# CONSTRUCTION OF SINGULAR RATIONAL SURFACES OF PICARD NUMBER ONE WITH AMPLE CANONICAL DIVISOR 

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In memory of the late Professor Hyo Chul Myung, the founder of KIAS


#### Abstract

Kollár gave a series of examples of rational surfaces of Picard number 1 with ample canonical divisor having cyclic singularities. In this paper, we construct several series of new examples in a geometric way, i.e., by blowing up several times inside a configuration of curves on the projective plane and then by contracting chains of rational curves. One series of our examples has the same singularities as Kollár's examples.


## 1. Introduction

A rational surface $S$ with quotient singularities has been studied extensively when its anti-canonical divisor $-K_{S}$ is ample or numerically trivial. In the former case the surface is called a log del Pezzo surface, and in the latter case the surface is called a $\log$ Enriques surface. On the other hand, when $K_{S}$ is ample, very little is known about the classification of such surfaces. Moreover, if in addition $S$ has Picard number $\rho(S)=1$, nothing seems to be known except the examples due to Keel and McKernan ([KM], Section 19) and Kollár ([K], Example 43). All these examples have exactly two quotient singularities. The examples of [KM] were obtained by blowing up several times inside a configuration of a smooth conic and 2 lines of $\mathbb{P}^{2}$ and then by contracting two chains of rational curves (see Remark 4.4).

Kollár's examples were constructed by contracting two rational curves on some well-chosen weighted projective hypersurfaces. We briefly review his construction. Let

$$
Y=Y\left(a_{1}, a_{2}, a_{3}, a_{4}\right):=\left(x_{1}^{a_{1}} x_{2}+x_{2}^{a_{2}} x_{3}+x_{3}^{a_{3}} x_{4}+x_{4}^{a_{4}} x_{1}=0\right)
$$

be a hypersurface in $\mathbb{P}\left(w_{1}, w_{2}, w_{3}, w_{4}\right)$, where $a_{i}$ and the weights $w_{i}$ satisfy a system of equations

$$
a_{1} w_{1}+w_{2}=a_{2} w_{2}+w_{3}=a_{3} w_{3}+w_{4}=a_{4} w_{4}+w_{1}=d
$$

[^0]with solutions
\[

$$
\begin{gathered}
w_{1}=\frac{1}{w^{*}}\left(a_{2} a_{3} a_{4}-a_{3} a_{4}+a_{4}-1\right), \quad w_{2}=\frac{1}{w^{*}}\left(a_{1} a_{3} a_{4}-a_{1} a_{4}+a_{1}-1\right) \\
w_{3}=\frac{1}{w^{*}}\left(a_{1} a_{2} a_{4}-a_{1} a_{2}+a_{2}-1\right), \quad w_{4}=\frac{1}{w^{*}}\left(a_{1} a_{2} a_{3}-a_{2} a_{3}+a_{3}-1\right) \\
d=\frac{1}{w^{*}}\left(a_{1} a_{2} a_{3} a_{4}-1\right)
\end{gathered}
$$
\]

where $w^{*}=\operatorname{gcd}\left(w_{1}, w_{2}, w_{3}, w_{4}\right)$. If $w^{*}=1$, then by Theorem 39 in $[\mathbf{K}, Y$ is a rational surface with 4 cyclic singularities at the coordinate vertices and with $H_{2}(Y, \mathbb{Q}) \cong \mathbb{Q}^{3}$. The two rational curves

$$
C_{1}:=\left(x_{1}=x_{3}=0\right), \quad C_{2}:=\left(x_{2}=x_{4}=0\right)
$$

are extremal rays for the $K_{Y}+(1-\epsilon)\left(C_{1}+C_{2}\right)$ minimal model program for $0<$ $\epsilon \ll 1$. Thus $C_{1}$ and $C_{2}$ are both contractible to quotient singularities, and we get a rational surface of Picard number 1,

$$
\pi: Y=Y\left(a_{1}, a_{2}, a_{3}, a_{4}\right) \rightarrow X=X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)
$$

If $a_{1}, a_{2}, a_{3}, a_{4} \geq 4$, then $K_{X}$ is ample by Theorem 39(5) in K. The surface $X$ has two cyclic singularities and no other singularities.

First we determine the types of singularities of $X$. Recall that a cyclic surface singularity $p$ can be uniquely determined by the dual graph of $f^{-1}(p)$

$$
\begin{array}{cc}
-n_{1} & -n_{2} \\
\circ & 0 \\
A_{1} & A_{2} \\
A_{2} & -n_{l} \\
0 & A_{l} \\
A_{l}
\end{array}
$$

where $f: X^{\prime} \rightarrow X$ is the minimal resolution and the $A_{j}$ 's are the irreducible components of $f^{-1}(p)$ with self-intersection $-n_{j}$. The dual graph can be represented by its Hirzebruch-Jung continued fraction

$$
\left[n_{1}, n_{2}, \ldots, n_{l}\right]=n_{1}-\frac{1}{n_{2}-\frac{1}{\ddots-\frac{1}{n_{l}}}}
$$

Theorem 1.1. Assume that $w^{*}=1$. Then the two cyclic singularities of the surface $X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ are of type

$$
\frac{1}{s_{1}}\left(w_{2}, w_{4}\right) \text { and } \frac{1}{s_{2}}\left(w_{1}, w_{3}\right)
$$

where $s_{1}=a_{4} w_{4}-w_{3}$ and $s_{2}=a_{3} w_{3}-w_{2}$. Their Hirzebruch-Jung continued fractions are

$$
[\underbrace{2, \ldots, 2}_{a_{4}-1}, a_{3}, a_{1}, \underbrace{2, \ldots, 2}_{a_{2}-1}] \text { and }[\underbrace{2, \ldots, 2}_{a_{3}-1}, a_{2}, a_{4}, \underbrace{2, \ldots, 2}_{a_{1}-1}] \text {, }
$$

respectively.
Remark 1.2. The condition $w^{*}=1$ gives some restriction on the choice of the integers $a_{i}$ for $X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$. In particular, it rules out the possibility that $a_{1}=$ $a_{2}=a_{3}=a_{4}$.

We recall that a normal projective surface $S$ with the same Betti numbers with $\mathbb{P}^{2}$ is called a $\mathbb{Q}$-homology projective plane. When a normal projective surface $S$ has quotient singularities only, $S$ is a $\mathbb{Q}$-homology projective plane if the second Betti number $b_{2}(S)=1$. For convenience, we adopt the following terminology:
a $\mathbb{Q}$-homology projective plane which is a rational surface is called a rational $\mathbb{Q}$ homology projective plane. The surfaces $X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ are rational $\mathbb{Q}$-homology projective planes if $w^{*}=1$.

Next, we give a geometric construction of a series of rational $\mathbb{Q}$-homology projective planes with ample canonical divisor having the same singularities as Kollár's examples.

Theorem 1.3. For each integers $a_{1}, a_{2}, a_{3}, a_{4} \geq 2$, there exists a rational $\mathbb{Q}$ homology projective plane $T=T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ with two cyclic singularities of type

$$
[\underbrace{2, \ldots, 2}_{a_{4}-1}, a_{3}, a_{1}, \underbrace{2, \ldots, 2}_{a_{2}-1}] \text { and }[\underbrace{2, \ldots, 2}_{a_{3}-1}, a_{2}, a_{4}, \underbrace{2, \ldots, 2}_{a_{1}-1}] \text {. }
$$

The surfaces can be constructed by blowing up several times inside four general lines of $\mathbb{P}^{2}$ and then by contracting two chains of rational curves. Moreover,
(1) if $a_{1}, a_{2}, a_{3}, a_{4} \geq 3$ and $a_{i}>3$ for some $i$, then $K_{T}$ is ample;
(2) if $a_{1}=a_{2}=a_{3}=a_{4}=3$, then $K_{T}$ is numerically trivial.

Remark 1.4. Note that for any choice of the numbers $a_{i} \geq 2, T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ exists. Thus in terms of types of singularities our examples $T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ properly include Kollár's examples $X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$.

Starting with different configurations of curves on $\mathbb{P}^{2}$ (see Sections 5 and 6 for construction), we also construct new series of rational $\mathbb{Q}$-homology projective planes with $K_{S}$ ample having one or three cyclic singularities.
Theorem 1.5. For each integer $b \geq 2$, there exists a rational $\mathbb{Q}$-homology projective plane $S:=S(b)$ with a unique cyclic singularity of type

$$
\frac{1}{27 b^{2}-36 b+4}\left(1,9 b^{2}-9 b+1\right)
$$

Moreover,
(1) if $b=2$, then $K_{S}$ is numerically trivial;
(2) if $b>2$, then $K_{S}$ is ample.

Theorem 1.6. For each integer $b \geq 2$, there exists a rational $\mathbb{Q}$-homology projective plane $S:=S(b)$ with three cyclic singularities of type

$$
A_{1}, \quad \frac{1}{7}(1,3), \frac{1}{3 b^{2}-2 b-2}\left(1,2 b^{2}-b-1\right)
$$

Moreover,
(1) if $b<5$, then $-K_{S}$ is ample;
(2) if $b=5$, then $K_{S}$ is numerically trivial;
(3) if $b>5$, then $K_{S}$ is ample.

Using different blow-ups of the same configurations, it is possible to construct more new examples with $K_{S}$ ample (see Remark 5.1, Remark 6.1). All of these examples have at most three cyclic singularities

The authors have shown that every rational $\mathbb{Q}$-homology projective plane with quotient singularities has at most 4 singularities HK1. It seems impossible, or very difficult, to construct a rational $\mathbb{Q}$-homology projective plane with $K_{S}$ ample having 4 cyclic singularities. We recall that there are rational $\mathbb{Q}$-homology projective planes with $K_{S}$ ample having an arbitrary number of rational singularities ([HK1], Introduction).

The original motivation of this study is the following conjecture, called the algebraic Montgomery-Yang problem:

Conjecture 1.7 ([区], Conjecture 30). Let $S$ be a $\mathbb{Q}$-homology projective plane with quotient singularities. If $\pi_{1}\left(S^{0}\right)=1$, then $S$ has at most 3 singular points. Here $S^{0}$ denotes the smooth locus of $S$.

Recently, the authors have confirmed Conjecture 1.7 when $S$ has at least one non-cyclic quotient singularity HK2 or when $S$ is a non-rational surface HK3] or when $S$ is a $\log$ del Pezzo surface [HK4]. Thus the only remaining case is the case where $S$ is a rational surface with $K_{S}$ ample having at worst cyclic singularities. Therefore, to solve the conjecture completely we need to study rational $\mathbb{Q}$-homology projective planes with $K_{S}$ ample having at worst cyclic singularities. The result in this paper is a step toward the goal.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

## 2. Preliminaries

We first collect some properties of Hirzebruch-Jung continued fractions for later use. For more details, we refer the reader to (HK3, Section 2).
Notation 2.1. For a fixed Hirzebruch-Jung continued fraction

$$
w=\left[n_{1}, n_{2}, \ldots, n_{l}\right]=n_{1}-\frac{1}{n_{2}-\frac{1}{\ddots-\frac{1}{n_{l}}}},
$$

we use the following notation:
(1) $|w|=q$, the order of the cyclic singularity corresponding to $w$, i.e., $w=\frac{q}{q_{1}}$ with $1 \leq q_{1}<q, \operatorname{gcd}\left(q, q_{1}\right)=1$. Note that $|w|$ is the absolute value of the determinant of the intersection matrix corresponding to $w$. We also define
(2) $u_{j}:=\left|\left[n_{1}, n_{2}, \ldots, n_{j-1}\right]\right|(2 \leq j \leq l+1), \quad u_{0}=0, u_{1}=1$,
(3) $v_{j}:=\left|\left[n_{j+1}, n_{j+2}, \ldots, n_{l}\right]\right| \quad(0 \leq j \leq l-1), \quad v_{l}=1, v_{l+1}=0$.

Note that $u_{l+1}=v_{0}=|w|$.
Lemma 2.2. $\left|\left[n_{1}, \ldots, n_{j-1}, n_{j}+c, n_{j+1}, \ldots, n_{l}\right]\right|=c v_{j} u_{j}+\left|\left[n_{1}, n_{2}, \ldots, n_{l}\right]\right|$.
Proof. By induction, it is enough to show that

$$
\left|\left[n_{1}, \ldots, n_{j-1}, n_{j}+1, n_{j+1}, \ldots, n_{l}\right]\right|=v_{j} u_{j}+\left|\left[n_{1}, n_{2}, \ldots, n_{l}\right]\right|
$$

which is exactly ([HK3], Lemma 2.4 (6)).
Notation 2.3. $[\cdots, 2 * a, \cdots]:=[\cdots, \underbrace{2, \ldots, 2}_{a}, \cdots]$.

## Lemma 2.4.

(1) $|[2 *(a-1), b]|=a b-a+1$,
(2) $|[2 *(a-1), b, c, 2 *(d-1)]|=a b c d-a b d-a c d+a b+c d-a-d+1$.

Proof. (1) Note that $[b, 2 *(a-1)]=b-\frac{a-1}{a}=\frac{a b-a+1}{a}$, so

$$
|[2 *(a-1), b]|=|[b, 2 *(a-1)]|=a b-a+1
$$

(2) It is easy to see that

$$
|[2 *(a-1), 2,2,2 *(d-1)]|=a+d+1
$$

By applying Lemma 2.2, we get

$$
\begin{aligned}
& |[2 *(a-1), b, 2,2 *(d-1)]| \\
& =|[2 *(a-1), 2+(b-2), 2,2 *(d-1)]| \\
& =(b-2)|[2,2 *(d-1)]| \cdot|[2 *(a-1)]|+|[2 *(a-1), 2,2,2 *(d-1)]| \\
& =(b-2)(d+1) a+(a+d+1)
\end{aligned}
$$

Again, by Lemma 2.2 and (1), we get

$$
\begin{aligned}
& |[2 *(a-1), b, c, 2 *(d-1)]| \\
& =|[2 *(a-1), b, 2+(c-2), 2 *(d-1)]| \\
& =(c-2)(a b-a+1) d+(b-2)(d+1) a+(a+d+1) \\
& =a b c d-a b d-a c d+a b+c d-a-d+1
\end{aligned}
$$

Now let $X$ be a projective surface with only cyclic singularities, and let $f$ : $X^{\prime} \rightarrow X$ be its minimal resolution. For each singular point $p$ of $X$, the dual graph of $f^{-1}(p)$ is of the form

$$
\stackrel{-n_{1, p}}{0,}-\stackrel{-n_{2, p}}{0}-\cdots-\begin{gathered}
-n_{l_{p}, p} \\
A_{1, p} \\
A_{2, p}
\end{gathered}-\cdots \begin{gathered}
A_{p, p}
\end{gathered}
$$

and corresponds to the Hirzebruch-Jung continued fraction

$$
w_{p}=\left[n_{1, p}, n_{2, p}, \ldots, n_{l_{p}, p}\right]
$$

where the $A_{j, p}$ 's are irreducible components of $f^{-1}(p)$. We omit the subscript $p$ if the meaning is clear from the context.

Lemma 2.5. If $X^{\prime}$ contains $a(-1)$-curve $E$, then

$$
E f^{*}\left(K_{X}\right)=-1+\sum_{p \in \operatorname{Sing}(S)} \sum_{j=1}^{l_{p}}\left(1-\frac{v_{j, p}+u_{j, p}}{\left|w_{p}\right|}\right) E A_{j, p}
$$

Proof. By Lemma 2.2 of [HK1, we have the adjunction formula

$$
K_{X^{\prime}}=f^{*}\left(K_{X}\right)-\sum_{p \in \operatorname{Sing}(S)} \sum_{j=1}^{l_{p}}\left(1-\frac{v_{j, p}+u_{j, p}}{\left|w_{p}\right|}\right) A_{j, p}
$$

Intersecting $E$ with both sides of the adjunction formula, we get the equality.
Now let $X$ be a $\mathbb{Q}$-homology projective plane with only cyclic singularities. Then we have the following criterion for ampleness of $K_{X}$.

## Lemma 2.6.

(1) $K_{X}$ is ample iff $E f^{*}\left(K_{X}\right)>0$ for all irreducible curves $E$ not contracted by $f$ iff $E f^{*}\left(K_{X}\right)>0$ for an irreducible curve $E$ not contracted by $f$.
(2) $K_{X}$ is numerically trivial iff $E f^{*}\left(K_{X}\right)=0$ for all irreducible curves $E$ not contracted by $f$ iff $E f^{*}\left(K_{X}\right)=0$ for an irreducible curve $E$ not contracted by $f$.
(3) $-K_{X}$ is ample iff $E f^{*}\left(K_{X}\right)<0$ for all irreducible curves $E$ not contracted by $f$ iff $E f^{*}\left(K_{X}\right)<0$ for an irreducible curve $E$ not contracted by $f$.

Proof. Since $X$ has Picard number 1, we have the trichotomy: $K_{X}$ is ample; $K_{X}$ is numerically trivial; $-K_{X}$ is ample. The assertion follows from $E f^{*}\left(K_{X}\right)=$ $f_{*}(E) K_{X}$.

## 3. Singularities of Kollár's examples

In this section, we will prove Theorem 1.1 by using the technique of unprojection (R). Let

$$
F=x_{1}^{a_{1}} x_{2}+x_{2}^{a_{2}} x_{3}+x_{3}^{a_{3}} x_{4}+x_{4}^{a_{4}} x_{1}
$$

be the weighted homogeneous polynomial defining

$$
Y=Y\left(a_{1}, a_{2}, a_{3}, a_{4}\right) \subset \mathbb{P}\left(w_{1}, w_{2}, w_{3}, w_{4}\right)
$$

Recall that on $Y$ there are two disjoint rational curves $C_{1}=\mathbb{P}\left(w_{2}, w_{4}\right)$ and $C_{2}=$ $\mathbb{P}\left(w_{1}, w_{3}\right)$. Write $F$ as

$$
F=A x_{1}+B x_{3},
$$

where

$$
A=x_{1}^{a_{1}-1} x_{2}+x_{4}^{a_{4}}, \quad B=x_{2}^{a_{2}}+x_{3}^{a_{3}-1} x_{4}
$$

By introducing a new variable

$$
y_{1}=\frac{A}{x_{3}}=-\frac{B}{x_{1}},
$$

we get an unprojection morphism $Y \rightarrow Y_{1}$ where

$$
Y_{1}=\left(x_{3} y_{1}=A, x_{1} y_{1}=-B\right) \subset \mathbb{P}\left(w_{1}, w_{2}, w_{3}, w_{4}, s_{1}\right)
$$

where

$$
s_{1}:=\operatorname{deg}\left(y_{1}\right)=a_{4} w_{4}-w_{3}=a_{2} w_{2}-w_{1} .
$$

This morphism contracts $C_{1}$ to the singular point

$$
p=(0,0,0,0,1) \in Y_{1}
$$

of $Y_{1}$. It is easy to see that near $p$

$$
Y_{1} \cong \mathbb{P}\left(w_{2}, w_{4}, s_{1}\right)
$$

Thus the singular point $p$ is of type $\frac{1}{s_{1}}\left(w_{2}, w_{4}\right)$.
Similarly, using another unprojection morphism contracting $C_{2}$, we see that the other singularity is of type $\frac{1}{s_{2}}\left(w_{1}, w_{3}\right)$, where

$$
s_{2}=a_{1} w_{1}-w_{4}=a_{3} w_{3}-w_{2}
$$

Now we determine the types of singularities in terms of a Hirzebruch-Jung continued fraction.

Notation 3.1. The expression $\gamma \underset{\bar{n}}{\overline{=}} \delta$ means that $\gamma$ is congruent to $\delta$ modulo $n$.
Write

$$
\frac{1}{s_{1}}\left(w_{2}, w_{4}\right)=\frac{1}{s_{1}}\left(1, t_{1}\right), \quad \frac{1}{s_{2}}\left(w_{1}, w_{3}\right)=\frac{1}{s_{2}}\left(1, t_{2}\right),
$$

where $t_{1} \underset{s_{1}}{\overline{=}} \alpha w_{4}$ and $t_{2} \equiv \beta w_{3}$ for some integers $\alpha$ and $\beta$ satisfying

$$
\alpha w_{2} \equiv 1, \quad \beta w_{1} \equiv 1
$$

The following lemma is immediate.

Lemma 3.2. For an integer $t$,
(1) $t \equiv \alpha w_{4}$ if and only if $t w_{2} \underset{s_{1}}{\equiv} w_{4}$,
(2) $t \underset{\bar{s}_{2}}{\equiv} \beta w_{3}$ if and only if $t w_{1} \underset{s_{2}}{\equiv} w_{3}$.

Lemma 3.3. If $w^{*}=1$, then the following equalities hold:
(1) $\left[2 *\left(a_{4}-1\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]=\frac{s_{1}}{t_{1}}$.
(2) $\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]=\frac{s_{2}}{t_{2}}$.

Proof. (1) Note that

$$
\begin{aligned}
s_{1} & =a_{4} w_{4}-w_{3} \\
& =a_{1} a_{2} a_{3} a_{4}-a_{1} a_{2} a_{4}-a_{2} a_{3} a_{4}+a_{1} a_{2}+a_{3} a_{4}-a_{2}-a_{4}+1 \\
& =\left|\left[2 *\left(a_{4}-1\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right|
\end{aligned}
$$

where the last equality follows from Lemma 2.4
Now it is enough to show that $\left|\left[2 *\left(a_{4}-2\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right|=t_{1}$, i.e.,

$$
\left|\left[2 *\left(a_{4}-2\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right| \underset{s_{1}}{\equiv} \alpha w_{4} .
$$

Since

$$
w_{2}=a_{1} a_{3} a_{4}-a_{1} a_{4}+a_{1}-1 \text { and } w_{4}=a_{1} a_{2} a_{3}-a_{2} a_{3}+a_{3}-1
$$

a direct computation shows that

$$
\begin{aligned}
\left|\left[2 *\left(a_{4}-2\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right| w_{2} & =w_{4}+\left(a_{1} a_{3} a_{4}-a_{1} a_{3}-a_{1} a_{4}+2 a_{1}-1\right) s_{1} \\
& \equiv w_{4}
\end{aligned}
$$

Now the assertion follows from Lemma 3.2.
(2) Similarly, note that

$$
\begin{aligned}
s_{2} & =a_{1} w_{1}-w_{4} \\
& =a_{1} a_{2} a_{3} a_{4}-a_{1} a_{2} a_{3}-a_{1} a_{3} a_{4}+a_{1} a_{4}+a_{2} a_{3}-a_{1}-a_{3}+1 \\
& =\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right| .
\end{aligned}
$$

A direct computation also shows that

$$
\begin{aligned}
\left|\left[2 *\left(a_{3}-2\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right| w_{1} & =w_{3}+\left(a_{2} a_{3} a_{4}-a_{2} a_{4}-a_{3} a_{4}+2 a_{4}-1\right) s_{2} \\
& \equiv w_{3} .
\end{aligned}
$$

Again the assertion follows from Lemma 3.2.
We have proved Theorem 1.1

## 4. Construction of $T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$

We construct rational $\mathbb{Q}$-homology projective planes with ample canonical divisor having the same singularities as Kollár's examples starting from the configuration of four general lines on $\mathbb{P}^{2}$.

Consider four general lines $L_{1}, L_{2}, L_{3}, L_{4}$ on $\mathbb{P}^{2}$. Choose four points among the six intersection points such that each $L_{i}$ passes through two of them.


By blowing up the four marked intersection points twice each, we get a rational surface $Z(2,2,2,2)$ having 12 rational curves such that

is their dual graph. Here, $\bullet$ means a $(-1)$-curve and $\circ$ means a $(-2)$-curve.
Let $Z\left(2+r_{1}, 2+r_{2}, 2+r_{3}, 2+r_{4}\right)$ be the surface obtained from $Z(2,2,2,2)$ by blowing up $r_{1}$ times at $E_{1} \cap L_{1}, r_{2}$ times at $E_{2} \cap L_{2}, r_{3}$ times at $E_{3} \cap L_{3}$ and $r_{4}$ times at $E_{4} \cap L_{4}$, respectively. Set

$$
\left(a_{1}, a_{2}, a_{3}, a_{4}\right):=\left(2+r_{1}, 2+r_{2}, 2+r_{3}, 2+r_{4}\right)
$$

Then $Z\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ has the following configuration of rational curves:


Here, • means a ( -1 -curve. By contracting the two maximal linear chains of rational curves denoted by white vertices,

$$
Z\left(a_{1}, a_{2}, a_{3}, a_{4}\right) \rightarrow T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)
$$

we get a rational $\mathbb{Q}$-homology projective plane $T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ with two cyclic quotient singularities. It has the same singularities as Kollár's example $X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ when $w^{*}=1$.

Proposition 4.1. (1) If $a_{1}, a_{2}, a_{3}, a_{4} \geq 3$ and $a_{i}>3$ for some $i$, then $K_{T}$ is ample.
(2) If $a_{1}=a_{2}=a_{3}=a_{4}=3$, then $K_{T}$ is numerically trivial.

Proof. Let $E:=E_{1}$ be the ( -1 )-curve meeting the component of self-intersection $-a_{1}$ of the upper chain and the rightmost component of the bottom chain. By Lemma 2.5, we see that

$$
\begin{aligned}
& E f^{*}\left(K_{T}\right) \\
& =1-\frac{\left|\left[2 *\left(a_{2}-1\right)\right]\right|+\left|\left[2 *\left(a_{4}-1\right), a_{3}\right]\right|}{\left|\left[2 *\left(a_{4}-1\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right|}-\frac{1+\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-2\right)\right]\right|}{\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right|}
\end{aligned}
$$

By Lemma 2.4 (1),

$$
\left|\left[2 *\left(a_{2}-1\right)\right]\right|+\left|\left[2 *\left(a_{4}-1\right), a_{3}\right]\right|=a_{2}+\left(a_{3} a_{4}-a_{4}+1\right)
$$

By Lemma 2.4 (2),

$$
\begin{aligned}
& \left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-2\right)\right]\right| \\
& \quad=\left(a_{1}-1\right) a_{2} a_{3} a_{4}-\left(a_{1}-1\right) a_{2} a_{3}-a_{4}\left(a_{1}-1\right) a_{3}+\left(a_{1}-1\right) a_{4} \\
& \quad \quad+a_{2} a_{3}-\left(a_{1}-1\right)-a_{3}+1 \\
& \quad=\left(a_{3} a_{2} a_{4} a_{1}-a_{3} a_{2} a_{1}-a_{3} a_{4} a_{1}+a_{3} a_{2}+a_{4} a_{1}-a_{3}-a_{1}+1\right) \\
& \quad-\left(a_{2} a_{3} a_{4}-a_{2} a_{3}-a_{3} a_{4}+a_{4}-1\right) \\
& \quad=\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right|-\left(a_{2} a_{3} a_{4}-a_{2} a_{3}-a_{3} a_{4}+a_{4}-1\right) .
\end{aligned}
$$

Thus

$$
\begin{aligned}
E f^{*}\left(K_{T}\right) & =-\frac{a_{3} a_{4}+a_{2}-a_{4}+1}{\left|\left[2 *\left(a_{4}-1\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right|}+\frac{a_{2} a_{3} a_{4}-a_{2} a_{3}-a_{3} a_{4}+a_{4}-2}{\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right|} \\
& =\frac{\alpha}{\left|\left[2 *\left(a_{4}-1\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right| \cdot\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right|}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha= & \left(a_{2} a_{3} a_{4}-a_{2} a_{3}-a_{3} a_{4}+a_{4}-2\right)\left|\left[2 *\left(a_{4}-1\right), a_{3}, a_{1}, 2 *\left(a_{2}-1\right)\right]\right| \\
& -\left(a_{3} a_{4}+a_{2}-a_{4}+1\right)\left|\left[2 *\left(a_{3}-1\right), a_{2}, a_{4}, 2 *\left(a_{1}-1\right)\right]\right|
\end{aligned}
$$

By Lemma 2.4 (2), $\alpha$ can be expanded and then can be factorized as follows (factorization can be done by a computer algebra system such as Maple):

$$
\alpha=\left(a_{2} a_{3} a_{4}-a_{3} a_{4}+a_{4}-1\right)\left\{\left(a_{1}-1\right)\left(a_{2}-1\right)\left(a_{3}-1\right)\left(a_{4}-1\right)-a_{1} a_{3}-a_{2} a_{4}+2\right\}
$$

Note first that

$$
a_{2} a_{3} a_{4}-a_{3} a_{4}+a_{4}-1>0
$$

if all $a_{i} \geq 2$.
Assume that $a_{i} \geq 3$ for every $i=1,2,3,4$. Then, it is easy to see that

- $a_{1} a_{3} \geq 2\left(a_{1}+a_{3}\right)-3$, where the equality holds iff $a_{1}=a_{3}=3$,
- $a_{2} a_{4} \geq 2\left(a_{2}+a_{4}\right)-3$, where the equality holds iff $a_{2}=a_{4}=3$,
- $\left(a_{1}-1\right)\left(a_{3}-1\right) \cdot\left(a_{2}-1\right)\left(a_{4}-1\right) \geq 2\left\{\left(a_{1}-1\right)\left(a_{3}-1\right)+\left(a_{2}-1\right)\left(a_{4}-1\right)\right\}$, where the equality holds iff $a_{1}=a_{2}=a_{3}=a_{4}=3$.
Thus

$$
\begin{aligned}
& \left(a_{1}-1\right)\left(a_{2}-1\right)\left(a_{3}-1\right)\left(a_{4}-1\right)-a_{1} a_{3}-a_{2} a_{4}+2 \\
& \quad \geq 2\left\{\left(a_{1}-1\right)\left(a_{3}-1\right)+\left(a_{2}-1\right)\left(a_{4}-1\right)\right\}-a_{1} a_{3}-a_{2} a_{4}+2 \\
& \quad=a_{1} a_{3}-2\left(a_{1}+a_{3}\right)+a_{2} a_{4}-2\left(a_{2}+a_{4}\right)+6 \\
& \quad \geq 0
\end{aligned}
$$

Here, both inequalities become equalities iff $a_{1}=a_{2}=a_{3}=a_{4}=3$. Now we apply Lemma 2.6 to get the assertions.

This completes the proof of Theorem 1.3 .

Remark 4.2. It is easy to check the following:
(1) When $a_{i}=a_{j}=2$ for $\{i, j\} \in\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}, K_{T}$ is ample iff $a_{k}, a_{l} \geq 6$ or $a_{k}=5, a_{l} \geq 7$ or $a_{k}=4, a_{l} \geq 10$, where $\{i, j, k, l\}=\{1,2,3,4\}$.
(2) When $a_{i}=a_{j}=2$ for $\{i, j\} \in\{\{1,3\},\{2,4\}\},-K_{T}$ is ample for all $a_{k}, a_{l} \geq$ 2 , where $\{i, j, k, l\}=\{1,2,3,4\}$.

Remark 4.3. It is easy to see that the map

$$
\pi: Y=Y\left(a_{1}, a_{2}, a_{3}, a_{4}\right) \rightarrow X=X\left(a_{1}, a_{2}, a_{3}, a_{4}\right)
$$

used for Kollár's construction in Section 1 is extended to the following commutative diagram:

where $X^{\prime}$ and $Y^{\prime}$ are minimal resolutions of $X$ and $Y$ respectively.
If $w^{*}=1$, then the map

$$
f: Z=Z\left(a_{1}, a_{2}, a_{3}, a_{4}\right) \rightarrow T=T\left(a_{1}, a_{2}, a_{3}, a_{4}\right)
$$

used for our construction can be extended to the following commutative diagram:

where $\tilde{Z}$ is a minimal resolution of $\tilde{T}$ and $\tilde{T}$ has the same singularities as $Y$.
To see this, assume, for simplicity, that $a_{1}=3, a_{2}=3, a_{3}=4$, and $a_{4}=4$. The minimal resolution $Z$ of $T$ has two chains of rational curves,

First, we blow up three times at the intersection point of $L_{3}$ and $L_{1}$ to get the following configuration of rational curves:
where the strict transform of $L_{i}$ is again denoted by $L_{i}$. Similarly, we blow up three times at the intersection point of $L_{2}$ and $L_{4}$ to get the following configuration of rational curves:

This gives the morphism $\tilde{Z} \rightarrow Z$. Now by contracting the four chains of rational curves with a self-intersection number at most -2 , we get a morphism $\tilde{Z} \rightarrow \tilde{T}$. It is easy to see that the types of the four singular points of $\tilde{T}$ and $Y$ are the same. By contracting the images of the two ( -1 )-curves $E_{1}$ and $E_{2}$, we get a morphism $\tilde{T} \rightarrow T$.

On the other hand, if $w^{*} \neq 1$, e.g., if $a_{1}=a_{2}=a_{3}=a_{4}$, then it is not possible to have such a diagram.

Remark 4.4. (1) If $a_{1}=a_{2}=a_{3}=a_{4}=2$, then $T$ is the unique Gorenstein $\log$ del Pezzo surface of Picard rank one with exactly two singular points of type $A_{4}$. It is a degree six hypersurface in $\mathbb{P}(1,1,2,3)$ ([HW $]$ ).
(2) In Section 19 of [KM], Keel and McKernan constructed 13 infinite series of rational $\mathbb{Q}$-homology projective planes with $K_{S}$ ample having two cyclic singularities starting from a union of a smooth conic, a secant line and a tangent line to the conic. In terms of singularity types, 11 series of them can also be obtained from the union of four lines used in this section and the remaining two series can also be obtained from the configuration in Section 5.

## 5. Examples with one cyclic singularity

In this section, we construct a series of new rational $\mathbb{Q}$-homology projective planes with $K_{S}$ ample starting from a different configuration of curves in $\mathbb{P}^{2}$.

Consider the following configuration of 4 lines and a nodal cubic curve in $\mathbb{P}^{2}$ :


The existence of the configuration can be checked as follows. Consider the plane nodal cubic curve

$$
C: y^{2}=x^{3}+x^{2}
$$

on $\mathbb{C}^{2}$. Let

$$
L_{1}: y=a x
$$

be a line passing through the node of $C$. If $a \neq \pm 1, L_{1}$ passes through $C$ at a point $p_{1}$ different from the origin. For $i=2,3,4$, we recursively define $L_{i}$ as the tangent line of $C$ at $p_{i-1}$, and $p_{i}$ as the another intersection point of $L_{i}$ and $C$. By explicitly calculating the coordinate of $p_{i}$, we see that there is a suitable number $a \neq \pm 1$ such that $p_{1}=p_{4}, p_{1} \neq p_{2}, p_{2} \neq p_{3}$ and $p_{3} \neq p_{1}$. Indeed,

- $p_{1}=\left(a^{2}-1, a\left(a^{2}-1\right)\right)$,
- $p_{2}=\left(\frac{\left(a^{2}-1\right)^{2}}{4 a^{2}},-\frac{\left(a^{2}-1\right)^{2}\left(a^{2}+1\right)}{8 a^{3}}\right)$,
- $p_{3}=\left(\frac{\left(a^{2}-1\right)^{4}}{16 a^{2}\left(a^{2}+1\right)^{2}}, \frac{\left(a^{2}-1\right)^{4}\left(a^{4}+6 a^{2}+1\right)}{64 a^{3}\left(a^{2}+1\right)^{3}}\right)$,
- $p_{4}=\left(\frac{\left(a^{2}-1\right)^{8}}{64 a^{2}\left(a^{2}+1\right)^{2}\left(a^{4}+6 a^{2}+1\right)^{2}},-\frac{\left(a^{2}-1\right)^{8}\left(1+28 a^{2}+70 a^{4}+28 a^{6}+a^{8}\right)}{512 a^{3}\left(a^{4}+6 a^{2}+1\right)^{3}\left(a^{2}+1\right)^{3}}\right)$.

By factorizing the components of the vector $p_{1}-p_{4}$, we see that the vector $p_{1}-p_{4}$ is equal to the vector

$$
\begin{aligned}
& \left(\frac{\left(3 a^{2}+1\right)\left(3 a^{6}+27 a^{4}+33 a^{2}+1\right)}{64 a^{2}\left(a^{2}+1\right)^{2}\left(a^{4}+6 a^{2}+1\right)^{2}}\right. \\
& \left.\quad \frac{\left(73 a^{16}+1168 a^{14}+6244 a^{12}+12064 a^{10}+9822 a^{8}+3024 a^{6}+372 a^{4}+1\right)}{512 a^{3}\left(a^{4}+6 a^{2}+1\right)^{3}\left(a^{2}+1\right)^{3}}\right)
\end{aligned}
$$

multiplied by the number $\left(a^{2}-1\right)\left(7 a^{6}+35 a^{4}+21 a^{2}+1\right)$. Thus for $a$ satisfying the polynomial equation $7 a^{6}+35 a^{4}+21 a^{2}+1=0$, we have $p_{1}=p_{4}, p_{1} \neq p_{2}, p_{2} \neq p_{3}$ and $p_{3} \neq p_{1}$. This shows the existence of such a configuration.

Now, by blowing up the node of $C$ once and the three other marked points $p_{1}, p_{2}, p_{3}$ three times each, we get a rational surface $Z(2)$ with the following configuration of 15 rational curves:


Here, $C$ and $L_{i}$ are the proper transforms of $C$ and $L_{i}$, a dotted curve is a ( -1 )curve and a solid curve is a $(-2)$-curve if it is not specified as a $(-3)$-curve. The surface $Z(2)$ contains the following Hirzebruch-Jung string of rational curves:

Blowing up $(b-2)$ times the marked point $P$, we get a surface $Z(b)$ with the following Hirzebruch-Jung string of rational curves:

Now by contracting these rational curves,

$$
Z(b) \rightarrow S(b)
$$

we get a rational $\mathbb{Q}$-homology projective plane $S(b)$ with a unique cyclic singularity of type

$$
\frac{1}{27 b^{2}-36 b+4}\left(1,9 b^{2}-9 b+1\right)
$$

Let $E$ be the exceptional curve of $Z(b) \rightarrow Z(b-1)$, i.e., the $(-1)$-curve with $E . A=E . B=1$. Then by Lemma 2.5

$$
\begin{aligned}
E f^{*}\left(K_{S}\right) & =1-\frac{3+|[2 * 7,3,2 *(b-2)]|}{|[3, b, 2 * 7,3,2 *(b-2)]|}-\frac{1+|[3, b, 2 * 7,3,2 *(b-3)]|}{|[3, b, 2 * 7,3,2 *(b-2)]|} \\
& =1-\frac{3+(9 b-1)}{27 b^{2}-36 b+4}-\frac{1+\left(27 b^{2}-63 b+37\right)}{27 b^{2}-36 b+4} \\
& =\frac{18(b-2)}{27 b^{2}-36 b+4} .
\end{aligned}
$$

Now Lemma 2.6 completes the proof of Theorem 1.5 ,
Remark 5.1. One can get more examples by blowing up not only the marked point $P$ but also the marked point $P^{\prime}$ or $P^{\prime \prime}$.
(1) Blowing up $(b-2)$ times the marked point $P$ and $(c-2)$ times the marked point $P^{\prime}$, we get a surface $Z(b, c)$ with the following Hirzebruch-Jung string of rational curves:

$$
[\overbrace{2, \ldots, 2}^{c-2}, 3, b, 2,2, c, 2,2,2,2,3, \overbrace{2, \ldots, 2}^{b-2}] .
$$

Here $b, c \geq 2$. The resulting rational $\mathbb{Q}$-homology projective plane $S(b, c)$ has ample canonical class if $b$ and $c$ are not small.
(2) Blowing up $(b-2)$ times the marked point $P$ and $(c-2)$ times the marked point $P^{\prime \prime}$, we get a surface $W(b, c)$ with the following Hirzebruch-Jung string of rational curves:

$$
\overbrace{2, \ldots, 2}^{c-2}, 3, b, 2,2,2,2,2, c, 2,3, \overbrace{2, \ldots, 2}^{b-2}] .
$$

Here $b, c \geq 2$. The resulting rational $\mathbb{Q}$-homology projective plane $T(b, c)$ has ample canonical class if $b$ and $c$ are not small.

## 6. Examples with three cyclic singularities

Consider the following configuration of 3 concurrent lines and a conic on $\mathbb{P}^{2}$ :


By blowing up the marked point $Q$ three times and the three other marked points twice each, we get a rational surface $Z(2)$ with the following configuration of 13 rational curves:


Here, $C$ and $L_{i}$ are the proper transforms of $C$ and $L_{i}$, a dotted line is a $(-1)$-curve and a solid line is a $(-2)$-curve except $L_{1}$, which is a $(-3)$-curve. Note that $Z(2)$ contains the following three Hirzebruch-Jung strings of rational curves:

Blowing up $(b-2)$ times the marked point $P$, we get a rational surface $Z(b)$ with the following three Hirzebruch-Jung strings of rational curves:

Contracting these rational curves,

$$
Z(b) \rightarrow S(b)
$$

we get a rational $\mathbb{Q}$-homology projective plane $S(b)$ with three cyclic singularities of type

$$
A_{1}, \quad \frac{1}{7}(1,3), \frac{1}{3 b^{2}-2 b-2}\left(1,2 b^{2}-b-1\right)
$$

Let $E$ be the exceptional curve of $Z(b) \rightarrow Z(b-1)$, i.e., the ( -1 )-curve with $E . C=E . B=1$. Then by Lemma 2.5

$$
E f^{*}\left(K_{S}\right)=1-\frac{(b+1)+3}{3 b^{2}-2 b-2}-\frac{1+\left(3 b^{2}-5 b+3\right)}{3 b^{2}-2 b-2}=\frac{2(b-5)}{3 b^{2}-2 b-2}
$$

Now Lemma 2.6 completes the proof of Theorem 1.6 .
Remark 6.1. One can get more examples by blowing up not only the marked point $P$ but also the marked points $P^{\prime}$ and $P^{\prime \prime}$.
(1) Blowing up $(b-2)$ times the marked point $P$ and $c$ times the marked point $P^{\prime \prime}$, we get a surface with the following three Hirzebruch-Jung strings of rational curves:

$$
[2], \quad[\overbrace{2, \ldots, 2}^{c}, 3,2,2], \quad[2,2+c, b, \overbrace{2, \ldots, 2}^{b}] .
$$

Here $b \geq 2, c \geq 0$. The resulting rational $\mathbb{Q}$-homology projective plane $V(b, c)$ has ample canonical class if $b$ and $c$ are not small.
(2) Blowing up $(b-2)$ times the marked point $P$, once the marked point $P^{\prime}$ and $c$ times the marked point $P^{\prime \prime}$, we get a surface with the following two Hirzebruch-Jung strings of rational curves:

$$
\overbrace{2, \ldots, 2}^{c}, 3,2,2,2,2], \quad[2,2+c, b+1, \overbrace{2, \ldots, 2}^{b}] .
$$

Here $b \geq 2, c \geq 0$. The resulting rational $\mathbb{Q}$-homology projective plane $Y(b, c)$ has 2 cyclic singularities, and its canonical class is ample if $b$ and $c$ are not small.

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## References

[HK1] D. Hwang and J. Keum, The maximum number of singular points on rational homology projective planes, J. Algebraic Geom. 20 (2011), 495-523.
[HK2] D. Hwang and J. Keum, Algebraic Montgomery-Yang Problem: the noncyclic case, Math. Ann. 350 (2011), no. 3, 721-754.
[HK3] D. Hwang and J. Keum, Algebraic Montgomery-Yang Problem: the non-rational surface case, arXiv:0906.0633.
[HK4] D. Hwang and J. Keum, Algebraic Montgomery-Yang Problem: the log del Pezzo surface case, preprint.
[HW] F. Hidaka and K. Watanabe, Normal Gorenstein surfaces with ample anti-canonical divisor, Tokyo J. Math. 4 (2) (1981), 319-330. MR 646042 (83h:14031)
[K] J. Kollár, Is there a topological Bogomolov-Miyaoka-Yau inequality?, Pure Appl. Math. Q. (Fedor Bogomolov special issue, part I), 4 (2008), no. 2, 203-236. MR2400877 (2009b:14086)
[KM] S. Keel and J. McKernan, Rational curves on quasi-projective surfaces, Mem. Amer. Math. Soc. 140 (1999), no. 669. MR 1610249 (99m:14068)
[R] M. Reid, Graded rings and birational geometry, Proc. of Algebraic Geometry Symposium (Kinosaki, Oct. 2000), K. Ohno (ed.), 1-72.

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