BASIC HYPERGEOMETRIC FUNCTIONS AND ORTHOGONAL LAURENT POLYNOMIALS

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ABSTRACT. A three-complex-parameter class of orthogonal Laurent polynomials on the unit circle associated with basic hypergeometric or q -hypergeometric functions is considered. To be precise, we consider the orthogonality properties of the sequence of polynomials $\{{}_2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q, q^{-c+d-1}z)\}_{n=0}^{\infty}$, where $0 < q < 1$ and the complex parameters b , c and d are such that $b \neq -1, -2, \dots$, $c - b + 1 \neq -1, -2, \dots$, $\operatorname{Re}(d) > 0$ and $\operatorname{Re}(c - d + 2) > 0$. Explicit expressions for the recurrence coefficients, moments, orthogonality and also asymptotic properties are given. By a special choice of the parameters, results regarding a class of Szegő polynomials are also derived.

1. INTRODUCTION

Given the double sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ of complex numbers, let the linear functional $\mathcal{M}$ on the space of Laurent polynomials be defined by

$$(1.1) \quad \mathcal{M}[z^{-n}] = \mu_n, \quad n = 0, \pm 1, \pm 2, \dots$$

The functional $\mathcal{M}$ can be referred to as a moment functional.

Let D_n , $n = 0, 1, \dots$, be the associated Toeplitz determinants as given by:

$$D_{-1} = 1, \quad D_0 = \mu_0 \quad \text{and} \quad D_n = \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \end{vmatrix}, \quad n \geq 1.$$

We consider the sequence of polynomials $\{Q_n\}$ that satisfies

$$\mathcal{M}[z^{-s}Q_n(z)] = \rho_n \delta_{n,s}, \quad 0 \leq s \leq n, \quad n \geq 1,$$

where Q_n , for any $n \geq 0$, is a monic polynomial of degree n . If the moment functional $\mathcal{M}$ is such that $D_n \neq 0$, $n \geq 0$, then we will refer to it as a semi-definite

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moment functional. In this case it is easily seen that the sequence of polynomials $\{Q_n\}$ exists uniquely and that

$$Q_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix} \quad \text{and} \quad \rho_n = \mathcal{M}[z^{-n}Q_n(z)] = \frac{D_n}{D_{n-1}}.$$

There have been different nomenclatures used with respect to such polynomials in recent years. The polynomials Q_n are related to *orthogonal Laurent polynomials* considered by, for example, Hendriksen and van Rossum [11] and Jones and Thron [15], in the sense that the Laurent polynomials

$$B_{2n}(z) = z^{-n}Q_{2n}(z), \quad B_{2n+1}(z) = z^{-n}Q_{2n+1}(z), \quad n \geq 0,$$

satisfy the orthogonality relations $\mathcal{M}[B_n(z)B_m(z)] = \delta_{n,m}\tilde{\rho}_n$, $n, m = 0, 1, 2, \dots$.

With the monic polynomials $\{\hat{Q}_n\}$ given by

$$\hat{Q}_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad n \geq 1,$$

we obtain the biorthogonality relations $\mathcal{M}[\hat{Q}_m(1/z)Q_n(z)] = \delta_{n,m}\rho_n$, $n, m = 0, 1, 2, \dots$. Hence, Zhedanov [32] calls such polynomials Laurent biorthogonal.

With respect to the moment functional $\mathcal{L}[z^n] = \mathcal{M}[z^{-n}] = \mu_n$, $n = 0, \pm 1, \pm 2, \dots$, the reciprocal polynomials $Q_n^\bullet(z) = z^n Q_n(1/z)$ satisfy the orthogonality relations $\mathcal{L}[z^{-n+s}Q_n^\bullet(z)] = \delta_{n,s}\rho_n$, $0 \leq s \leq n$. Polynomials satisfying such orthogonality relations have been referred to as *L-orthogonal polynomials* in some earlier contributions, including [1], of one of the present authors. We remark that Zhedanov [32] uses the definition $\mathcal{L}[z^n] = \mu_n$, $n = 0, \pm 1, \pm 2, \dots$, for his moment functional and $\mathcal{L}[\hat{Q}_m(z)Q_n(1/z)] = \delta_{n,m}\rho_n$, $n, m = 0, 1, 2, \dots$.

In a recent manuscript [17], $\{Q_n\}$ has been called a sequence of monic *Szegő type* polynomials when $\mathcal{M}$ is such that $D_n \neq 0$ and $\mu_{-n} = \mu_n$ for $n \geq 0$. In this case the Zhedanov [32] biorthogonality can be written as $\mathcal{M}[Q_m(1/z)Q_n(z)] = \delta_{n,m}\rho_n$, $n, m = 0, 1, 2, \dots$.

However, if $\mathcal{M}$ is such that $D_n > 0$ and $\mu_{-n} = \overline{\mu}_n$, $n \geq 0$, then this moment functional is known as a positive definite moment functional and the sequence of polynomials $\{S_n\} = \{Q_n\}$ are known as monic *Szegő polynomials*. Now we must have $\mathcal{M}[f] = \int_{\mathcal{C}} f(z)d\mu(z)$, where $\mu(z) = \mu(e^{i\theta})$ is a positive measure on the unit circle $\mathcal{C} = \{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Since the integration is along the unit circle, $\int_{\mathcal{C}} z^{-j}S_n(z)d\mu(z) = \int_{\mathcal{C}} \bar{z}^j S_n(z)d\mu(z)$ and the associated sequence of monic Szegő polynomials $\{S_n\}$ are usually defined by

$$\int_{\mathcal{C}} \overline{S_m(z)} S_n(z)d\mu(z) = \int_0^{2\pi} \overline{S_m(e^{i\theta})} S_n(e^{i\theta})d\mu(e^{i\theta}) = \kappa_n^{-2} \delta_{nj}, \quad m, n = 0, 1, 2, \dots,$$

where $\kappa_n^{-2} = \|S_n\|^2 = \int_{\mathcal{C}} |S_n(z)|^2 d\mu(z)$.

With his publications [28] and [29], Szegő introduced these *orthogonal polynomials on the unit circle* in the early 20th century. Many interesting results on these

polynomials can be found in his classical book [30], the first edition of which was published in 1939. Since then, these polynomials which bear the name of Szegő were extensively studied by many. We cite, for example, [5], [6], [7], [10], [19], [20], [22], [25] and [27] as some of the very recent contributions. The recent publications of the two excellent volumes [23] and [24] by Simon have given a boost to the interest in studying these polynomials. We also cite the recent book [13] by Ismail containing a nice chapter on these orthogonal polynomials on the unit circle.

Some information on the Szegő polynomials with respect to the measure $d\mu(e^{i\theta}) = [e^{-\theta}]^\eta [\sin^2(\theta/2)]^\lambda d\theta$, defined for $\eta, \lambda \in \mathbb{R}$ and $\lambda > -1/2$, are provided in [27]. It was shown that these Szegő polynomials are constant multiples of the hypergeometric polynomials ${}_2F_1(-n, b+1; b+\bar{b}+1; 1-z)$, $n \geq 1$, where $\eta = \mathcal{I}m(b)$ and $\lambda = \mathcal{R}e(b)$.

Results used in [27] have an important root in the paper [11] of Hendriksen and van Rossum, where these authors look at T-fractions and orthogonal Laurent polynomials originating from three-term contiguous relations satisfied by the hypergeometric functions ${}_2F_1(a, b; c; z)$.

In this paper, using a three-term contiguous relation satisfied by q -hypergeometric functions ${}_2\Phi_1(q^a, q^b; q^c; q, z)$, we obtain information on the three-parameter class of orthogonal Laurent polynomials $\{z^{-\lfloor n/2 \rfloor} Q_n^{(b,c,d)}(z)\}_{n=0}^\infty$ on the unit circle $\mathcal{C} = \{z = e^{i\theta} : 0 < \theta < 2\pi\}$, where the monic polynomials $Q_n^{(b,c,d)}$, $n \geq 0$, are given by

$$Q_n^{(b,c,d)}(z) = \frac{(q^{c-b+1}; q)_n}{(q^{b+1}; q)_n} q^{n(b-d+1)} {}_2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q, q^{-c+d-1}z),$$

with $0 < q < 1$ and the three complex parameters b , c and d are such that $b \neq -1, -2, \dots$, $c - b + 1 \neq -1, -2, \dots$, $\mathcal{R}e(d) > 0$ and $\mathcal{R}e(c + 2 - d) > 0$. The orthogonality is with respect to the semi-definite moment functional $\mathcal{M}^{(b,c,d)}$ given by

$$\mathcal{M}^{(b,c,d)}[f(z)] = \frac{\tau^{(b,c)}}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{(q^{-b+d}e^{i\theta}; q)_\infty (q^{b-d+1}e^{-i\theta}; q)_\infty}{(q^d e^{i\theta}; q)_\infty (q^{c+2-d}e^{-i\theta}; q)_\infty} d\theta.$$

Here the constant $\tau^{(b,c)}$, defined in Theorem 3.5, is such that $\mathcal{M}^{(b,c,d)}[1] = 1$. By considering separately the real and imaginary parts of b , c and d , and neglecting $\mathcal{I}m(d)$, which induces only a rotation, we can also consider $\{z^{-\lfloor n/2 \rfloor} Q_n^{(b,c,d)}(z)\}_{n=0}^\infty$ as a five-real-parameter class of orthogonal Laurent polynomials.

The class of polynomials considered here is somewhat different and broader than the class of orthogonal Laurent polynomials $\{z^{-\lfloor n/2 \rfloor} P_n(-z, \alpha, \beta)\}_{n=0}^\infty$ that follows from Pastro [21], where

$$P_n(z, \alpha, \beta) = {}_2\Phi_1(\tilde{q}^{-n}, \tilde{q}^\alpha; \tilde{q}^{2-\beta-n}; \tilde{q}, \tilde{q}^{-\beta+3/2}z),$$

with $|\tilde{q}| < 1$, $\alpha > 1/2$ and $\beta > 1/2$. Pastro shows that the polynomials $P_n(-z, \alpha, \beta)$ are the Laurent biorthogonal polynomials with respect to the semi-definite moment functional given by

$$\tilde{\mathcal{M}}^{(\alpha,\beta)}[f(z)] = \int_0^{2\pi} f(e^{i\theta}) \frac{(\tilde{q}^{1/2}e^{i\theta}; \tilde{q})_\infty (\tilde{q}^{1/2}e^{-i\theta}; \tilde{q})_\infty}{(\tilde{q}^{\alpha-1/2}e^{i\theta}; \tilde{q})_\infty (\tilde{q}^{\beta-1/2}e^{-i\theta}; \tilde{q})_\infty} d\theta.$$

These Pastro polynomials can be considered as belonging to a class with the three real parameters α , β and, say, ϑ if one assumes the $\tilde{q}$ to be such that $\tilde{q} = |\tilde{q}|e^{i\vartheta}$. An

explicit expression for the moments $\tilde{\mathcal{M}}^{(\alpha,\beta)}[z^{-n}]$ is found in Vinet and Zhedanov [31].

Note that the moment functional $\tilde{\mathcal{M}}^{(\alpha,\beta)}$ can only be made positive definite with the choice $-1 < \tilde{q} < 1$ and $\alpha = \beta > 1/2$. Thus with this choice, Pastro [21] recovers the class of real Szegő polynomials previously described by Askey [4, p. 806].

By a special choice of the parameters b , c and d we also obtain in the present manuscript information regarding the class of (complex and real) Szegő polynomials $S_n^{(\lambda,\eta,\phi)}$ characterized by the reflection coefficients

$$a_n^{(\lambda,\eta,\phi)} = \frac{(q^{\lambda+i\eta}; q)_n}{(q^{\lambda+1-i\eta}; q)_n} q^{n[\frac{1}{2}-i(\eta+\phi)]}, \quad n \geq 1,$$

where $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$. The parameter ϕ , which comes from $\mathcal{I}m(d)$, induces only a rotation and can be made equal to zero without any loss of generality.

The polynomials obtained by taking $\eta = \phi = 0$ and $\lambda > -1/2$, for example, coincide with the real Szegő polynomials of [21] and [4], obtained when $0 < \tilde{q} < 1$ and $\alpha = \beta > 1/2$.

The paper is organized as follows. In Section 2 we present some fundamental results on three-term recurrence relations, continued fractions and basic hypergeometric functions, which we will be using in later sections. In Section 3 we define the monic q -hypergeometric polynomials $Q_n^{(b,c,d)}(z)$ and obtain their orthogonality and asymptotic properties. In Section 4, in addition to discussing when the polynomials $Q_n^{(b,c,d)}(z)$ coincide with the Szegő polynomials $S_n^{(\lambda,\eta,\phi)}$ mentioned above, we also obtain explicitly the associated Szegő function.

2. SOME PRELIMINARY RESULTS

Let $\{Q_n\}$ be the sequence of polynomials given by the three-term recurrence relation

$$(2.1) \quad Q_{n+1}(z) = (z + \beta_{n+1})Q_n(z) - \alpha_{n+1}zQ_{n-1}(z), \quad n \geq 1,$$

with $Q_0(z) = 1$ and $Q_1(z) = z + \beta_1$.

Lemma 2.1. *Let $\beta_1 \neq 0$ and $\alpha_{n+1} \neq 0$ for $n \geq 1$. Given any sequence $\{h_n\}$ of arbitrary complex numbers h_n (or complex functions $h_n(z)$), let the sequence of functions $\{G_n(h_n; z)\}$ be such that $G_1(h_1; z) = \frac{\beta_1}{z + \beta_1 - h_1}$ and*

$$G_n(h_n; z) = \frac{\beta_1}{z + \beta_1} - \frac{\alpha_2 z}{z + \beta_2} - \dots - \frac{\alpha_n z}{z + \beta_n - h_n}, \quad n \geq 2.$$

Then

$$G_n(h_n; z) - G_n(0; z) = \frac{\beta_1 \alpha_2 \alpha_3 \cdots \alpha_n h_n z^{n-1}}{Q_n(z)[Q_n(z) - h_n Q_{n-1}(z)]}.$$

Proof. Let the sequence of polynomials $\{R_n\}$ be such that

$$R_{n+1}(z) = (z + \beta_{n+1})R_n(z) - \alpha_{n+1}zR_{n-1}(z), \quad n \geq 1,$$

with $R_0(z) = 0$, $R_1(z) = \beta_1$. Then from basic results on continued fractions (see, for example, [14, 18])

$$G_n(h_n; z) - G_n(0; z) = \frac{R_n(z) - h_n R_{n-1}(z)}{Q_n(z) - h_n Q_{n-1}(z)} - \frac{R_n(z)}{Q_n(z)}, \quad n \geq 1.$$

Hence,

$$G_n(h_n; z) - G_n(0; z) = \frac{h_n[R_n(z)Q_{n-1}(z) - Q_n(z)R_{n-1}(z)]}{Q_n(z)[Q_n(z) - h_nQ_{n-1}(z)]}, \quad n \geq 1.$$

Therefore, the lemma follows from $R_n(z)Q_{n-1}(z) - Q_n(z)R_{n-1}(z) = \beta_1\alpha_2\alpha_3 \cdots \alpha_n z^{n-1}$. $\square$

As a particular case of this lemma, if one takes $h_n = \alpha_{n+1}z/(z + \beta_{n+1})$, then

$$(2.2) \quad G_{n+1}(0; z) - G_n(0; z) = \frac{\beta_1\alpha_2\alpha_3 \cdots \alpha_{n+1}}{Q_n(z)Q_{n+1}(z)} z^n, \quad n \geq 1.$$

Lemma 2.2. *In the three-term recurrence relation (2.1), if*

$$\beta_n \neq 0 \quad \text{and} \quad \alpha_{n+1} \neq 0, \quad n \geq 1,$$

then there exists a semi-definite moment functional $\mathcal{M}$ such that the polynomials Q_n satisfy

$$\mathcal{M}[z^{-s}Q_n(z)] = \delta_{n,s}\rho_n, \quad 0 \leq s \leq n, \quad n \geq 1,$$

where $\rho_n = \frac{\alpha_2 \cdots \alpha_{n+1}}{\beta_2 \cdots \beta_{n+1}}$. Moreover, the associated moments $\mu_n = \mathcal{M}[z^{-n}]$, $n = 0, \pm 1, \pm 2, \dots$ are such that $L_0(z) = \sum_{j=0}^{\infty} \mu_j z^j$, $L_{\infty}(z) = -\sum_{j=1}^{\infty} \mu_{-j} z^{-j}$, where

$$(2.3) \quad \begin{aligned} L_0(z) - G_n(0; z) &= \rho_n \frac{1}{Q_n(0)} z^n + O(z^{n+1}), \\ L_{\infty}(z) - G_n(0; z) &= \rho_n Q_{n+1}(0) \frac{1}{z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right). \end{aligned}$$

Proof. First note that $Q_n(0) = \beta_1\beta_2 \cdots \beta_n \neq 0$. Now from (2.2) by considering the expansions of $G_n(0; z)$ about the origin and infinity there exist power series $L_0(z) = \sum_{j=0}^{\infty} \mu_j z^j$ and $L_{\infty}(z) = -\sum_{j=1}^{\infty} \mu_{-j} z^{-j}$ such that (2.3) holds.

With respect to these power series coefficients, if we define the moment functional $\mathcal{M}$ by (1.1), then the lemma follows from the linear system on the coefficients of Q_n and R_n obtained from (2.3). $\square$

For $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$ and $0 < |q| < 1$, the ${}_2\Phi_1$ q -hypergeometric or the ${}_2\Phi_1$ basic hypergeometric function (hypergeometric function with base q) may be defined by

$${}_2\Phi_1(q^a, q^b; q^c; q, z) = \sum_{n=0}^{\infty} \frac{(q^a; q)_n (q^b; q)_n}{(q^c; q)_n (q; q)_n} z^n,$$

for $|z| < 1$ and by analytic continuation for other values of $z \in \mathbb{C}$. Here, $(q^a; q)_0 = 1$ and $(q^a; q)_n = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n-1})$, $n \geq 1$.

For more information regarding q -hypergeometric functions, we refer to, for example, Andrews, Askey and Roy [2], Gasper and Rahman [8], Koekoek and Swart-touw [16] and Slater [26].

Two “distinct” q -hypergeometric functions ${}_2\Phi_1(q^{a_1}, q^{a_2}; q^{a_3}; q, z)$ and ${}_2\Phi_1(q^{\tilde{a}_1}, q^{\tilde{a}_2}; q^{\tilde{a}_3}; q, z)$ may be called contiguous if $|a_i - \tilde{a}_i| = 0$ or 1 for at least one $i \in \{1, 2, 3\}$. There are interesting relations between contiguous q -hypergeometric functions called contiguous relations.

Lemma 2.3. *If $c \neq 0, -1, -2, \dots$, then*

$$\begin{aligned} {}_2\Phi_1(q^a, q^{b+1}; q^c; q, z) &= \left(1 + \frac{1 - q^{a-b}}{1 - q^c} q^b z\right) {}_2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z) \\ &\quad - \frac{(1 - q^{a+1})(1 - q^{c-b})}{(1 - q^c)(1 - q^{c+1})} q^b z {}_2\Phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, z). \end{aligned}$$

Proof. From contiguous relations obtained by Heine (see [8, p. 22]), we consider the following:

$$\begin{aligned} {}_2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z) &= {}_2\Phi_1(q^{a+1}, q^b; q^c; q, z) \\ &\quad + \frac{(1 - q^{a+1})(1 - q^{c-b})}{(1 - q^c)(1 - q^{c+1})} q^b z {}_2\Phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, z), \\ {}_2\Phi_1(q^{a+1}, q^b; q^c; q, z) &= {}_2\Phi_1(q^a, q^{b+1}; q^c; q, z) \\ &\quad - \frac{(1 - q^{a-b})}{(1 - q^c)} q^b z {}_2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z), \end{aligned}$$

which hold for $c \neq 0, -1, -2, \dots$. Substitution for ${}_2\Phi_1(q^{a+1}, q^b; q^c; q, z)$ in the first relation using the other gives the required result. $\square$

We will be using the q -binomial theorem (see [16, Eq. (0.5.2)])

$$(2.4) \quad {}_2\Phi_1(q^a, q^c; q^c; q, z) = {}_1\Phi_0(q^a; q, z) = \frac{(q^a z; q)_\infty}{(z; q)_\infty},$$

which holds for $c \neq 0, -1, -2, \dots$ and $|z| < 1$, and the Heine transformation formula (see [16, Eq. (0.6.3)])

$$(2.5) \quad {}_2\Phi_1(q^a, q^b; q^c; q, z) = \frac{(q^{a+b-c} z; q)_\infty}{(z; q)_\infty} {}_2\Phi_1(q^{-a+c}, q^{-b+c}; q^c; q, q^{a+b-c} z),$$

which holds for $c \neq 0, -1, -2, \dots$ and $|z| < \min\{1, |q^{c-a-b}|\}$. We will also be needing the polynomial identities (see [16, Eq. (0.6.19)])

$$(2.6) \quad \begin{aligned} &{}_2\Phi_1(q^{-n}, q^b; q^c; q, z) \\ &= \frac{(q^b; q)_n}{(q^c; q)_n} q^{-n(n+1)/2} (-z)^n {}_2\Phi_1(q^{-n}, q^{-c-n+1}; q^{-b-n+1}; q, q^{c-b+n+1} z^{-1}), \end{aligned}$$

for $n \geq 0$, which hold when $c \neq 0, -1, -2, \dots$ and $b \neq -n+1, -n+2, -n+3, \dots$.

3. q -ORTHOGONAL LAURENT POLYNOMIALS

From now on we restrict the value of q to be such that $0 < q < 1$. Then for any $b \in \mathbb{C}$ we have

$$\overline{q^b} = q^{\bar{b}} \quad \text{and} \quad |q^b| = q^{\operatorname{Re}(b)}.$$

With $b, c, d \in \mathbb{C}$ and $c - b + 1 \neq 0, -1, -2, \dots$, let

$$F_n^{(b,c,d)}(z) = \frac{{}_2\Phi_1(q^{n+1}, q^{-b}; q^{c-b+n+2}; q, q^d z)}{{}_2\Phi_1(q^n, q^{-b}; q^{c-b+n+1}; q, q^d z)}, \quad n \geq 0.$$

Then from Lemma 2.3,

$$F_n^{(b,c,d)}(z) = \frac{1}{1 + g_{n+1}^{(b,c,d)} z - f_{n+2}^{(b,c,d)} z F_{n+1}^{(b,c,d)}(z)}, \quad n \geq 0,$$

where

$$g_n^{(b,c,d)} = \frac{1 - q^{b+n}}{1 - q^{c-b+n}} q^{-b+d-1}, \quad f_{n+1}^{(b,c,d)} = \frac{(1 - q^n)(1 - q^{c+n+1})}{(1 - q^{c-b+n})(1 - q^{c-b+n+1})} q^{-b+d-1},$$

for $n \geq 1$. This leads to the continued fraction expansion

$$(3.1) \quad F_0^{(b,c,d)}(z) = \cfrac{1}{1 + g_1^{(b,c,d)} z} - \cfrac{f_2^{(b,c,d)} z}{1 + g_2^{(b,c,d)} z} - \cdots - \cfrac{f_n^{(b,c,d)} z}{1 + g_n^{(b,c,d)} z - f_{n+1}^{(b,c,d)} z F_n^{(b,c,d)}(z)}.$$

Also assuming $b \neq -1, -2, \dots$, this can be written in the equivalent form

$$(3.2) \quad F_0^{(b,c,d)}(z) = \cfrac{\beta_1^{(b,c,d)}}{z + \beta_1^{(b,c,d)}} - \cfrac{\alpha_2^{(b,c,d)} z}{z + \beta_2^{(b,c,d)}} - \cdots - \cfrac{\alpha_n^{(b,c,d)} z}{z + \beta_n^{(b,c,d)}} - \cfrac{\alpha_{n+1}^{(b,c,d)} z F_n^{(b,c,d)}(z)}{\beta_{n+1}^{(b,c,d)}},$$

where $\beta_n^{(b,c,d)} = 1/g_n^{(b,c,d)}$ and $\alpha_{n+1}^{(b,c,d)} = f_{n+1}^{(b,c,d)}/(g_n^{(b,c,d)} g_{n+1}^{(b,c,d)})$, $n \geq 1$.

Theorem 3.1. *With $b \neq -1, -2, \dots$ and $c - b + 1 \neq -1, -2, \dots$, let the sequence of monic polynomials $\{Q_n^{(b,c,d)}\}$ be given by*

$$(3.3) \quad Q_{n+1}^{(b,c,d)}(z) = (z + \beta_{n+1}^{(b,c,d)})Q_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} z Q_{n-1}^{(b,c,d)}(z), \quad n \geq 1,$$

with $Q_0^{(b,c,d)}(z) = 1$ and $Q_1^{(b,c,d)}(z) = z + \beta_1^{(b,c,d)}$, where

$$\beta_n^{(b,c,d)} = \frac{1 - q^{c-b+n}}{1 - q^{b+n}} q^{b-d+1}, \quad \alpha_{n+1}^{(b,c,d)} = \frac{(1 - q^n)(1 - q^{c+n+1})}{(1 - q^{b+n})(1 - q^{b+n+1})} q^{b-d+1}, \quad n \geq 1.$$

Then the polynomials $Q_n^{(b,c,d)}$ satisfy the orthogonality relations

$$(3.4) \quad \mathcal{M}^{(b,c,d)}[z^{-s} Q_n^{(b,c,d)}(z)] = \delta_{n,s} \rho_n^{(b,c)}, \quad 0 \leq s \leq n, \quad n \geq 1,$$

with respect to the semi-definite moment functional

$$\mathcal{M}^{(b,c,d)}[z^{-j}] = \frac{(q^{-b}; q)_j}{(q^{c-b+2}; q)_j} q^{jd}, \quad j = 0, \pm 1, \pm 2, \dots$$

Here,

$$\rho_n^{(b,c)} = \frac{\alpha_2^{(b,c,d)} \cdots \alpha_{n+1}^{(b,c,d)}}{\beta_2^{(b,c,d)} \cdots \beta_{n+1}^{(b,c,d)}} = \frac{(q; q)_n (q^{c+2}; q)_n}{(q^{b+1}; q)_n (q^{c-b+2}; q)_n}.$$

Proof. We first prove the theorem for $c - b + 1 \neq 0, -1, -2, \dots$ and $b \neq -1, -2, \dots$. With these restrictions $\beta_n^{(b,c,d)} \neq 0$ and $\alpha_{n+1}^{(b,c,d)} \neq 0$, $n \geq 1$, and hence from Lemma 2.2 there exists a semi-definite moment functional such that (3.4) holds.

To obtain the values of $\mu_j^{(b,c,d)} = \mathcal{M}^{(b,c,d)}[z^{-j}]$, $j = 0, \pm 1, \pm 2, \dots$, let us consider the functions

$$G_n^{(b,c,d)}(z) = \cfrac{\beta_1^{(b,c,d)}}{z + \beta_1^{(b,c,d)}} - \cfrac{\alpha_2^{(b,c,d)} z}{z + \beta_2^{(b,c,d)}} - \cdots - \cfrac{\alpha_n^{(b,c,d)} z}{z + \beta_n^{(b,c,d)}}, \quad n \geq 1.$$

Then from Lemma 2.1 and the continued fraction expansion (3.2),

$$\begin{aligned} F_0^{(b,c,d)}(z) - G_n^{(b,c,d)}(z) &= \frac{\beta_1^{(b,c,d)} \alpha_2^{(b,c,d)} \cdots \alpha_n^{(b,c,d)} \alpha_{n+1}^{(b,c,d)} z^n F_n^{(b,c,d)}(z)}{Q_n^{(b,c,d)}(z) [\beta_{n+1}^{(b,c,d)} Q_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} z F_n^{(b,c,d)}(z) Q_{n-1}^{(b,c,d)}(z)]} \\ &= \rho_n^{(b,c)} \frac{1}{Q_n^{(b,c,d)}(0)} z^n + O(z^{n+1}), \quad \text{for } n \geq 1. \end{aligned}$$

Since $F_0^{(b,c,d)}(z) = {}_2\Phi_1(q, q^{-b}; q^{c-b+2}; q, q^d z)$, from the latter part of Lemma 2.2,

$$\mu_j^{(b,c,d)} = \frac{(q^{-b}; q)_j}{(q^{c-b+2}; q)_j} q^{jd}, \quad j = 0, 1, 2, \dots,$$

thus giving the results for the positive moments.

From (3.1), by realizing that $g_n^{(c-b,c,c+2-d)} = \beta_n^{(b,c,d)}$ and $f_{n+1}^{(c-b,c,c+2-d)} = \alpha_{n+1}^{(b,c,d)}$, $n \geq 1$, we also obtain the continued fraction expansion

$$\begin{aligned} \frac{\beta_1^{(b,c,d)}}{z} F_0^{(c-b,c,c+2-d)}(z^{-1}) &= \cfrac{\beta_1^{(b,c,d)}}{z + \beta_1^{(b,c,d)}} - \cfrac{\alpha_2^{(b,c,d)} z}{z + \beta_2^{(b,c,d)}} - \cdots - \cfrac{\alpha_n^{(b,c,d)} z}{z + \beta_n^{(b,c,d)}} - \cfrac{\alpha_{n+1}^{(b,c,d)} F_n^{(c-b,c,c+2-d)}(z^{-1})}{1}. \end{aligned}$$

Hence, again from Lemma 2.1,

$$\begin{aligned} \frac{\beta_1^{(b,c,d)}}{z} F_0^{(c-b,c,c+2-d)}(z^{-1}) - G_n^{(b,c,d)}(z) &= \frac{\beta_1^{(b,c,d)} \alpha_2^{(b,c,d)} \cdots \alpha_n^{(b,c,d)} \alpha_{n+1}^{(b,c,d)} z^{n-1} F_n^{(c-b,c,c+2-d)}(z^{-1})}{Q_n^{(b,c,d)}(z) [\beta_{n+1}^{(b,c,d)} Q_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} z F_n^{(c-b,c,c+2-d)}(z^{-1}) Q_{n-1}^{(b,c,d)}(z)]} \\ &= \rho_n^{(b,c)} Q_{n+1}^{(b,c,d)}(0) \frac{1}{z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right), \quad \text{for } n \geq 1. \end{aligned}$$

Since $F_0^{(c-b,c,c+2-d)}(z^{-1}) = {}_2\Phi_1(q, q^{-c+b}; q^{b+2}; q, q^{c+2-d} z^{-1})$, from the latter part of Lemma 2.2,

$$\mu_{-j}^{(b,c,d)} = \frac{(q^{-c+b-1}; q)_j}{(q^{b+1}; q)_j} q^{j(c+2-d)}, \quad j = 1, 2, 3, \dots$$

Thus, using $(a, q)_n = (a; q)_\infty / (aq^n; q)_\infty$, for $n = 0, \pm 1, \pm 2, \dots$, we also obtain the results for the negative moments. This concludes the theorem for $c - b + 1 \neq 0, -1, -2, \dots$ and $b \neq -1, -2, \dots$.

Now to extend the results for $c - b + 1 \neq -1, -2, \dots$ and $b \neq -1, -2, \dots$, we need to prove the theorem for $c - b + 1 = 0$ and $b \neq -1, -2, \dots$.

If $b \neq -1, -2, \dots$, then $\beta_1^{(b,b-1,d)} = 0$ and $\beta_{n+1}^{(b,b-1,d)} = \alpha_{n+1}^{(b,b-1,d)} \neq 0$ for $n \geq 1$. Hence, $Q_n^{(b,b-1,d)}(z) = z^n$, $n \geq 0$ and

$$\mathcal{M}^{(b,b-1,d)}[z^{-s} Q_n^{(b,b-1,d)}(z)] = \mathcal{M}^{(b,b-1,d)}[z^{n-s}] = \frac{(q^{-b}; q)_{-n+s}}{(q; q)_{-n+s}} q^{(-n+s)d}.$$

Since

$$\frac{(q^{-b}; q)_{-n+s}}{(q; q)_{-n+s}} q^{(-n+s)d} = 0 \quad \text{if } s < n \quad \text{and} \quad \rho_n^{(b,b-1)} = \frac{(q^{-b}; q)_0}{(q; q)_0} q^{(0)d} = 1,$$

the validity of the theorem when $c - b + 1 = 0$ and $b \neq -1, -2, \dots$ is confirmed. This concludes the theorem. $\square$

The same explicit expression for the moments, when the moment functional is considered as in Pastro [21], is obtained in [31].

From the three-term recurrence relation (3.3) it follows that

$$Q_n^{(b,c,d)}(0) = \beta_1^{(b,c,d)} \beta_2^{(b,c,d)} \dots \beta_n^{(b,c,d)} = \frac{(q^{c-b+1}; q)_n}{(q^{b+1}; q)_n} q^{n(b-d+1)}, \quad n \geq 1.$$

Theorem 3.2. *Let $b \neq -1, -2, \dots$ and $c - b + 1 \neq -1, -2, \dots$. Then*

$$\begin{aligned} a) \quad & \lim_{n \rightarrow \infty} \beta_n^{(b,c,d)} = q^{b-d+1}, \quad \lim_{n \rightarrow \infty} \alpha_n^{(b,c,d)} = q^{b-d+1}, \\ b) \quad & \lim_{n \rightarrow \infty} q^{-n(b-d+1)} Q_n^{(b,c,d)}(0) = (1-q)^{-c+2b} \frac{\Gamma_q(b+1)}{\Gamma_q(c-b+1)}, \\ c) \quad & \lim_{n \rightarrow \infty} \rho_n^{(b,c)} = \frac{\Gamma_q(b+1) \Gamma_q(c-b+2)}{\Gamma_q(c+2)}. \end{aligned}$$

Proof. Part a) of this theorem is clear. To obtain parts b) and c) we use the definition

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}$$

of the q -gamma function. $\square$

Theorem 3.3. *Let $b \neq -1, -2, \dots$ and $c - b + 1 \neq -1, -2, \dots$. Then the monic polynomials $Q_n^{(b,c,d)}$, $n \geq 0$, given by the recurrence relation (3.3) have the explicit representation*

$$(3.5) \quad Q_n^{(b,c,d)}(z) = \frac{(q^{c-b+1}; q)_n}{(q^{b+1}; q)_n} q^{n(b-d+1)} {}_2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q, q^{-c+d-1}z).$$

Proof. From (3.3) it is easily verified that the reciprocal (or inverse) polynomials

$$Q_n^{*(b,c,d)}(z) = z^n \overline{Q_n^{(b,c,d)}(1/\bar{z})} \quad \text{and} \quad Q_n^{\bullet(b,c,d)}(z) = z^n Q_n^{(b,c,d)}(1/z), \quad n \geq 0,$$

satisfy the three-term recurrence relations

$$(3.6) \quad \begin{aligned} Q_{n+1}^{*(b,c,d)}(z) &= (1 + \beta_{n+1}^{(\bar{b}, \bar{c}, \bar{d})} z) Q_n^{*(b,c,d)}(z) - \alpha_{n+1}^{(\bar{b}, \bar{c}, \bar{d})} z Q_{n-1}^{*(b,c,d)}(z), \\ Q_{n+1}^{\bullet(b,c,d)}(z) &= (1 + \beta_{n+1}^{(b,c,d)} z) Q_n^{\bullet(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} z Q_{n-1}^{\bullet(b,c,d)}(z), \end{aligned} \quad n \geq 1,$$

with $Q_0^{*(b,c,d)}(z) = Q_0^{\bullet(b,c,d)}(z) = 1$, $Q_1^{*(b,c,d)}(z) = 1 + \beta_1^{(\bar{b}, \bar{c}, \bar{d})} z$ and $Q_1^{\bullet(b,c,d)}(z) = 1 + \beta_1^{(b,c,d)} z$. This means that

$$\begin{aligned} Q_n^{*(b,c,d)}(z) &= {}_2\Phi_1(q^{-n}, q^{\bar{c}-\bar{b}+1}; q^{-\bar{b}-n}; q, q^{-\bar{d}+1}z), \\ Q_n^{\bullet(b,c,d)}(z) &= {}_2\Phi_1(q^{-n}, q^{c-b+1}; q^{-b-n}; q, q^{-d+1}z), \end{aligned} \quad n \geq 1,$$

which we can easily verify from Lemma 2.3. Hence, application of the transformation (2.6) in $Q_n^{\bullet(b,c,d)}$, for example, gives the required results of the theorem. $\square$

Note that by the application of the transform (2.5) in $Q_n^{*(b,c,d)}$, for example, we can also write that

$$\frac{(q^{-\bar{d}+1}z; q)_\infty}{(q^{\bar{c}-\bar{d}+2}z; q)_\infty} Q_n^{*(b,c,d)}(z) = {}_2\Phi_1(q^{-\bar{b}}, q^{-\bar{c}-n-1}; q^{-\bar{b}-n}; q, q^{\bar{c}-\bar{d}+2}z), \quad n \geq 1,$$

provided that $|z| < q^{-\mathcal{R}e(c-d+2)}$. This can also be directly verified from Lemma 2.3 and (3.6).

Theorem 3.4. *Let $b \neq -1, -2, \dots$, $c - b + 1 \neq -1, -2, \dots$ and*

$$\sigma = \min\{q^{-\mathcal{R}e(c-d+2)}, q^{-\mathcal{R}e(b-d+1)}\}.$$

Then uniformly on compact subsets of $|z| < \sigma$,

$$\lim_{n \rightarrow \infty} Q_n^{*(b,c,d)}(z) = \frac{(q^{\bar{c}-\bar{d}+2}z; q)_\infty}{(q^{\bar{b}-\bar{d}+1}z; q)_\infty}.$$

Proof. Since

$$\lim_{n \rightarrow \infty} \frac{(q^{-\bar{c}-n-1}; q)_j}{(q^{-\bar{b}-n}; q)_j} q^{(\bar{c}-\bar{d}+2)j} = q^{(\bar{b}-\bar{d}+1)j},$$

using Lebesgue's dominated convergence theorem and then (2.4), we obtain

$$\lim_{n \rightarrow \infty} {}_2\Phi_1(q^{-\bar{b}}, q^{-\bar{c}-n-1}; q^{-\bar{b}-n}; q, q^{\bar{c}-\bar{d}+2}z) = {}_1\Phi_0(q^{-\bar{b}}; q, q^{\bar{b}-\bar{d}+1}z) = \frac{(q^{-\bar{d}+1}z; q)_\infty}{(q^{\bar{b}-\bar{d}+1}z; q)_\infty},$$

uniformly on compact subsets of $|z| < \sigma$. Thus, the result of the theorem follows. $\square$

Theorem 3.5. *In addition to $b \neq -1, -2, \dots$ and $c - b + 1 \neq -1, -2, \dots$, if one also assumes that*

$$\mathcal{R}e(c+2) > \mathcal{R}e(d) > 0,$$

then the polynomials $Q_n^{(b,c,d)}$, $n \geq 0$, given by (3.5), satisfy the orthogonality relations

$$\frac{\tau^{(b,c)}}{2\pi i} \int_C z^{-s} Q_n^{(b,c,d)}(z) \frac{(q^{-b+d}z; q)_\infty (q^{b-d+1}/z; q)_\infty}{(q^d z; q)_\infty (q^{c-d+2}/z; q)_\infty} \frac{1}{z} dz = \delta_{n,s} \rho_n^{(b,c)}, \quad 0 \leq s \leq n.$$

Here, $\rho_n^{(b,c)}$ are as in Theorem 3.1 and

$$\tau^{(b,c)} = \frac{(q; q)_\infty (q^{c+2}; q)_\infty}{(q^{c-b+2}; q)_\infty (q^{b+1}; q)_\infty}.$$

Proof. Let us consider the following identity of Ramanujan:

$$\sum_{n=-\infty}^{\infty} \frac{(\alpha; q)_n}{(\beta; q)_n} x^n = \frac{(q; q)_\infty (\frac{\beta}{\alpha}; q)_\infty (\alpha x; q)_\infty (\frac{q}{\alpha x}; q)_\infty}{(\beta; q)_\infty (\frac{q}{\alpha}; q)_\infty (\frac{\beta}{\alpha x}; q)_\infty (x; q)_\infty},$$

which holds for $|\beta\alpha^{-1}| < |x| < 1$. Simple proofs of this identity can be found in [3] and [12].

In our case, since $0 < q < 1$, with the assumptions of the theorem if we take $x = q^d z$, $\alpha = q^{-b}$ and $\beta = q^{c-b+2}$, then

$$(3.7) \quad \sum_{n=-\infty}^{\infty} \frac{(q^{-b}; q)_n}{(q^{c-b+2}; q)_n} q^{nd} z^n = \tau^{(b,c)} \frac{(q^{-b+d}z; q)_\infty (q^{b-d+1}/z; q)_\infty}{(q^d z; q)_\infty (q^{c+2-d}/z; q)_\infty},$$

which holds for $|q^{c+2-d}| < |z| < |q^{-d}|$, where $|q^{c+2-d}| < 1$ and $|q^{-d}| > 1$.

Hence, multiplying by z^{-j-1} and integrating along the unit circle we obtain from Laurent's theorem

$$\frac{(q^{-b}; q)_j}{(q^{c-b+2}; q)_j} q^{jd} = \frac{\tau^{(b,c)}}{2\pi i} \int_{\mathcal{C}} z^{-j-1} \frac{(q^{-b+d}z; q)_{\infty} (q^{b-d+1}/z; q)_{\infty}}{(q^d z; q)_{\infty} (q^{c+2-d}/z; q)_{\infty}} dz, \quad j = 0, \pm 1, \pm 2, \dots$$

Thus, the moment functional in Theorem 3.1 satisfies

$$(3.8) \quad \mathcal{M}^{(b,c,d)}[z^{-j}] = \frac{\tau^{(b,c)}}{2\pi i} \int_{\mathcal{C}} z^{-j-1} \frac{(q^{-b+d}z; q)_{\infty} (q^{b-d+1}/z; q)_{\infty}}{(q^d z; q)_{\infty} (q^{c+2-d}/z; q)_{\infty}} dz,$$

for $j = 0, \pm 1, \pm 2, \dots$, which completes the proof of the theorem. $\square$

As a particular case, letting $b = 0$ and $c + 2 \neq 0, -1, -2, \dots$ we have $\beta_1^{(0,c,d)} = \frac{1-q^{c+1}}{1-q} q^{-d+1}$ and

$$\beta_{n+1}^{(0,c,d)} = \alpha_{n+1}^{(0,c,d)} = \frac{1-q^{c+n+1}}{1-q^{n+1}} q^{-d+1}, \quad n \geq 1.$$

Moreover, $\mu_0^{(0,c,d)} = 1$,

$$\mu_j^{(0,c,d)} = 0 \quad \text{and} \quad \mu_{-j}^{(0,c,d)} = \frac{(q^{-c-1}; q)_j}{(q; q)_j} q^{j(c+2-d)}, \quad j = 1, 2, \dots$$

Furthermore, the following corollary holds.

Corollary 3.5.1. *If $\operatorname{Re}(c+2) > \operatorname{Re}(d) > 0$, then the sequence of polynomials $\{\mathcal{Q}_n^{(0,c,d)}\}$ given by*

$$\mathcal{Q}_n^{(0,c,d)}(z) = \frac{(q^{c+1}; q)_n}{(q; q)_n} q^{n(-d+1)} {}_2\Phi_1(q^{-n}, q; q^{-c-n}; q, q^{-c+d-1}z), \quad n \geq 1,$$

apart from satisfying the three-term recurrence relation

$$\mathcal{Q}_{n+1}^{(0,c,d)}(z) = \left(z + \frac{1-q^{c+n+1}}{1-q^{n+1}} q^{-d+1}\right) \mathcal{Q}_n^{(0,c,d)}(z) - \frac{1-q^{c+n+1}}{1-q^{n+1}} q^{-d+1} z \mathcal{Q}_{n-1}^{(0,c,d)}(z),$$

for $n \geq 1$, with $\mathcal{Q}_0^{(0,c,d)}(z) = 1$ and $\mathcal{Q}_1^{(0,c,d)}(z) = z + \frac{1-q^{c+1}}{1-q} q^{-d+1}$, satisfies the orthogonality relations

$$\frac{1}{2\pi i} \int_{\mathcal{C}} z^{-s} \mathcal{Q}_n^{(0,c,d)}(z) \frac{(q^{-d+1}/z; q)_{\infty}}{(q^{c+2-d}/z; q)_{\infty}} \frac{1}{z} dz = \delta_{n,s}, \quad 0 \leq s \leq n.$$

Moreover, uniformly on compact subsets of $|z| < \min\{q^{-\operatorname{Re}(c-d+2)}, q^{-\operatorname{Re}(-d+1)}\}$,

$$\lim_{n \rightarrow \infty} \mathcal{Q}_n^{*(0,c,d)}(z) = \frac{(q^{\bar{c}-\bar{d}+2}z; q)_{\infty}}{(q^{-\bar{d}+1}z; q)_{\infty}}.$$

As another particular case, letting $c = b$ and $b + 1 \neq 0, -1, -2, \dots$, we have

$$\beta_n^{(b,b,d)} = \alpha_{n+1}^{(b,b,d)} = \frac{1-q^n}{1-q^{b+n}} q^{b-d+1}, \quad n \geq 1.$$

Moreover,

$$\mu_j^{(b,b,d)} = \frac{(q^{-b}; q)_j}{(q^2; q)_j} q^{jd}, \quad j = 0, \pm 1, \pm 2, \dots$$

Furthermore, the following corollary can be stated.

Corollary 3.5.2. *If $b+1 \neq 0$ and $\operatorname{Re}(b+2) > \operatorname{Re}(d) > 0$, then the sequence of polynomials $\{\mathcal{Q}_n^{(b,b,d)}\}$ given by*

$$\mathcal{Q}_n^{(b,b,d)}(z) = \frac{(q; q)_n}{(q^{b+1}; q)_n} q^{n(b-d+1)} \sum_{j=0}^n \frac{(q^{b+1}; q)_j}{(q; q)_j} q^{-j(b-d+1)} z^j, \quad n \geq 1,$$

satisfies the orthogonality relations

$$\frac{1}{2\pi i} \frac{(1-q)}{(1-q^{b+1})} \int_{\mathcal{C}} z^{-s} \mathcal{Q}_n^{(b,b,d)}(z) \frac{(q^{-b+d}z; q)_{\infty}}{(q^d z; q)_{\infty}} \frac{z - q^{b-d+1}}{z^2} dz = \delta_{n,s}, \quad 0 \leq s \leq n.$$

Moreover, uniformly on compact subsets of $|z| < q^{-\operatorname{Re}(b-d+1)}$,

$$\lim_{n \rightarrow \infty} \mathcal{Q}_n^{*(b,b,d)}(z) = \frac{1}{(1 - q^{b-d+1}z)}.$$

4. q -SZEGŐ POLYNOMIALS

From (3.8) the moment functional $\mathcal{M}^{(b,c,d)}$ is easily seen to be positive definite if $b \neq -1, -2, \dots$, $c - b + 1 \neq -1, -2, \dots$, $\operatorname{Re}(c+2) > \operatorname{Re}(d) > 0$, $\overline{-b+d} = b-d+1$ and $\overline{d} = c+2-d$. That is, with the restrictions

$$c = b + \bar{b} - 1, \quad d + \bar{d} = b + \bar{b} + 1 \quad \text{and} \quad \operatorname{Re}(b) > -1/2,$$

the moment functional $\mathcal{M}^{(b,c,d)}$ is positive definite, and hence the polynomials $\mathcal{Q}_n^{(b,c,d)}$ are the associated Szegő polynomials.

Hence, setting

$$b = \lambda - i\eta, \quad c = 2\lambda - 1 \quad \text{and} \quad d = \frac{1}{2} + \lambda + i\phi,$$

if $\lambda > -1/2$, our special case of Ramanujan identity (3.7) becomes

$$\sum_{n=-\infty}^{\infty} \frac{(q^{-\lambda+i\eta}; q)_n}{(q^{\lambda+1+i\eta}; q)_n} q^{n(\frac{1}{2}+\lambda+i\phi)} z^n = \tilde{\tau}(\lambda, \eta) \frac{(q^{\frac{1}{2}+i(\eta+\phi)}z; q)_{\infty} (q^{\frac{1}{2}-i(\eta+\phi)}/z; q)_{\infty}}{(q^{\frac{1}{2}+\lambda+i\phi}z; q)_{\infty} (q^{\frac{1}{2}+\lambda-i\phi}/z; q)_{\infty}},$$

which holds for $q^{\lambda+1/2} < |z| < q^{-\lambda-1/2}$, where

$$\tilde{\tau}(\lambda, \eta) = \frac{(q; q)_{\infty} (q^{2\lambda+1}; q)_{\infty}}{(q^{1+\lambda+i\eta}; q)_{\infty} (q^{1+\lambda-i\eta}; q)_{\infty}}.$$

This means that we can write

$$\begin{aligned} \mathcal{M}^{(\lambda-i\eta, 2\lambda-1, \lambda+i\phi+1/2)}[z^{-j}] &= \frac{(q^{-\lambda+i\eta}; q)_j}{(q^{\lambda+1+i\eta}; q)_j} q^{j(\frac{1}{2}+\lambda+i\phi)} \\ &= \int_{\mathcal{C}} z^{-j} d\mu^{(\lambda, \eta, \phi)}(z) = \int_0^{2\pi} e^{-ij\theta} \omega^{(\lambda, \eta, \phi)}(\theta) d\theta, \end{aligned}$$

for $j = 0, \pm 1, \pm 2, \dots$, where $\omega^{(\lambda, \eta, \phi)}(\theta) d\theta = d\mu^{(\lambda, \eta, \phi)}(e^{i\theta})$, with

$$\frac{d\mu^{(\lambda, \eta, \phi)}(z)}{dz} = \frac{\tilde{\tau}(\lambda, \eta)}{2\pi i} \frac{1}{z} \frac{(q^{\frac{1}{2}+i(\eta+\phi)}z; q)_{\infty} (q^{\frac{1}{2}-i(\eta+\phi)}/z; q)_{\infty}}{(q^{\frac{1}{2}+\lambda+i\phi}z; q)_{\infty} (q^{\frac{1}{2}+\lambda-i\phi}/z; q)_{\infty}}$$

and

$$(4.1) \quad \omega^{(\lambda, \eta, \phi)}(\theta) = \frac{\tilde{\tau}(\lambda, \eta)}{2\pi} \frac{(q^{\frac{1}{2}+i(\eta+\phi)}e^{i\theta}; q)_{\infty} (q^{\frac{1}{2}-i(\eta+\phi)}e^{-i\theta}; q)_{\infty}}{(q^{\frac{1}{2}+\lambda+i\phi}e^{i\theta}; q)_{\infty} (q^{\frac{1}{2}+\lambda-i\phi}e^{-i\theta}; q)_{\infty}}.$$

As expected, $\omega^{(\lambda, \eta, \phi)}(\theta)$ is a positive weight function in $[0, 2\pi]$.

Adopting the notation $S_n^{(\lambda, \eta, \phi)}(z) = Q_n^{(\lambda - i\eta, 2\lambda - 1, \frac{1}{2} + \lambda + i\phi)}(z)$ we can write the following:

$$S_{n+1}^{(\lambda, \eta, \phi)}(z) = \left(z + \frac{1 - q^{\lambda + i\eta + n}}{1 - q^{\lambda - i\eta + n + 1}} q^{\frac{1}{2} - i(\eta + \phi)} \right) S_n^{(\lambda, \eta, \phi)}(z) \\ - \frac{(1 - q^n)(1 - q^{2\lambda + n})}{(1 - q^{\lambda - i\eta + n})(1 - q^{\lambda - i\eta + n + 1})} q^{\frac{1}{2} - i(\eta + \phi)} z S_{n-1}^{(\lambda, \eta, \phi)}(z), \quad n \geq 1,$$

with $S_0^{(\lambda, \eta, \phi)}(z) = 1$ and $S_1^{(\lambda, \eta, \phi)}(z) = \left(z + \frac{1 - q^{\lambda + i\eta}}{1 - q^{\lambda - i\eta + 1}} q^{\frac{1}{2} - i(\eta + \phi)} \right)$. Moreover,

$$(4.2) \quad S_n^{(\lambda, \eta, \phi)}(0) = \frac{(q^{\lambda + i\eta}; q)_n}{(q^{1 + \lambda - i\eta}; q)_n} q^{n[\frac{1}{2} - i(\eta + \phi)]}, \quad n \geq 1.$$

Hence, in particular, using Theorems 3.3 and 3.5 we have

Theorem 4.1. *If $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$, then the polynomials*

$$S_n^{(\lambda, \eta, \phi)}(z) = \frac{(q^{\lambda + i\eta}; q)_n q^{n[\frac{1}{2} - i(\eta + \phi)]}}{(q^{\lambda + 1 - i\eta}; q)_n} {}_2\Phi_1(q^{-n}, q^{\lambda + 1 - i\eta}; q^{-\lambda - n + 1 - i\eta}; q, q^{\frac{1}{2} - \lambda + i\phi} z)$$

are the monic Szegő polynomials satisfying

$$\int_0^{2\pi} \overline{S_n^{(\lambda, \eta, \phi)}(e^{i\theta})} S_m^{(\lambda, \eta, \phi)}(e^{i\theta}) \omega^{(\phi, \eta, \lambda)}(\theta) d\theta = [\kappa_n^{(\lambda, \eta)}]^{-2} \delta_{n,m}, \quad n, m = 0, 1, 2, \dots,$$

with respect to the weight function $\omega^{(\lambda, \eta, \phi)}(\theta)$ given by (4.1). Here,

$$[\kappa_n^{(\lambda, \eta)}]^{-2} = \rho_n^{(\lambda - i\eta, 2\lambda - 1)} = \frac{(q; q)_n (q^{2\lambda + 1}; q)_n}{(q^{\lambda + 1 + i\eta}; q)_n (q^{\lambda + 1 - i\eta}; q)_n}.$$

Moreover, these polynomials satisfy the Szegő recurrence relation

$$S_n^{*(\lambda, \eta, \phi)}(z) = \overline{a_n^{(\lambda, \eta, \phi)}} z S_{n-1}^{(\lambda, \eta, \phi)}(z) + S_{n-1}^{*(\lambda, \eta, \phi)}(z), \quad n \geq 1,$$

where the reflection (or Verblunsky) coefficients $a_n^{(\lambda, \eta, \phi)} = S_n^{(\lambda, \eta, \phi)}(0)$ are given by (4.2).

Now using Theorem 3.4 we can state the following. Let $\lambda, \eta, \phi \in \mathbb{R}$, $\lambda > -1/2$ and $\sigma = \min\{q^{-1/2}, q^{-\lambda - 1/2}\}$. Then uniformly on compact subsets of $|z| < \sigma$,

$$(4.3) \quad \lim_{n \rightarrow \infty} S_n^{*(\lambda, \eta, \phi)}(z) = \frac{(q^{\lambda + \frac{1}{2} + i\phi} z; q)_\infty}{(q^{\frac{1}{2} + i(\eta + \phi)} z; q)_\infty}.$$

Moreover,

$$\sum_{n=1}^{\infty} |a_n^{(\lambda, \eta, \phi)}|^2 = |1 - q^{\lambda + i\eta}|^2 \sum_{n=1}^{\infty} \frac{q^n}{|1 - q^{n + \lambda + i\eta}|^2} \leq |1 - q^{\lambda + i\eta}|^2 \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^{n + \lambda})^2} < \infty.$$

This last result means that the Szegő condition

$$\frac{1}{2\pi} \int_0^{2\pi} \log(\omega^{(\lambda, \eta, \phi)}(\theta)) d\theta > -\infty$$

holds and we can now give an expression for the associated Szegő function

$$D^{(\lambda, \eta, \phi)}(z) = \exp \left(\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\omega^{(\lambda, \eta, \phi)}(\theta)) d\theta \right).$$

Theorem 4.2. *If $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$, then for $|z| < 1$,*

$$D^{(\lambda, \eta, \phi)}(z) = \sqrt{\frac{\Gamma_q(2\lambda + 1)}{\Gamma_q(\lambda + 1 - i\eta) \Gamma_q(\lambda + 1 + i\eta)}} \frac{(q^{\frac{1}{2} + i(\eta + \phi)} z; q)_\infty}{(q^{\lambda + \frac{1}{2} + i\phi} z; q)_\infty}.$$

Proof. Since, $\kappa_n^{(\lambda, \eta)} S_n^{*(\lambda, \eta, \phi)} \rightarrow [D^{(\lambda, \eta, \phi)}(z)]^{-1}$ for $|z| < 1$ (see [23, p. 144]), the result follows from part c) of Theorem 3.2 and from (4.3). $\square$

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