RELATIONS BETWEEN METRICS OF ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER SPHERE

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ABSTRACT. We give a simplified proof that Wilhelm’s metric on the Gromoll-Meyer sphere has positive curvature almost everywhere. We determine that its zero locus coincides with that of the almost positively curved metric of Eschenburg and Kerin.

INTRODUCTION

$G = \mathrm{Sp}(2) = \{Q \in M_2(\mathbb{H}) \mid Q^*Q = I\}$ is endowed with a bi-invariant metric with non-negative curvature on which $H = \mathrm{Sp}(1)$ acts freely and isometrically by

$$q \cdot Q \equiv qQ \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix}.$$

Gromoll and Meyer, [4], observed that the orbit space $M = G//H$ is an exotic 7-sphere and inherits non-negative sectional curvature by O’Neill’s formula, [5]. Although M has points with strictly positive curvature, Wilhelm, [6], identified an open set in M having zero curvature at each point and gave a different metric on this exotic 7-sphere with positive curvature almost everywhere. Subsequently Eschenburg and Kerin, [3], gave a clear proof that there was a second biquotient metric on $M = G//H$ with almost positive curvature. We will see that these two almost positively curved metrics have the same zero curvature locus in diffeomorphism type.

By fully exploiting the machinery for biquotients we achieve a more concise and accessible proof that Wilhelm’s metric has positive curvature almost everywhere than the original argument of [6]. We give a description of the zero locus in this framework in Propositions 2.3, 2.4, 4.1, 4.2 and 4.3. Interest has been revived in this metric in connection with deformations of it proposed by Wilhelm and Petersen to put positive sectional curvature on this exotic 7-sphere.

1. PRELIMINARIES

Proposition 1.1. *Let $(G, \langle \cdot, \cdot \rangle)$ be a compact Lie group with a bi-invariant Riemannian metric. Then the sectional curvature of G is non-negative. Moreover the zero planes at each point are precisely spanned by $x, y \in \mathfrak{g}$ with $[x, y] = 0$.*

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Definition 1.1. Consider the orbit space of a free isometric action of one compact Lie group, H , on another $(G, \langle, \rangle)$, equipped with a bi-invariant metric defined via two representations, $\varrho_L, \varrho_R : H \rightarrow G$, by $h \cdot g \equiv \varrho_L(h)g\varrho_R(h)^{-1}$, via multiplication on the left and multiplication by the inverse on the right. We call such a Riemannian with its natural metric a biquotient, $G//H$.

The above is a slight abuse of the standard terminology.

Proposition 1.2. *A biquotient, $G//H$, has non-negative curvature and has zero curvature at $H \cdot g$ if and only if $x, y \in \mathfrak{g}$ at g span a plane normal to $H \cdot g$ and $[x, y] = 0$.*

Recall Cheeger's trick of [1].

Proposition 1.3. *Let K be a compact Lie group that acts isometrically on a Riemannian manifold, M , via $\cdot$. Then $K \times M$ carries an isometric action of K by $(k(\tilde{k}, m) = (\tilde{k}k^{-1}, k \cdot m)$ and M is diffeomorphic to the orbit space, $K \times M/K$.*

Proof. Consider maps, $f : M \rightarrow K \times M/K$ defined by $f(m) = K(1, m)$ and $g(K(\tilde{k}, m)) = \tilde{k}m$. g is well defined since $\tilde{k} \cdot m = (\tilde{k}k^{-1}) \cdot (km)$. Moreover $g(f(m)) = g(K(1, m)) = m$ and $f(g(K(\tilde{k}, m))) = f(\tilde{k} \cdot m) = K(1, \tilde{k} \cdot m) = K((\tilde{k}^{-1})^{-1}, \tilde{k}^{-1} \cdot (\tilde{k} \cdot m)) = K(\tilde{k}, m)$. Hence f and g are inverses. Both maps are smooth, hence the diffeomorphism. $\square$

We use two variants of Cheeger's trick. A similar variant was used to construct examples with almost positive curvature in [7].

Proposition 1.4. *Let K be a compact Lie group that acts isometrically on a Lie group G via two representations, $\varrho_L, \varrho_R : K \rightarrow G$, by $k \cdot g \equiv \varrho_L(k)g\varrho_R(k)^{-1}$, via multiplication on the left and multiplication by the inverse on the right. Moreover let $\varrho : H \rightarrow K$ be a representation so that the action $h \odot g \equiv \varrho(h) \cdot g$ is free. Then the orbit space of the action of $H \times K$ on $K \times G$ defined by $(h, k) \cdot (\tilde{k}, g) \equiv (\varrho(h)\tilde{k}k^{-1}, k \cdot g)$ is diffeomorphic to the biquotient, $G//H$.*

Proof. Consider maps

$$\begin{aligned} f : G//H &\rightarrow K \times G//H \times K, & f(H \odot g) &\equiv H \times K \cdot (1, g), \\ \tilde{f} : K \times G//H \times K &\rightarrow G//H, & \tilde{f}(H \times K \cdot (\tilde{k}, g)) &\equiv H \odot (\tilde{k} \cdot g). \end{aligned}$$

Verify that f is well defined,

$$\begin{aligned} H \times K(1, h \odot g) &= H \times K(1, \varrho(h) \cdot g) = H \times K(\varrho(h^{-1})1(\varrho(h)^{-1})^{-1}, \varrho(h)^{-1} \cdot (\varrho(h) \cdot g)) \\ &= H \times K(1, g). \end{aligned}$$

Verify that $\tilde{f}$ is well defined,

$$\begin{aligned} H \odot ((h, k) \cdot (\tilde{k}, g)) &= H \odot ((\varrho(h)\tilde{k}k^{-1}) \cdot (k \cdot g)) = H \odot (\varrho(h) \cdot (\tilde{k} \cdot g)) \\ &= H \odot (h \odot (\tilde{k} \cdot g)) = H \odot (\tilde{k} \cdot g). \end{aligned}$$

Verify that f and $\tilde{f}$ are mutual inverses,

$$\begin{aligned} \tilde{f}(f(H \odot g)) &= \tilde{f}(H \times K(1, g)) = H \odot (1 \cdot g) = H \odot g, \\ f(\tilde{f}(H \times K \cdot (\tilde{k}, g))) &= f(H \odot (\tilde{k} \cdot g)) = H \times K \cdot (1, \tilde{k} \cdot g) \\ &= H \times K \cdot (1(\tilde{k}^{-1})^{-1}, \tilde{k}^{-1} \cdot (\tilde{k} \cdot g)) = H \times K \cdot (\tilde{k}, g). \end{aligned}$$

Moreover f and $\tilde{f}$ are smooth. $\square$

Proposition 1.5. *Consider representations, $\varrho_L : K \rightarrow G$, $\varrho_R : H \rightarrow G$ and $\varrho : H \rightarrow K$ such that the isometric group action defined by $h \odot g \equiv \varrho_L(\varrho(h))g\varrho_R(h)^{-1}$ via multiplication on the left and multiplication by the inverse on the right is fixed point free. Then the orbit space of the action of $H \times H \times K$ on $H \times K \times G$ defined by $(h, h_1, k) \cdot (\tilde{h}, \tilde{k}, g) \equiv (h\tilde{h}h_1^{-1}, \varrho(h_1)\tilde{k}k^{-1}, \varrho_L(k)g\varrho_R(h)^{-1})$ is diffeomorphic to the biquotient $G//H$.*

Proof. Consider maps

$$\begin{aligned} f : G//H &\rightarrow H \times K \times G//H \times H \times K, & f(H \odot g) &\equiv H \times H \times K \cdot (1, 1, g), \\ \tilde{f} : H \times K \times G//H \times H \times K &\rightarrow G//H, & \tilde{f}(H \times H \times K \cdot (\tilde{h}, \tilde{k}, g)) &\equiv H \odot (\varrho_L(\varrho(\tilde{h})\tilde{k})g). \end{aligned}$$

Verify that f is well defined,

$$\begin{aligned} H \times H \times K \cdot (1, 1, h \odot g) &= H \times H \times K \cdot (1, 1, \varrho_L(\varrho(h))\varrho_R(h)^{-1}) \\ &= H \times H \times K \cdot (h1h^{-1}, \varrho(h)1\varrho(h)^{-1}, \varrho_L(\varrho(h))g\varrho_R(h)^{-1}) \\ &= H \times H \times K \cdot (h, h, \varrho(h)) \cdot (1, 1, g) = H \times H \times K \cdot (1, 1, g). \end{aligned}$$

To verify that $\tilde{f}$ is well defined, note that

$$(h, h_1, k) \cdot (\tilde{h}, \tilde{k}, g) = (h\tilde{h}h_1^{-1}, \varrho(h_1)\tilde{k}k^{-1}, \varrho_L(k)g\varrho_R(h)^{-1}) \text{ and thus}$$

$$\begin{aligned} H \odot (\varrho_L(\varrho(h\tilde{h}h_1^{-1}))\varrho(h_1)\tilde{k}k^{-1})\varrho_L(k)g\varrho_R(h)^{-1} &= H \odot (\varrho_L(\varrho(h)\tilde{k})g\varrho_R(h)^{-1}) \\ &= H \odot (h \odot g) = H \odot g. \end{aligned}$$

Verify that f and $\tilde{f}$ are mutual inverses,

$$\begin{aligned} \tilde{f}(f(H \odot g)) &= \tilde{f}(H \times H \times K(1, 1, g)) = H \odot g \text{ and} \\ f(\tilde{f}(H \times H \times K \cdot (\tilde{h}, \tilde{k}, g))) &= f(H \odot (\varrho_L(\varrho(\tilde{h})\tilde{k})g)) = H \times H \times K \cdot (1, 1, \varrho_L(\varrho(\tilde{h})\tilde{k})g) \\ &= H \times H \times K \cdot (1, \tilde{h}^{-1}, (\varrho(\tilde{h})\tilde{k})^{-1}) \cdot (1, 1, \varrho_L(\varrho(\tilde{h})\tilde{k})g) \\ &= (H \times H \times K) \cdot (\tilde{h}, \tilde{k}, g). \end{aligned}$$

Moreover f and $\tilde{f}$ are smooth. $\square$

2. WILHELM'S METRIC

Let $G = \mathrm{Sp}(2)$, $K = \mathrm{Sp}(1)^4$ with $\varrho_L((p, q, r, s)) = \mathrm{diag}(p, q)$ and $\varrho_R((p, q, r, s)) = \mathrm{diag}(r, s)$. Consider the bi-invariant metric on K given by $\ell_1^2 \langle \cdot, \cdot \rangle \oplus \ell_2^2 \langle \cdot, \cdot \rangle \oplus \nu_1^2 \langle \cdot, \cdot \rangle \oplus \nu_2^2 \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is standard on $\mathrm{Sp}(1)$ and $\langle M, M' \rangle' = \mathrm{Re}(\mathrm{tr}(M^* M'))$ on G . Let $H = \mathrm{Sp}(1)$ and $\varrho(q) = (q, q, q, 1)$. Proposition 1.4 gives that the orbit space of $K \times G//H \times K$ is diffeomorphic to the exotic 7-sphere, $M = G//H$. In [6] an isometric metric was realised by applying Proposition 1.3 to $M = G$ and submersing to $G//H$. The isometry arises by virtue of the fact that the actions of $H \times 1$ and $1 \times K$ commute on $K \times G$.

2.1. Horizontal vectors. With no loss of generality, restrict to coset representatives, $P = ((1, 1, 1, 1), Q) \in K \times G$. Let $((\alpha, \beta, \gamma, \delta), M) \in \mathfrak{k} \oplus T_Q G = T_P(K \times G)$. Recall that

$$\mathfrak{g} = \mathfrak{sp}(2) = \left\{ \begin{pmatrix} u & x \\ -\bar{x} & v \end{pmatrix} : u, v \in \mathrm{Im}(\mathbb{H}), x \in \mathbb{H} \right\}.$$

Define $M_L, M_R \in \mathfrak{g}$ via

$$M = (l_Q)_*(M_L) = Q \begin{pmatrix} u_L & x_L \\ -\bar{x}_L & v_L \end{pmatrix} = (r_Q)_*(M_R) = \begin{pmatrix} u_R & x_R \\ -\bar{x}_R & v_R \end{pmatrix} Q.$$

Proposition 2.1. *The horizontal distribution at $((1, 1, 1, 1), Q)$ is*

$$\mathcal{H} = \{(\ell_1^{-2}u_R, \ell_2^{-2}v_R, -\nu_1^{-2}u_L, -\nu_2^{-2}v_L, M) \in \mathfrak{k} \oplus T_Q G : u_L = u_R + v_R\}.$$

Proof. Let $\epsilon \in \text{Im}(\mathbb{H})$ be arbitrary and $Z = ((\alpha, \beta, \gamma, \delta), (r_Q)_*(M_R))$ be horizontal.

Z is perpendicular to the fundamental field of $(0, (\epsilon, 0, 0, 0))$.

So $\ell_1^2 \langle \alpha, -\epsilon \rangle + \langle \text{diag}(\epsilon, 0), M_R \rangle' = 0$, i.e., $-\ell_1^2 \langle \alpha, \epsilon \rangle + \langle \epsilon, u_R \rangle = 0$. Hence $\alpha = \ell_1^{-2}u_R$. Likewise consider that Z is perpendicular to fundamental fields of $(0, (0, \epsilon, 0, 0, 0))$, $(0, (0, 0, \epsilon, 0))$, and $(0, (0, 0, 0, \epsilon))$ in turn to deduce $\beta = \ell_2^{-2}v_R$, $\gamma = -\nu_1^{-2}u_L$ and $\delta = -\nu_2^{-2}v_L$.

Z is perpendicular to the fundamental field of $(\epsilon, (0, 0, 0, 0)) \perp ((\alpha, \beta, \gamma, \delta), M)$. Hence $(\epsilon, \epsilon, \epsilon, 0) \perp ((\alpha, \beta, \gamma, \delta), M)$. So $\ell_1^2 \langle \alpha, \epsilon \rangle + \ell_2^2 \langle \beta, \epsilon \rangle + \nu_1^2 \langle \gamma, \epsilon \rangle = 0$. Hence $\ell_1^2 \alpha + \ell_2^2 \beta + \nu_1^2 \gamma = 0$. Thus $u_L = u_R + v_R$. $\square$

2.2. Reducing the number of planes. It follows from Proposition 1.2 that zero planes, $\text{span}\{M^1, M^2\}$, are characterized by the conditions $[u_R^1, u_R^2] = 0$, $[v_R^1, v_R^2] = 0$, $[u_L^1, u_L^2] = 0$, $[v_L^1, v_L^2] = 0$ and $[M_R^1, M_R^2] = 0$.

To begin consider the 3 conditions, $[u_R^1, u_R^2] = 0$, $[v_R^1, v_R^2] = 0$ and $[M_R^1, M_R^2] = 0$; cf. [2]. The top left entry of $[M_R^1, M_R^2]$ is $[u_R^1, u_R^2] - x_R^1 \bar{x}_R^2 + x_R^2 \bar{x}_R^1$; since $[u_R^1, u_R^2] = 0$ and $[M_R^1, M_R^2] = 0$, it follows that $\text{Im}(x_R^2 \bar{x}_R^1) = 0$. That is, $x_R^2 \bar{x}_R^1 \in \mathbb{R}$; namely, x_R^1 and x_R^2 are linearly dependent. So without loss of generality, by taking a linear combination of M^1, M^2 , one may assume that $x_R^1 = 0$. Put $x_R^2 = x \in \mathbb{H}$ and put $u_R^1 = u$ and $v_R^1 = v$. The top right entry of $[M_R^1, M_R^2]$ is $ux - xv$; hence $[M^1, M^2] = 0$ gives $ux - xv = 0$. This splits into two cases: (1) $x = 0$ or (2) $x \neq 0$ and $u = xvx^{-1}$.

- (1) If $x = 0$, then one may assume without loss that $M_R^1 = \text{diag}(u, 0)$ and $M_R^2 = \text{diag}(0, v)$ since conditions $[u_R^1, u_R^2] = 0$ and $[v_R^1, v_R^2] = 0$ give that u_R^1, u_R^2 and v_R^1, v_R^2 are linearly dependent.
- (2) On the other hand if $x \neq 0$ and $u = xvx^{-1}$, then since $M^1 \neq 0$, $u \neq 0$ one may assume without loss of generality that $u_R^2 = 0$. The linear dependence of v and v_R^2 give $v_R^2 = tv$ for some $t \in \mathbb{R}$. Hence without loss one may assume that $M_R^1 = \text{diag}(xvx^{-1}, v)$ and $M_R^2 = \begin{pmatrix} 0 & x \\ -\bar{x} & tv \end{pmatrix}$.

It suffices to consider cases (1) and (2) below:

- (1) $M_R^1 = \text{diag}(u, 0), \quad M_R^2 = \text{diag}(0, v),$
- (2) $M_R^1 = \text{diag}(xvx^{-1}, v), \quad M_R^2 = \begin{pmatrix} 0 & x \\ -\bar{x} & tv \end{pmatrix}.$

2.3. Horizontal zero planes. Assume the zero plane, $\text{span}\{M_R^1, M_R^2\}$, is horizontal; then $u_L^1 = u_R^1 + v_R^1$ and $u_L^2 = u_R^2 + v_R^2$. In case (1) it follows from the above that $u_L^1 = u + 0 = u$ and $u_L^2 = 0 + v = v$. Recall that $M_L = \text{Ad}(Q^{-1})M_R$. Let $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; thus $u_L^1 = \bar{a}ua$ and $u_L^2 = \bar{c}vc$. Hence $\bar{a}ua = u$ and $\bar{c}vc = v$. Thus $a = \pm 1$ and $c = \pm 1$, a contradiction, since $Q \in \text{Sp}(2)$. Hence no horizontal zero planes arise for (1).

For (2) consider the two remaining conditions for zero curvature, $[u_L^1, u_L^2] = 0$ and $[v_L^1, v_L^2] = 0$. The horizontal condition gives $u_L^1 = xvx^{-1} + v$ and $u_L^2 = tv$.

Hence the condition $[u_L^1, u_L^2] = t[xvx^{-1}, v] = 0$ splits into two cases, (i) $t = 0$ and (ii) $t \neq 0$. Hence

Proposition 2.2. *Every zero plane in $\mathcal{H}_{(e,Q)}$ may be written in the form described in Proposition 2.3 or Proposition 2.4.*

2.4. The $t = 0$ case. Throughout, $\hat{x}$ denotes the unit length quaternion, $\frac{x}{|x|}$, for $x \neq 0$.

Proposition 2.3. *$Q \in \text{Sp}(2)$ has zero planes, $\text{span}\{(*, M^1), (*, M^2)\} \subset \mathcal{H}_{(e,Q)}$, of the form*

$$M_R^1 = \text{diag}(xvx^{-1}, v), \quad M_R^2 = \begin{pmatrix} 0 & x \\ -\bar{x} & 0 \end{pmatrix},$$

where

$$(3) \quad pv\bar{p} = qv\bar{q} + v$$

for appropriately chosen $p, q \in \text{Sp}(1)$. Namely

- (i) if $a\bar{c} \neq 0$, $|\text{Im}(\hat{c})| \geq \frac{1}{2}$ and $|\text{Im}(\hat{a}\hat{c})| \geq \frac{\sqrt{3}}{2}$, pick $p = \hat{c}^{-1}$ and $q = \hat{a}\hat{c}$ and $x = a\bar{c}$;
- (ii) if $a = 0$ and $|\text{Im}(c)| \geq \frac{1}{2}$, pick $p = \bar{c}$ and $q = \hat{x}$ for some x with $|\text{Im}(\hat{x})| \geq \frac{\sqrt{3}}{2}$;
- (iii) if $c = 0$ and $|\text{Im}(a)| \geq \frac{1}{2}$, pick $p = \bar{a}\hat{x}$ and $q = \hat{x}$ for some x with $|\text{Im}(\hat{x})| \geq \frac{\sqrt{3}}{2}$.

A solution (p, q, v) for (3) must have

- (a) v is a scalar multiple of $\text{Im}(\bar{q}pq) \times \text{Im}(p\bar{q})$ provided this quaternion is nonzero,
- (b) v is a scalar multiple of $\text{Im}(q) \times \text{Im}(\bar{q}pq)$ if $\text{Im}(\bar{q}pq) \times \text{Im}(p\bar{q}) = 0$ provided $\text{Im}(p)$ and $\text{Im}(q)$ are linearly independent,
- (c) if $\text{Im}(p)$ and $\text{Im}(q)$ are linearly dependent, then $p = e^{\alpha\frac{\pi}{6}}$ and $q = \pm e^{\alpha\frac{\pi}{3}}$ for some purely imaginary unit quaternion, α , with $v \perp \alpha$.

Proof. It follows that a plane of form (2) with $t = 0$ has

$$M_L^2 = \text{Ad}(Q^{-1}) \begin{pmatrix} 0 & x \\ -\bar{x} & 0 \end{pmatrix} = \begin{pmatrix} \bar{a}xc - \bar{c}\bar{x}a & * \\ * & \bar{b}xd - \bar{d}xb \end{pmatrix}.$$

Thus the horizontal condition gives $u_L^2 = 2\text{Im}(\bar{a}xc) = 0$. Hence $\bar{a}xc$ is real, so either $a\bar{c} = -b\bar{d} = 0$ or x is a scalar multiple of $a\bar{c} = -b\bar{d}$. In either case $v_L^2 = 2\text{Im}(\bar{b}xd) = 0$. So $[u_L^1, u_L^2] = [v_L^1, v_L^2] = 0$ and

$$M_L^1 = \text{Ad}(Q^{-1}) \begin{pmatrix} xvx^{-1} & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \bar{a}xvx^{-1}a + \bar{c}vc & * \\ * & \bar{b}xvx^{-1}b + \bar{d}vd \end{pmatrix}.$$

Hence the horizontal equation is $\bar{a}xvx^{-1}a + \bar{c}vc = xvx^{-1} + v$.

This gives three possibilities:

- (i) $a\bar{c} \neq 0$ and x is a scalar multiple of $a\bar{c}$,
- (ii) $a = 0$ and $\bar{c}vc = xvx^{-1} + v$ with $|c| = 1$, giving (3) with $p = \bar{c}$ and $q = \hat{x}$,
- (iii) $c = 0$ and $\bar{a}xvx^{-1}a = xvx^{-1} + v$ with $|a| = 1$, giving (3) with $p = \bar{a}\hat{x}$ and $q = \hat{x}$.

In the case that $a\bar{c} \neq 0$ and x is a scalar multiple of $a\bar{c}$, it follows that

$$\begin{aligned}\frac{1}{|a|^2|c|^2}\bar{a}a\bar{c}vc\bar{c}a + \bar{c}vc &= \frac{1}{|a|^2|c|^2}a\bar{c}v\bar{c}a + v, \\ \frac{|a|^2}{|c|^2}\bar{c}vc + \bar{c}vc &= \frac{1}{|a|^2|c|^2}a\bar{c}v\bar{c}a + v, \\ \frac{1}{|c|^2}\bar{c}vc &= \frac{1}{|a|^2|c|^2}a\bar{c}v\bar{c}a + v, \\ c^{-1}vc &= a\bar{c}v(a\bar{c})^{-1} + v.\end{aligned}$$

This gives (3) with $p = \hat{c}^{-1}$ and $q = \hat{a}\bar{c}$. Solutions to (3) must obey $|\operatorname{Im}(p)| \geq \frac{1}{2}$ and $|\operatorname{Im}(q)| \geq \frac{\sqrt{3}}{2}$. This follows easily as a corollary to Proposition 4.1; see Remark 4.3. This gives the restrictions on the magnitude of the imaginary parts and covers all contingencies, (i), (ii) and (iii).

Consider the nature of a solution to condition (3) in general. Right multiplying by p , it follows that $pv = qv\bar{q}p + vp$. Taking the real part, since $\operatorname{Re}(pv) = \operatorname{Re}(vp)$ it follows that $\operatorname{Re}(v\bar{q}pq) = \operatorname{Re}(qv\bar{q}p) = 0$. Thus $v \perp \operatorname{Im}(\bar{q}pq)$. Put $w = qv\bar{q}$. Note that $w \perp \operatorname{Im}(p)$. Then $p\bar{q}wq\bar{p} = \bar{q}wq + w$. So by the same logic, setting $p' = p\bar{q}$ and $q' = \bar{q}$, it follows that $w \perp \operatorname{Im}(qp\bar{q}^2)$, or rather $v \perp \operatorname{Im}(p\bar{q})$. Thus it follows that v must be a scalar multiple of $v' = \operatorname{Im}(\bar{q}pq) \times \operatorname{Im}(p\bar{q})$ provided $v' \neq 0$, giving (a).

If not, then $\operatorname{Im}(p)$ and $\operatorname{Im}(qp\bar{q}^2)$ are linearly dependent. Hence

$$\lambda \operatorname{Im}(p) = \operatorname{Im}((qp\bar{q})\bar{q}) = -\operatorname{Re}(qp\bar{q})\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(qp\bar{q}) - \operatorname{Im}(\operatorname{Im}(qp\bar{q})\operatorname{Im}(q)).$$

Conjugating by $\bar{q}$ and rearranging,

$$\operatorname{Im}(\operatorname{Im}(p)\operatorname{Im}(q)) = \operatorname{Re}(q)\operatorname{Im}(p) - \operatorname{Re}(q)\operatorname{Im}(q) - \lambda\bar{q}\operatorname{Im}(p)q.$$

Without loss put $q = e^{i\theta}$ and $\operatorname{Im}(p) = \sin(\eta)(\cos(\phi)i + \sin(\phi)j)$ to find

$$\begin{aligned}\sin(\eta)\cos(\phi)\sin(\theta)k &= \sin(\eta)\cos(\phi)(\cos(\theta) - \lambda\cos(2\theta))i \\ &\quad + (\cos(\theta)\sin(\eta)\sin(\phi) - \cos(\eta)\sin(\theta) - \lambda\sin(\eta)\sin(\phi))j \\ &\quad - \lambda\sin(\eta)\cos(\phi)\sin(2\theta)k.\end{aligned}$$

Hence $\cos(\theta) = \lambda\cos(2\theta)$ and $\sin\theta = -\lambda\sin(2\theta)$, i.e. $e^{i\theta} = \lambda e^{-2i\theta}$. Hence $\lambda = \pm 1$ and $e^{3i\theta} = \pm 1$; hence $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. Recall that $pv\bar{p} = qv\bar{q} + v$; hence $|v|^2 = 2|v|^2 + 2\langle qv\bar{q}, v \rangle$, i.e. $\langle qv\bar{q}, v \rangle / |v|^2 = -\frac{1}{2}$, so the angle between v and $qv\bar{q}$ is $\frac{2\pi}{3}$. But since $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, it follows that $v \perp \operatorname{Im}(q)$. Since $v \perp \operatorname{Im}(\bar{q}pq)$, it follows that v is a scalar multiple of

$$v'' = \operatorname{Im}(q) \times \operatorname{Im}(\bar{q}pq) = \bar{q}\operatorname{Im}(\operatorname{Im}(q)\operatorname{Im}(p))q = \bar{q}(\operatorname{Im}(q) \times \operatorname{Im}(p))q$$

if $v'' \neq 0$, that is, if $\operatorname{Im}(p)$ and $\operatorname{Im}(q)$ are linearly independent, giving (b).

If not, then both $\operatorname{Im}(p)$ and $\operatorname{Im}(q)$ are linearly dependent, or rather are multiples of unit length $\alpha \in \operatorname{Im}(\mathbb{H})$, and hence $p = e^{\alpha\frac{\pi}{6}}$ and $q = \pm e^{\alpha\frac{\pi}{3}}$ with $v \perp \alpha$, giving (c). $\square$

2.5. The $t \neq 0$ case.

Proposition 2.4. $Q \in \operatorname{Sp}(2)$ has zero planes, $\operatorname{span}\{(*, M^1), (*, M^2)\} \subset \mathcal{H}_{(e, Q)}$, of the form:

$$M_R^1 = \operatorname{diag}(-v, v), \quad M_R^2 = \begin{pmatrix} 0 & x \\ -\bar{x} & v \end{pmatrix}$$

for Q with $|a|^2 = |c|^2 = \frac{1}{2}$ precisely when

- (i) $a\bar{c}$ is not real and $2a\operatorname{Im}(a\bar{c})\bar{c} - a\bar{c}\operatorname{Im}(a\bar{c})$ is purely imaginary and perpendicular to $a\bar{c}$, in which case one takes $v = \operatorname{Im}(a\bar{c})$ and $x = 2a\operatorname{Im}(a\bar{c})\bar{c} - a\bar{c}\operatorname{Im}(a\bar{c})$;
- (ii) $a = \pm c$ and $|\operatorname{Im}(c)| \geq \frac{1}{2\sqrt{2}}$, in which case one takes v such that $\operatorname{Re}(cv\bar{c}) = -\frac{1}{4}|v|^2$. If $a = c$, then one takes $x = 2cv\bar{c} - \frac{1}{2}v$; if $a = -c$, then one takes $x = -2cv\bar{c} + \frac{1}{2}v$.

Proof. Recall that a zero plane necessarily has the form (2). xvx^{-1} and v are linearly dependent; hence $xvx^{-1} = \pm v$. Suppose $xvx^{-1} = v$. Then $\bar{a}va + \bar{c}vc = 2v$; however $|\bar{a}va + \bar{c}vc| \leq (|a|^2 + |c|^2)|v| = |v|$, giving a contradiction. Hence $xvx^{-1} = -v$; thus it follows from the horizontal equation that $u_L^1 = -\bar{a}va + \bar{c}vc = 0$. This is only possible if $|a|^2 = |c|^2 = \frac{1}{2}$, in which case $va\bar{c} = a\bar{c}v$. Since $|b|^2 = |d|^2 = \frac{1}{2}$ and $b\bar{d} = -a\bar{c}$ it follows that $v_L^1 = -\bar{b}vb + \bar{d}vd = 0$, giving $[u_L^1, u_L^2] = [v_L^1, v_L^2] = 0$. Since $xvx^{-1} = -v$ it follows that $\operatorname{Re}(x) = 0$ and $x \perp v$. Then

$$M_L^2 = \operatorname{Ad}(Q^{-1}) \begin{pmatrix} 0 & x \\ -\bar{x} & tv \end{pmatrix} = \begin{pmatrix} \bar{a}xc - \bar{c}\bar{x}a + t\bar{c}vc & * \\ * & \bar{b}xd - \bar{d}\bar{x}b + t\bar{d}vd \end{pmatrix}.$$

The horizontal condition now requires that $\bar{a}xc - \bar{c}\bar{x}a + t\bar{c}vc = tv$, or rather

$$(4) \quad \bar{a}xc - \bar{c}\bar{x}a = t(v - \bar{c}vc).$$

$a(4)\bar{c}$ yields $|a|^2|c|^2x - a\bar{c}\bar{x}a\bar{c} = t(av\bar{c} - a\bar{c}|c|^2v)$ or $\frac{1}{4}x + a\bar{c}xa\bar{c} = t(av\bar{c} - \frac{1}{2}a\bar{c}v)$, i.e. $x + pxp = t(4av\bar{c} - pv)$ where $p = 2a\bar{c}$ has $|p| = 1$. Recall that $va\bar{c} = a\bar{c}v$; thus $vp = pv$. Assume for $\operatorname{Im}(p) \neq 0$ that v is a scalar multiple of $\operatorname{Im}(p)$. It follows that $x \perp \operatorname{Im}(p)$, and hence $x + p\bar{p}x = t(4av\bar{c} - pv)$, or rather $2x = t(4av\bar{c} - 2a\bar{c}v)$. If $p = \pm 1$, substitution gives $2x = t(4av\bar{c} - 2a\bar{c}v)$ again. Hence $x = t(2av\bar{c} - a\bar{c}v)$. With no loss of generality take $t = 1$ to get

$$(5) \quad x = 2av\bar{c} - a\bar{c}v.$$

Assume $a\bar{c}$ is not real. Without loss let $v = \operatorname{Im}(a\bar{c})$. The condition that $x = 2av\bar{c} - a\bar{c}v$ is purely imaginary and perpendicular to v becomes that $2a\operatorname{Im}(a\bar{c})\bar{c} - a\bar{c}\operatorname{Im}(a\bar{c})$ is purely imaginary and perpendicular to $a\bar{c}$, giving (i).

It follows from Proposition 4.2 that when $a = \pm c$, (5) has solutions precisely when $|\operatorname{Im}(c)| \geq \frac{1}{2\sqrt{2}}$; see also Remark 4.4. If $a = c$, then (5) becomes $x = 2cv\bar{c} - |c|^2v = 2cv\bar{c} - \frac{1}{2}v$. If $a = -c$, then (5) becomes $x = -2cv\bar{c} + |c|^2v = -2cv\bar{c} + \frac{1}{2}v$. In both cases $\operatorname{Re}(x) = 0$. The condition that v is perpendicular to x means that v is restricted by $\langle cv\bar{c}, v \rangle = \frac{1}{4}|v|^2$, completing (ii). $\square$

3. ESCHENBURG AND KERIN'S METRIC

Let $G = \operatorname{Sp}(2)$, $H = \operatorname{Sp}(1)$ and $K = \operatorname{Sp}(1)^2$. Let $\varrho_L(p, q) = \operatorname{diag}(p, q)$, $\varrho_R(q) = \operatorname{diag}(q, 1)$ and $\varrho(q) = (q, q)$. By Proposition 1.5 we have that $G//H$ is diffeomorphic to $H \times K \times G//H \times H \times K$. Take the bi-invariant metric on $H \times K = \operatorname{Sp}(1)^3$ given by $s\langle, \rangle \oplus t(\langle, \rangle \oplus \langle, \rangle)$. Observe that the metric induced by Proposition 1.5 is isometric to *Eschenburg and Kerin's metric* of [3] via the equivariant isometry defined by the inverse on $\operatorname{Sp}(2)$, $Q \mapsto Q^{-1} = Q^*$.

3.1. Horizontal vectors. With no loss of generality consider a coset representative of the form $P = (1, 1, 1, Q) \in H \times K \times G$.

Let $(\alpha, \beta, \gamma, M) \in \mathfrak{h} \oplus \mathfrak{k} \oplus T_Q G = T_P(H \times K \times G)$. Define M_L and M_R as previously.

Proposition 3.1. *The horizontal distribution at $(1, 1, 1, Q)$ is*

$$\mathcal{H} = \{(t^{-1}u_L, s^{-1}u_R, s^{-1}v_R, M) \in T_P(H \times K \times G) : u_L = u_R + v_R\}.$$

Proof. Let $\epsilon \in \text{Im}(\mathbb{H})$ be arbitrary. Consider the field induced from $(0, 0, \epsilon, 0)$, i.e. $(0, -\epsilon, 0, \text{diag}(\epsilon, 0)Q)$. Hence the horizontal vector satisfies $s\langle -\epsilon, \beta \rangle + \langle \epsilon, u_R \rangle = 0$, i.e. $\beta = s^{-1}u_R$. Likewise from the field $(0, 0, 0, \epsilon)$ we find that $\gamma = s^{-1}v_R$.

Consider the field induced from $(0, \epsilon, 0, 0)$, i.e. $(-\epsilon, \epsilon, \epsilon, 0)$; the horizontal vector satisfies $t\langle -\epsilon, \alpha \rangle + s\langle \epsilon, \beta \rangle + s\langle \epsilon, \gamma \rangle = 0$, i.e. $\alpha = t^{-1}s(\beta + \gamma) = t^{-1}(u_R + v_R)$.

Consider the field induced from $(\epsilon, 0, 0, 0)$, i.e. $(\epsilon, 0, 0, Q\text{diag}(-\epsilon, 0))$; the horizontal vector satisfies $t\langle \epsilon, \alpha \rangle + \langle u_L, -\epsilon \rangle = 0$, i.e. $\alpha = t^{-1}u_L$. Hence $u_L = u_R + v_R$. $\square$

3.2. Zero planes. It follows from Proposition 1.2 that zero planes, $\text{span}\{M^1, M^2\}$, are characterized by the conditions $[u_R^1, u_R^2] = 0$, $[v_R^1, v_R^2] = 0$, and $[M_R^1, M_R^2] = 0$ and $[u_L^1, u_L^2] = 0$ with $u_L^i = u_R^i + v_R^i$. The Wilhelm metric has in addition $[v_L^1, v_L^2] = 0$. But from the earlier analysis of the Wilhelm metric, given the first three conditions one can assume without loss either (1) or (2), and in either case the third condition $[u_L^1, u_L^2] = 0$ then implies the fourth condition that $[v_L^1, v_L^2] = 0$. So in fact both Eschenburg and Kerin's metric and Wilhelm's metric have equivalent zero plane loci.

4. STRUCTURE OF THE ZERO LOCUS

Propositions 2.3 and 2.4 allow one to determine if a given $[Q] \in M$ is in the zero locus. In contrast a useful parametrisation follows from Propositions 4.1 and 4.2.

Remark 4.1. Proposition 4.1 addresses the general form of a solution to the condition (3) in Proposition 2.3.

Proposition 4.1. *The solutions, $p, q \in \text{Sp}(1)$ and $v \in \text{Im}(\mathbb{H})$ with $|v| = 1$ to*

$$pv\bar{p} = qv\bar{q} + v$$

are parametrised by $(p, q, v) = (e^{w\frac{\pi}{6}}e^{v\theta}, e^{w\frac{\pi}{3}}e^{v\phi}, v)$, where $\theta, \phi \in \mathbb{R}$ and $w \in \text{Im}(\mathbb{H})$ with $|w| = 1$ and $w \perp v$ and $v \in \text{Im}(\mathbb{H})$ with $|v| = 1$.

Proof. Write $qv\bar{q} = -(qv\bar{q}v)v$. Note that $\langle -qv\bar{q}v, v \rangle = \text{Re}(qv\bar{q}) = 0$ since $\text{Re}(v) = 0$. Note also $|-qv\bar{q}v| = 1$. Recall that $\langle qv\bar{q}, v \rangle = -\frac{1}{2}$. Thus $\text{Re}(-qv\bar{q}v) = -\frac{1}{2}$. Hence $-qv\bar{q}v = e^{w\frac{2\pi}{3}}$ for $w = \pm \text{Im}(\widehat{qv\bar{q}v})$. It follows that $qv\bar{q} = e^{w\frac{2\pi}{3}}v = e^{w\frac{\pi}{3}}ve^{-w\frac{\pi}{3}}$. Hence $(e^{-w\frac{\pi}{3}}q)v(e^{-w\frac{\pi}{3}}q)^{-1} = v$ and thus $e^{-w\frac{\pi}{3}}q = e^{v\phi}$, or rather $q = e^{w\frac{\pi}{3}}e^{v\phi}$.

$qv\bar{q} + v = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}w\right)v + v = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}w\right)v = e^{w\frac{\pi}{3}}v = e^{w\frac{\pi}{6}}ve^{-w\frac{\pi}{6}}$. So $pv\bar{p} = e^{w\frac{\pi}{6}}ve^{-w\frac{\pi}{6}}$, or rather $e^{-w\frac{\pi}{6}}pv\bar{p}e^{w\frac{\pi}{6}} = v$. Hence $e^{-w\frac{\pi}{6}}p = e^{v\theta}$, or rather $p = e^{w\frac{\pi}{6}}e^{v\theta}$. Substitution reveals that $(p, q, v) = (e^{w\frac{\pi}{6}}e^{v\theta}, e^{w\frac{\pi}{3}}e^{v\phi}, v)$ solves $pv\bar{p} = qv\bar{q} + v$. $\square$

Remark 4.2. Proposition 4.2 addresses the general form of a solution to condition (4) in Proposition 2.4. Equation (6) in Proposition 4.2 below is equivalent to (4) with $a = \frac{1}{\sqrt{2}}p$, $c = \frac{1}{\sqrt{2}}q$ and $t = 1$.

Proposition 4.2. *The solutions $p, q \in \text{Sp}(1)$, $v, x \in \text{Im}(\mathbb{H})$ and $x \perp v$ with $|v| = 1$ to*

$$(6) \quad \bar{p}xq + \bar{q}xp = 2v - \bar{q}vq$$

such that $vp\bar{q} = p\bar{q}v$ are parametrised by

$$(p, q, x, v) = \left(e^{v\theta} e^{w\frac{\pi}{6}} e^{v\phi}, e^{w\frac{\pi}{6}} e^{v\phi}, \frac{\sqrt{3}}{2} e^{v\theta} wv, v \right),$$

where $v, w \in \text{Im}(\mathbb{H})$ with $|v| = |w| = 1$ and $v \perp w$ and $\theta, \phi \in \mathbb{R}$.

Proof. $p(6)\bar{q}$ is $x + p\bar{q}xp\bar{q} = 2pv\bar{q} - p\bar{q}v$. It follows that $2x + p\bar{q}v = 2pv\bar{q}$. Now $p\bar{q} = e^{v\theta}$ since $vp\bar{q} = p\bar{q}v$. Hence $p\bar{q}v \perp x$ since both $x \in \text{Im}(\mathbb{H})$ and $x \perp v$. This in conjunction with the fact that $x + \frac{1}{2}p\bar{q}v = pv\bar{q}$ implies $|x|^2 + \frac{1}{4} = 1$. Hence $|x|^2 = \frac{3}{4}$. So $x = \frac{\sqrt{3}}{2}\hat{x}$. Thus

$$pv\bar{q} = \frac{1}{2}e^{v\theta}v + \frac{\sqrt{3}}{2}\hat{x}.$$

Let $r = \frac{1}{2}e^{v\theta} - \frac{\sqrt{3}}{2}\hat{x}v$. Then $pv\bar{q} = rv$. Hence $e^{v\theta}qv\bar{q} = rv$. Thus $qv\bar{q} = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}e^{-v\theta}\hat{x}v\right)v = e^{w\frac{\pi}{3}}v = e^{w\frac{\pi}{6}}ve^{-w\frac{\pi}{6}}$, where $w = -e^{-v\theta}\hat{x}v$. Therefore $e^{-w\frac{\pi}{6}}qv(e^{-w\frac{\pi}{6}}q)^{-1} = v$. Hence $e^{-w\frac{\pi}{6}}q = e^{v\phi}$, or rather $q = e^{w\frac{\pi}{6}}e^{v\phi}$. Consequently $p = e^{v\theta}e^{w\frac{\pi}{6}}e^{v\phi}$ and $\hat{x} = e^{v\theta}wv$. Again substitution reveals that such data gives a solution to (6). $\square$

Remark 4.3. Note in Proposition 4.1 that p, q are such that

$$\text{Im}(p) = \frac{1}{2}e^{-v\theta}w + \frac{\sqrt{3}}{2}\sin(\theta)v, \quad \text{Im}(q) = \frac{\sqrt{3}}{2}e^{-v\phi}w + \frac{1}{2}\sin(\phi)v.$$

Thus $|\text{Im}(p)|^2 = \frac{1}{4} + \frac{3}{4}\sin^2(\theta)$ and $|\text{Im}(q)|^2 = \frac{3}{4} + \frac{1}{4}\sin^2(\phi)$.

Remark 4.4. Note in Proposition 4.2 that p is such that

$$\text{Im}(p) = \frac{\sqrt{3}}{2}v\sin(\theta + \phi) + \frac{1}{2}e^{v(\theta - \phi)}w.$$

Hence $|\text{Im}(p)|^2 = \frac{1}{4} + \frac{3}{4}\sin^2(\theta + \phi)$. Similarly it follows that $|\text{Im}(q)|^2 = \frac{1}{4} + \frac{3}{4}\sin^2(\phi)$.

Proposition 4.3. Any $[Q] \in M = G//H$ in the zero locus up to the isometric action of $\text{Sp}(1)$ via

$$r \odot [Q] \equiv \left[Q \begin{pmatrix} 1 & 0 \\ 0 & \bar{r} \end{pmatrix} \right]$$

can be represented by $Q \in \text{Sp}(2)$ of one of the forms

$$(7) \quad Q = \begin{pmatrix} e^{j\frac{\pi}{3}}e^{i(\phi - \theta)}e^{-j\frac{\pi}{6}}\cos(\psi) & -\sin(\psi) \\ e^{-i\theta}e^{-j\frac{\pi}{6}}\sin(\psi) & e^{-i\phi}e^{-j\frac{\pi}{3}}\cos(\psi) \end{pmatrix},$$

$$(8) \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta}e^{j\frac{\pi}{6}}e^{i\phi} & -1 \\ e^{j\frac{\pi}{6}}e^{i\phi} & e^{-i\theta} \end{pmatrix},$$

where $\theta, \phi, \psi \in \mathbb{R}$.

Proof. The proof splits into the $t = 0$ case and the $t \neq 0$ case.

In the $t = 0$ case since $|a|^2 + |c|^2 = 1$ put $a = \hat{a}\cos(\psi)$ and $c = \hat{c}\sin(\psi)$. Generically from Propositions 2.3 and 4.1 it follows that $\hat{c} = \bar{p} = e^{-v\theta}e^{-w\frac{\pi}{6}}$ and $\hat{a} = \hat{a}\hat{c} = q\bar{p} = e^{w\frac{\pi}{3}}e^{v(\phi - \theta)}e^{-w\frac{\pi}{6}}$. Without loss assume $b = -\sin(\psi)$ since $b\bar{r}$ is real for appropriate r . It follows that $b\bar{d} = -a\bar{c} = -\cos(\psi)\sin(\psi)q$. Hence $d = \cos(\psi)\bar{q} = \cos(\psi)e^{-v\phi}e^{-w\frac{\pi}{3}}$.

In the $t \neq 0$ case, $|a|^2 = |c|^2 = \frac{1}{2}$. Let $a = \frac{1}{\sqrt{2}}p$ and $c = \frac{1}{\sqrt{2}}q$ and as before let $b = -\frac{1}{\sqrt{2}}$ be real. Then $d = \frac{1}{\sqrt{2}}q\bar{p}$. It follows from Propositions 2.4 and 4.2 that $a = \frac{1}{\sqrt{2}}e^{v\theta}e^{w\frac{\pi}{6}}e^{v\phi}$, $c = \frac{1}{\sqrt{2}}e^{w\frac{\pi}{6}}e^{v\phi}$ and $d = \frac{1}{\sqrt{2}}e^{-v\theta}$.

Let $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = s \cdot Q = \begin{pmatrix} sa\bar{s} & sb \\ sc\bar{s} & sd \end{pmatrix}$ and $(v', w') = (sv\bar{s}, sw\bar{s})$.

In the $t = 0$ case, Q takes the form $\begin{pmatrix} e^{w\frac{\pi}{6}}e^{v(\phi-\theta)}e^{-w\frac{\pi}{6}}\cos(\psi) & * \\ e^{-v\theta}e^{-w\frac{\pi}{6}}\sin(\psi) & * \end{pmatrix}$. Thus

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} se^{w\frac{\pi}{6}}e^{v(\phi-\theta)}e^{-w\frac{\pi}{6}}\cos(\psi)\bar{s} & * \\ se^{-v\theta}e^{-w\frac{\pi}{6}}\sin(\psi)\bar{s} & * \end{pmatrix} = \begin{pmatrix} e^{w'\frac{\pi}{6}}e^{v'(\phi-\theta)}e^{-w'\frac{\pi}{6}}\cos(\psi) & * \\ e^{-v'\theta}e^{-w'\frac{\pi}{6}}\sin(\psi) & * \end{pmatrix}.$$

Similarly in the $t \neq 0$ case Q takes the form $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{v\theta}e^{w\frac{\pi}{6}}e^{v\phi} & * \\ e^{v\frac{\pi}{6}}e^{w\phi} & * \end{pmatrix}$. Thus

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} s(e^{v\theta}e^{w\frac{\pi}{6}}e^{v\phi})\bar{s} & * \\ s(e^{v\frac{\pi}{6}}e^{w\phi})\bar{s} & * \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{v'\theta}e^{w'\frac{\pi}{6}}e^{v'\phi} & * \\ e^{v'\frac{\pi}{6}}e^{w'\phi} & * \end{pmatrix}.$$

It follows from the transitivity of the action on the two orthonormal frames that one may assume without loss that $(v, w) = (i, j)$ in either case from the outset. $\square$

Remark 4.5. The two parametrised components of the zero locus in Proposition 4.3 are 6-dimensional and 5-dimensional in $M = G//H$ for forms (7) and (8) respectively. Since $M = G//H$ is 7-dimensional the manifold has almost positive sectional curvature.

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