Monotone Convergence Theorem implies (0.2) with $C \lesssim B$.

Necessity. For arbitrary $x \in (0, b)$ we put $\lambda := (Tf)(x)$. Then $(0, x] \subset \{y \in (0, b) \mid (Tf)(y) > t\}$ for all $t \in (0, \lambda)$, and

$$\begin{aligned} C\|f\|_{L^{r,s}(v)} &\geq \|Tf\|_{L^{p,\infty}(w)} \\ &\geq \sup_{t \in (0, \lambda)} t (Tf)_*(t)^{\frac{1}{p}} \geq \lambda \left(\int_0^x w \right)^{\frac{1}{p}} \geq \left(\int_0^x w \right)^{\frac{1}{p}} \tilde{u}(x) \int_0^x f \cdot \frac{\sigma}{v} \cdot v. \end{aligned}$$

This implies $C \gtrsim B$. $\square$

By using results of the paper [6] and Lemma 1.2 it is possible to characterize the inequality (0.2) in the case of $r = s$, $q < \infty$. We note here a case which is not contained in Theorem 1.3 and follows from [6, Theorem 4.4] and Lemma 1.2.

Theorem 1.5. *Let $s = r$, $p \in (0, \infty)$, $1 < q < r < \infty$, $\frac{1}{\nu} = \frac{1}{q} - \frac{1}{r}$. Let u be a continuous function on $(0, b)$, $\sigma \equiv 1$, $w, v \in \mathfrak{M}^+$, $\rho(x) = w(x) \left(\int_0^x w \right)^{\frac{q}{p}-1}$, $\tilde{u}(x) = \sup_{x \leq \tau < b} u(\tau)$. The inequality (0.2) holds iff*

$$\int_0^b \left(\int_t^b \tilde{u}(s)^q \rho(s) ds \right)^{\frac{\nu}{r}} \tilde{u}(t)^q \left(\int_0^t v^{1-r'} \right)^{\frac{\nu}{r'}} \rho(t) dt < \infty$$

and

$$\int_0^b \left(\int_t^b \rho \right)^{\frac{\nu}{r}} \sup_{t \leq \tau < b} \left[\tilde{u}(\tau)^\nu \left(\int_0^\tau v^{1-r'} \right)^{\frac{\nu}{r'}} \right] \rho(t) dt < \infty.$$

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