ON THE LINEAR INDEPENDENCY OF MONOIDAL NATURAL TRANSFORMATIONS

KENICHI SHIMIZU

(Communicated by Lev Borisov)

ABSTRACT. Let $F, G : \mathcal{I} \rightarrow \mathcal{C}$ be monoidal functors from a monoidal category $\mathcal{I}$ to a linear abelian rigid monoidal category $\mathcal{C}$ over an algebraically closed field $\mathbf{k}$. Then the set $\text{Nat}(F, G)$ of natural transformations $F \rightarrow G$ is naturally a vector space over $\mathbf{k}$. Under certain assumptions, we show that the set of monoidal natural transformations $F \rightarrow G$ is linearly independent as a subset of $\text{Nat}(F, G)$.

As a corollary, we can show that the group of monoidal natural automorphisms on the identity functor on a finite tensor category is finite. We can also show that the set of pivotal structures on a finite tensor category is finite.

1. INTRODUCTION

Monoidal categories [3] arise in many contexts in mathematics. In this paper, we prove a basic fact on monoidal natural transformations between monoidal functors.

Our terminology basically follows that of Mac Lane [3]. Recall that a monoidal functor [3, XI.2] is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories equipped with a natural transformation $\varphi_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ ($X, Y \in \mathcal{C}$) and a morphism $\varphi_0 : \mathbf{1} \rightarrow F(\mathbf{1})$ in $\mathcal{D}$ satisfying certain axioms and that it is said to be strong if φ and φ_0 are isomorphisms. In this paper, by a monoidal functor we always mean a strong monoidal functor.

Throughout, we work over an algebraically closed field $\mathbf{k}$. By a *tensor category* over $\mathbf{k}$ we mean a $\mathbf{k}$ -linear abelian monoidal category $\mathcal{C}$ which is rigid [1, §2.1] and satisfies the following conditions:

- The unit object $\mathbf{1} \in \mathcal{C}$ is simple.
- The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is $\mathbf{k}$ -linear in both variables.
- Every object of $\mathcal{C}$ is of finite length.
- Every hom-set in $\mathcal{C}$ is finite-dimensional over $\mathbf{k}$.

Let $\mathcal{C}$ be a tensor category over $\mathbf{k}$ and let $\mathcal{I}$ be a skeletally small monoidal category. If $F, G : \mathcal{I} \rightarrow \mathcal{C}$ are functors, then the set $\text{Nat}(F, G)$ of natural transformations $F \rightarrow G$ is naturally a vector space over $\mathbf{k}$. Now we suppose that both F and G are monoidal functors. Then we can consider the set $\text{Nat}_{\otimes}(F, G)$ of monoidal natural transformations. Our main result in this paper is the following:

Theorem 1.1. $\text{Nat}_{\otimes}(F, G) \subset \text{Nat}(F, G)$ is linearly independent.

Received by the editors October 21, 2010 and, in revised form, February 11, 2011.

2010 *Mathematics Subject Classification.* Primary 18D10.

Key words and phrases. Monoidal category, monoidal functor, finite tensor category.

©2011 American Mathematical Society
 Reverts to public domain 28 years from publication

We will prove Theorem 1.1 in Section 2. In what follows, we give some applications of this theorem. The following criterion for finite-dimensionality of $\text{Nat}(F, G)$ is important:

Lemma 1.2. *Let $F, G : \mathcal{A} \rightarrow \mathcal{B}$ be right exact $\mathbf{k}$ -linear functors between $\mathbf{k}$ -linear abelian categories. Suppose that $\mathcal{A}$ has a projective generator P and that every object of $\mathcal{A}$ is of finite length. Then the linear map*

$$(-)|_P : \text{Nat}(F, G) \rightarrow \text{Hom}_{\mathcal{B}}(F(P), G(P)), \quad f \mapsto f|_P$$

is injective. In particular, if every hom-set in $\mathcal{B}$ is finite-dimensional, then

$$\dim_{\mathbf{k}} \text{Nat}(F, G) \leq \dim_{\mathbf{k}} \text{Hom}_{\mathcal{B}}(F(P), G(P)) < \infty.$$

Proof. Let $f \in \text{Nat}(F, G)$ and suppose that $f|_P = 0$. By the assumption, for every $X \in \mathcal{A}$, there exists an exact sequence $P^{\oplus m} \rightarrow P^{\oplus n} \rightarrow X \rightarrow 0$. By applying F and G to this sequence, we have a commutative diagram

$$\begin{array}{ccccccc} F(P)^{\oplus m} & \longrightarrow & F(P)^{\oplus n} & \longrightarrow & F(X) & \longrightarrow & 0 \\ (f|_P)^{\oplus m} \downarrow & & (f|_P)^{\oplus n} \downarrow & & f|_X \downarrow & & \\ G(P)^{\oplus m} & \longrightarrow & G(P)^{\oplus n} & \longrightarrow & G(X) & \longrightarrow & 0 \end{array}$$

in $\mathcal{B}$ with exact rows. Since $f|_P = 0$, $f|_X = 0$. This implies the injectivity. $\square$

A $\mathbf{k}$ -linear abelian category $\mathcal{A}$ is said to be *finite* if it is $\mathbf{k}$ -linearly equivalent to the category of finitely generated modules over a finite-dimensional $\mathbf{k}$ -algebra. We present some applications of Theorem 1.1 to *finite tensor categories* [2]. By a *tensor functor* we mean a $\mathbf{k}$ -linear monoidal functor between tensor categories. By Theorem 1.1 and Lemma 1.2, we have the following:

Corollary 1.3. *Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two right exact tensor functors between tensor categories $\mathcal{C}$ and $\mathcal{D}$. Suppose that $\mathcal{C}$ is finite. Then $\text{Nat}_{\otimes}(F, G)$ is finite.*

Let $\text{Aut}_{\otimes}(F) \subset \text{Nat}_{\otimes}(F, F)$ denote the group of monoidal natural automorphisms on a monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$. As an immediate consequence of Corollary 1.3, we have the following:

Corollary 1.4. *Let $F : \mathcal{C} \rightarrow \mathcal{C}$ be a right exact tensor endofunctor on a finite tensor category $\mathcal{C}$. Then $\text{Aut}_{\otimes}(F)$ is finite. In particular, $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$ is finite.*

We will give some remarks on the structure of $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$ in Section 3. We note that the finiteness of $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$ is well known in the case where $\mathcal{C}$ is the category $\text{Rep}(H)$ of finite-dimensional representations of a finite-dimensional Hopf algebra H . In fact, $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$ is then isomorphic to the group of central grouplike elements of H .

There is another important corollary. A *pivotal structure* on a rigid monoidal category $\mathcal{C}$ is an element of $\text{Piv}(\mathcal{C}) := \text{Nat}_{\otimes}(\text{id}_{\mathcal{C}}, (-)^{**})$, where $(-)^* : \mathcal{C} \rightarrow \mathcal{C}$ is the left duality functor.

Corollary 1.5. *If $\mathcal{C}$ is a finite tensor category, then $\text{Piv}(\mathcal{C})$ is finite.*

Also this corollary is well known in the case where $\mathcal{C} = \text{Rep}(H)$ for some finite-dimensional Hopf algebra H . In fact, $\text{Piv}(\mathcal{C})$ then is in one-to-one correspondence between grouplike elements $g \in H$ such that $S^2(x) = gxg^{-1}$ for all $x \in H$, where S is the antipode of H .

Remark 1.6. We should remark that every finite tensor category is equivalent as a tensor category to the category of finite-dimensional representations of a weak quasi-Hopf algebra [2]. It will be interesting to interpret Corollaries 1.4 and 1.5 in terms of “grouplike elements” of weak quasi-Hopf algebras (cf. [4]).

2. PROOF OF THEOREM 1.1

Let $\mathcal{C}$ be a tensor category over $\mathbf{k}$. Without loss of generality, we may suppose $\mathcal{C}$ to be strict. Let $V \in \mathcal{C}$ be an object. The tensor product of $\mathcal{C}$ defines two $\mathbf{k}$ -linear endofunctors $V \otimes (-)$ and $(-) \otimes V$ on $\mathcal{C}$. Both these functors are exact since $\mathcal{C}$ is rigid [1].

Lemma 2.1. *If $V \neq 0$, then $V \otimes (-)$ and $(-) \otimes V$ are faithful.*

Proof. Suppose that $V \neq 0$. Then the evaluation morphism $d_V : V^* \otimes V \rightarrow \mathbf{1}$ is an epimorphism. Indeed, otherwise, since $\mathbf{1}$ is simple, $d_V = 0$. It follows from the rigidity axiom that $\text{id}_V = 0$, and hence $V = 0$, a contradiction.

Let $X, Y \in \mathcal{C}$ be objects. Consider the linear map

$$(-)^{\natural} : \text{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y) \rightarrow \text{Hom}_{\mathcal{C}}(V^* \otimes V \otimes X, Y)$$

given by $\varphi^{\natural} = (d_V \otimes \text{id}_Y) \circ (\text{id}_{V^*} \otimes \varphi)$. If $f : X \rightarrow Y$ is a morphism in $\mathcal{C}$, then

$$(\text{id}_V \otimes f)^{\natural} = f \circ (d_V \otimes \text{id}_X).$$

If $\text{id}_V \otimes f = 0$, then, by the above equation, we have $f \circ (d_V \otimes \text{id}_X) = 0$. As we observed above, d_V is an epimorphism. Since the tensor product of $\mathcal{C}$ is exact, also $d_V \otimes \text{id}_X$ is an epimorphism. Therefore, we conclude that $f = 0$. This means that the functor $V \otimes (-)$ is faithful.

The faithfulness of $(-) \otimes V$ can be proved in a similar way. $\square$

Let $\mathcal{I}$ be a skeletally small category and let $F, G : \mathcal{I} \rightarrow \mathcal{C}$ be functors. Then the set $\text{Nat}(F, G)$ of natural transformations $F \rightarrow G$ is a vector space over $\mathbf{k}$. Let $V, W \in \mathcal{C}$ be objects. The tensor product of $\mathcal{C}$ defines a linear map

$$\text{Hom}_{\mathcal{C}}(V, W) \otimes_{\mathbf{k}} \text{Nat}(F, G) \rightarrow \text{Nat}(V \otimes F(-), W \otimes G(-)).$$

Lemma 2.2. *The above map is injective.*

In the case where F and G are constant functors sending all objects of $\mathcal{I}$ respectively to $X \in \mathcal{C}$ and $Y \in \mathcal{C}$, Lemma 2.2 states that the map

$$\text{Hom}_{\mathcal{C}}(V, W) \otimes_{\mathbf{k}} \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(V \otimes X, W \otimes Y)$$

induced from the tensor product is injective. We need to consider a bit more general situation than this case.

Proof. Let $f_1, \dots, f_m \in \text{Nat}(F, G)$ be linearly independent elements. It suffices to show that if $c_1, \dots, c_m : V \rightarrow W$ are morphisms in $\mathcal{C}$ such that

$$(2.1) \quad c_1 \otimes f_1|_X + \dots + c_m \otimes f_m|_X = 0$$

for all $X \in \mathcal{I}$, then $c_i = 0$ for $i = 1, \dots, m$.

We show the above claim by induction on the length $\ell(V)$ of V . If $\ell(V) = 0$, our claim is obvious. Suppose that $\ell(V) \geq 1$. Then there exists a simple subobject of V , say L . Let K be the image of the morphism

$$(c_1|_L, c_2|_L, \dots, c_m|_L) : L^{\oplus m} \rightarrow W.$$

Since L is simple, $K \cong L^{\oplus n}$ for some $n \geq 0$. Let $p_j : K \cong L^{\oplus n} \rightarrow L$ be the j -th projection. Then $p_j \circ c_i|_L = \lambda_{ij} \text{id}_L$ for some $\lambda_{ij} \in \mathbf{k}$. By (2.1),

$$\text{id}_L \otimes \left(\sum_{i=1}^m \lambda_{ij} f_i|_X \right) = (p_j \otimes \text{id}_{G(X)}) \circ \left(\sum_{i=1}^m c_i|_L \otimes f_i|_X \right) = 0.$$

Lemma 2.1 yields that $\sum_{i=1}^m \lambda_{ij} f_i|_X = 0$ for all $X \in \mathcal{I}$. By the linear independence of the f_i 's, we have that $\lambda_{ij} = 0$ for all i and j . This means that $c_i|_L = 0$ for all i .

Let $p : V \rightarrow V/L$ be the projection. By the above observation, for each i , there exists a morphism $\bar{c}_i : V/L \rightarrow W$ such that $c_i = \bar{c}_i \circ p$. By (2.1),

$$\sum_{i=1}^m \bar{c}_i \otimes f_i|_X = \left(\sum_{i=1}^m c_i \otimes f_i|_X \right) \circ (p \otimes \text{id}_{F(X)}) = 0$$

for all $X \in \mathcal{I}$. Note that $\ell(V/L) = \ell(V) - 1$. By the induction hypothesis, $\bar{c}_i = 0$ for all i . Therefore, $c_i = \bar{c}_i \circ p = 0$. $\square$

Now we generalize Lemma 2.2. Let $\mathcal{I}_i$ ($i = 1, 2$) be skeletally small categories and let $E_i : \mathcal{I}_i \rightarrow \mathcal{C}$ ($i = 1, 2$) be functors. We denote by $E_1 \otimes E_2$ the functor

$$E_1 \otimes E_2 : \mathcal{I}_1 \times \mathcal{I}_2 \xrightarrow{E_1 \times E_2} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

Lemma 2.3. *Let $F_i, G_i : \mathcal{I}_i \rightarrow \mathcal{C}$ ($i = 1, 2$) be functors. Then the map*

$$\text{Nat}(F_1, G_1) \otimes_{\mathbf{k}} \text{Nat}(F_2, G_2) \rightarrow \text{Nat}(F_1 \otimes F_2, G_1 \otimes G_2)$$

induced from the tensor product is injective.

Proof. Let $f_1, \dots, f_m \in \text{Nat}(F_2, G_2)$ be linearly independent elements. Suppose that $c_1, \dots, c_m \in \text{Nat}(F_1, G_1)$ are elements such that

$$c_1|_X \otimes f_1|_Y + \dots + c_m|_X \otimes f_m|_Y = 0$$

for all $(X, Y) \in \mathcal{I}_1 \times \mathcal{I}_2$. If we fix $X \in \mathcal{I}_1$, we can apply Lemma 2.2 and obtain that $c_i|_X = 0$ for $i = 1, \dots, m$. By letting X run through all objects of $\mathcal{I}_1$, we have that $c_i = 0$ for $i = 1, \dots, m$. Thus the map under consideration is injective. $\square$

Proof of Theorem 1.1. Now we prove Theorem 1.1. Our proof is based on a proof of the linear independence of grouplike elements in a coalgebra over a field. Recall the assumptions: $\mathcal{I}$ is a skeletally small monoidal category, $\mathcal{C}$ is a tensor category over $\mathbf{k}$, and F and G are monoidal functors from $\mathcal{I}$ to $\mathcal{C}$.

We first note that $0 \notin \text{Nat}_{\otimes}(F, G)$. Indeed, if $g \in \text{Nat}_{\otimes}(F, G)$, $g|_{\mathbf{1}} : F(\mathbf{1}) \rightarrow G(\mathbf{1})$ must be an isomorphism. Since $F(\mathbf{1}) \cong \mathbf{1} \neq 0$, $g|_{\mathbf{1}} \neq 0$, and hence $g \neq 0$.

Suppose to the contrary that $\text{Nat}_{\otimes}(F, G)$ is linearly dependent. Then there exist elements $g_1, \dots, g_m \in \text{Nat}_{\otimes}(F, G)$ and $\lambda_1, \dots, \lambda_m \in \mathbf{k}$ such that

$$g := \lambda_1 g_1 + \dots + \lambda_m g_m \in \text{Nat}_{\otimes}(F, G)$$

and $g \neq g_i$ for $i = 1, \dots, m$. We may suppose that $g_1, \dots, g_m$ are linearly independent. Since $g \neq 0$, we may also suppose that $\lambda_h \neq 0$ for some h .

By the definition of monoidal natural transformations, we have

$$\sum_{j=1}^m \lambda_j g_j|_X \otimes g_j|_Y = g|_X \otimes g|_Y = \sum_{i,j=1}^m \lambda_i \lambda_j g_i|_X \otimes g_j|_Y$$

for all $X, Y \in \mathcal{I}$. By Lemma 2.3, $\sum_{i=1}^m \lambda_j \lambda_i g_i = \lambda_j g_j$ for each j . By the linear independence of the g_i 's, we have that $\lambda_i = 0$ for $i \neq h$. Therefore, $g = g_h$. This is a contradiction. $\square$

3. SOME REMARKS ON $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$

3.1. Bound of the order. Let $\mathcal{C}$ be a finite tensor category over $\mathbf{k}$. In this section, we give some remarks on the structure of the group $G(\mathcal{C}) = \text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$. We first note that by the definition of natural transformations, $G(\mathcal{C})$ is abelian.

Let I be the set of isomorphism classes of simple objects of $\mathcal{C}$. For each $i \in I$, we fix $S_i \in i$. Let P_i be the projective cover of S_i . Then $P = \bigoplus_{i \in I} P_i$ is a projective generator. As every object of $\mathcal{C}$ is of finite length, $\mathcal{C}$ is $\mathbf{k}$ -linearly equivalent to the category of finite-dimensional right $\text{End}_{\mathcal{C}}(P)$ -modules. Hence the map

$$\text{Nat}(\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}}) \rightarrow Z(\text{End}_{\mathcal{C}}(P)), \quad \eta \mapsto \eta|_P$$

is an isomorphism of $\mathbf{k}$ -algebras (cf. Lemma 1.2).

Theorem 1.1 states that $G(\mathcal{C}) \subset \text{Nat}(\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}})$ is linearly independent. Therefore, we have the following bound on the order of $G(\mathcal{C})$.

Proposition 3.1. $|G(\mathcal{C})| \leq \dim_{\mathbf{k}} Z(\text{End}_{\mathcal{C}}(P))$.

In the case where $\mathcal{C} = \text{Rep}(H)$ for some finite-dimensional Hopf algebra H , this proposition is obvious since the right-hand side of the inequality is equal to the dimension of the center of H .

3.2. Values on simple objects. Let us consider the map

$$\text{Nat}(\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}}) \rightarrow \prod_{i \in I} \text{End}_{\mathcal{C}}(S_i), \quad \eta \mapsto (\eta|_{S_i})_{i \in I}.$$

This map is not injective anymore unless $\mathcal{C}$ is semisimple. Since $\mathbf{k}$ is algebraically closed, we can identify $\text{End}_{\mathcal{C}}(S_i)$ with $\mathbf{k}$. The above map induces a group homomorphism

$$\varphi : G(\mathcal{C}) \rightarrow \prod_{i \in I} \text{End}_{\mathcal{C}}(S_i)^{\times} \cong \text{Map}(I, \mathbf{k}^{\times}), \quad g \mapsto (i \mapsto g|_{S_i}).$$

We show that φ is injective if $\mathbf{k}$ is of characteristic zero. To describe the kernel of φ , we introduce some subgroups of $G(\mathcal{C})$. If $p = \text{char}(\mathbf{k}) > 0$, then we set

$$\begin{aligned} G(\mathcal{C})_p &= \{g \in G(\mathcal{C}) \mid g^{p^k} = 1 \text{ for some } k \geq 0\}, \\ G(\mathcal{C})'_p &= \{g \in G(\mathcal{C}) \mid \text{the order of } g \text{ is relatively prime to } p\}. \end{aligned}$$

Otherwise we set $G(\mathcal{C})_p = \{1\}$ and $G(\mathcal{C})'_p = G(\mathcal{C})$. By the fundamental theorem of finite abelian groups, we have a decomposition $G(\mathcal{C}) = G(\mathcal{C})_p \times G(\mathcal{C})'_p$.

Lemma 3.2. *Let $X \in \mathcal{C}$ be an indecomposable object.*

- (a) *If $g \in G(\mathcal{C})_p$, then $g|_X$ is unipotent.*
- (b) *If $g \in G(\mathcal{C})'_p$, then $g|_X = \lambda \cdot \text{id}_X$ for some $\lambda \in \mathbf{k}^{\times}$.*

Proof. Let $g \in G(\mathcal{C})$. As X is indecomposable, $\text{End}_{\mathcal{C}}(X)$ is a local algebra. Since $\mathbf{k}$ is algebraically closed, $g|_X$ can be written uniquely in the form

$$(3.1) \quad g|_X = \lambda \text{id}_X + r \quad (\lambda \in \mathbf{k}^{\times}, r \in \mathfrak{m}),$$

where $\mathfrak{m}$ is the maximal ideal of $\text{End}_{\mathcal{C}}(X)$. If $r \neq 0$, then there exists $k \geq 1$ such that $r \in \mathfrak{m}^{k-1}$ but $r \notin \mathfrak{m}^k$. By the binomial formula, we have

$$(g|_X)^n \equiv \lambda^n \text{id}_X + nr \pmod{\mathfrak{m}^k}$$

for every $n \geq 0$. This implies that if $(g|_X)^n = \text{id}_X$, then $\lambda^n = 1$ and $nr = 0$.

(a) Suppose that $g \in G(\mathcal{C})'_p$. Since the claim is obvious for $p = 0$, we assume that $p > 0$. Then the order of $g|_X$ is a power of p . Thus, by the above observation, $\lambda = 1$. This implies that $g|_X$ is unipotent.

(b) Suppose that $g \in G(\mathcal{C})'_p$. Then the order of $g|_X$ is nonzero in $\mathbf{k}$. Thus, by the above observation, $r = 0$. This implies that $g|_X = \lambda \text{id}_X$. $\square$

In what follows, we denote λ in equation (3.1) by $\lambda_g(X)$. We note the following easy but important property of $\lambda_g(X)$.

Lemma 3.3. *Let X and Y be indecomposable objects of $\mathcal{C}$. If X and Y belong to the same block, then $\lambda_g(X) = \lambda_g(Y)$ for every $g \in G(\mathcal{C})'_p$.*

Here, a *block* is an equivalence class of indecomposable objects of $\mathcal{C}$ under the weakest equivalence relation such that two indecomposable objects X and Y of $\mathcal{C}$ are equivalent whenever $\text{Hom}_{\mathcal{C}}(X, Y) \neq 0$.

Proof. Let $g \in G(\mathcal{C})'_p$. By the definition of blocks, it is sufficient to prove this in the case when there exists a nonzero morphism $f : X \rightarrow Y$. By the naturality of g ,

$$\lambda_g(X)f = (g|_X) \circ f = f \circ (g|_Y) = \lambda_g(Y)f.$$

Since $f \neq 0$, we have $\lambda_g(X) = \lambda_g(Y)$. $\square$

Now we have a description of the kernel of φ as follows:

Proposition 3.4. *$\text{Ker}(\varphi) = G(\mathcal{C})_p$. In particular, φ is injective if $p = 0$.*

Proof. Let $g \in G(\mathcal{C})_p$. Then, by Lemma 3.2, $g|_S = \text{id}_S$ for every simple object S of $\mathcal{C}$ and hence $g \in \text{Ker}(\varphi)$. This implies that $G(\mathcal{C})_p \subset \text{Ker}(\varphi)$.

Next let $g \in G(\mathcal{C})'_p \cap \text{Ker}(\varphi)$. Let X be an indecomposable object of $\mathcal{C}$ and fix a simple subobject S of X . Then, by Lemma 3.3, $\lambda_g(X) = \lambda_g(S)$. On the other hand, $\lambda_g(S) = 1$ since $g \in \text{Ker}(\varphi)$. This implies that $g|_X = \text{id}_X$ for all indecomposable objects X and hence $g = 1$. Therefore $G(\mathcal{C})'_p \cap \text{Ker}(\varphi) = \{1\}$.

Now we recall that $G(\mathcal{C}) = G(\mathcal{C})_p \times G(\mathcal{C})'_p$. Our claim follows immediately from the above observations. $\square$

It is interesting to characterize the image of φ . Let N_{ij}^k ($i, j, k \in I$) be the multiplicity of S_k as a composition factor of $S_i \otimes S_j$. In the case where $\mathcal{C}$ is semisimple, it is known that the image of φ is the set of functions $\lambda : I \rightarrow \mathbf{k}^\times$ such that

$$(3.2) \quad \lambda(i)\lambda(j) = \lambda(k) \text{ whenever } N_{ij}^k \neq 0 \quad (i, j, k \in I).$$

In general, the image of φ is smaller than the set of such functions.

Proposition 3.5. *The image of φ is the set of all functions satisfying (3.2) and the following condition:*

$$(3.3) \quad \lambda(i) = \lambda(j) \text{ whenever } S_i \text{ and } S_j \text{ belong to the same block} \quad (i, j \in I).$$

Proof. We remark that if an indecomposable object $X \in \mathcal{C}$ has S_i as a composition factor, then S_i and X belong to the same block. Indeed, then there exists a nonzero morphism $P_i \rightarrow X$ and hence P_i and X belong to the same block. On the other hand, since S_i is a quotient of P_i , S_i and P_i belong to the same block. Therefore the claim follows.

Let $\lambda = \varphi(g)$. By Proposition 3.4, we may assume $g \in G(\mathcal{C})'_p$. (3.3) follows from Lemma 3.3. Thus we check that λ satisfies (3.2). Let $i, j \in I$. By the definition of monoidal natural transformations,

$$g|_{S_i \otimes S_j} = g|_{S_i} \otimes g|_{S_j} = \lambda(i)\lambda(j) \text{id}_{S_i \otimes S_j}.$$

Suppose that $N_{ij}^k \neq 0$. This means that $S_i \otimes S_j$ has S_k as a composition factor. Let X be an indecomposable direct summand of $S_i \otimes S_j$ having S_k as a composition factor. By the above equation, $g|_X = \lambda(i)\lambda(j) \text{id}_X$. On the other hand, since X and S_k belong to the same block, $g|_X = \lambda(k) \text{id}_X$. Therefore $\lambda(k) = \lambda(i)\lambda(j)$.

Conversely, given a function $\lambda : I \rightarrow \mathbf{k}^\times$ satisfying (3.2) and (3.3), we define a natural automorphism $g : \text{id}_{\mathcal{C}} \rightarrow \text{id}_{\mathcal{C}}$ as follows: If X is an indecomposable object of $\mathcal{C}$, then $g|_X = \lambda(i) \text{id}_X$, where $i \in I$ is such that X has S_i as a composition factor. As λ satisfies (3.3), this does not depend on the choice of i . We can extend g to all objects of $\mathcal{C}$, since they are direct sums of indecomposable objects.

Now we need to show that $g \in G(\mathcal{C})$, that is, $g|_{X \otimes Y} = g|_X \otimes g|_Y$ for all objects $X, Y \in \mathcal{C}$. We may assume that X and Y are indecomposable. Suppose that

$$X = \sum_{i \in I} m_i S_i \quad (m_i \in \mathbb{Z}_{\geq 0}) \quad \text{and} \quad Y = \sum_{j \in I} n_j S_j \quad (n_j \in \mathbb{Z}_{\geq 0})$$

in the Grothendieck ring $K(\mathcal{C})$. Then

$$X \cdot Y = \sum_{k \in I} \left(\sum_{i, j \in I} m_i n_j N_{ij}^k \right) S_k$$

in $K(\mathcal{C})$. This equation means that if $X \otimes Y$ has S_k as a composition factor, then there exist $i, j \in I$ such that X has S_i as a composition factor, Y has S_j as a composition factor and $N_{ij}^k \neq 0$. By the definition of g and (3.2), we have that $g|_{X \otimes Y} = g|_X \otimes g|_Y$.

It is obvious that $\lambda = \varphi(g)$. The proof is completed. $\square$

The following theorem is a direct consequence of Propositions 3.4 and 3.5.

Theorem 3.6. *If $p = 0$, then φ gives an isomorphism between $G(\mathcal{C})$ and the group of functions $\lambda : I \rightarrow \mathbf{k}^\times$ satisfying (3.2) and (3.3).*

ACKNOWLEDGEMENT

The author is supported by Grant-in-Aid for JSPS Fellows.

REFERENCES

- [1] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of University Lecture Series. American Mathematical Society, Providence, RI, 2001. MR1797619 (2002d:18003)
- [2] P. Etingof and V. Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4(3):627–654, 782–783, 2004. MR2119143 (2005j:18006)

- [3] S. Mac Lane. *Categories for the working mathematician*, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998. MR1712872 (2001j:18001)
- [4] D. Nikshych. On the structure of weak Hopf algebras. *Adv. Math.*, 170(2):257–286, 2002. MR1932332 (2003f:16063)

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, IBARAKI, 305-8571, JAPAN
E-mail address: `shimizu@math.tsukuba.ac.jp`