A-HYPERGEOMETRIC SYSTEMS THAT COME FROM GEOMETRY

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ABSTRACT. In recent work, Beukers characterized A -hypergeometric systems having a full set of algebraic solutions. He accomplished this by (1) determining which A -hypergeometric systems have a full set of polynomial solutions modulo p for almost all primes p and (2) showing that these systems come from geometry. He then applied a fundamental theorem of N. Katz, which says that such systems have a full set of algebraic solutions. In this paper we establish some connections between nonresonant A -hypergeometric systems and de Rham-type complexes, which leads to a determination of which A -hypergeometric systems come from geometry. We do not use the fact that the system is irreducible or find integral formulas for its solutions.

1. INTRODUCTION

Let $A = \{a^{(1)}, \dots, a^{(N)}\} \subseteq \mathbb{Z}^n$ with $a^{(j)} = (a_1^{(j)}, \dots, a_n^{(j)})$. We shall assume throughout this paper that these lattice points generate $\mathbb{Z}^n$ as an abelian group. Let L be the corresponding lattice of relations,

$$L = \left\{ (l_1, \dots, l_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N l_j a^{(j)} = 0 \right\},$$

and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. The A -hypergeometric system is the system of partial differential equations in the variables $\lambda_1, \dots, \lambda_N$ consisting of the operators (we write ∂_j for $\partial/\partial\lambda_j$)

$$\square_l = \prod_{l_j > 0} \partial_j^{l_j} - \prod_{l_j < 0} \partial_j^{-l_j}$$

for all $l \in L$ and the operators

$$Z_{i,\alpha} = \sum_{j=1}^N a_i^{(j)} \lambda_j \partial_j - \alpha_i$$

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for $i = 1, \dots, n$. We denote by $\mathcal{D} = \mathbb{C}\langle \lambda_1, \dots, \lambda_N, \partial_1, \dots, \partial_N \rangle$ the ring of differential operators in the λ_j . The associated hypergeometric $\mathcal{D}$ -module is

$$\mathcal{M}_\alpha = \mathcal{D} \left/ \left(\sum_{l \in L} \mathcal{D}\square_l + \sum_{i=1}^n \mathcal{D}Z_{i,\alpha} \right) \right.$$

Let $\mathbb{C}[\lambda] = \mathbb{C}[\lambda_1, \dots, \lambda_N]$ be the polynomial ring in N variables and let X be a smooth variety over $\mathbb{C}[\lambda]$. Let G be a finite group acting on $X/\mathbb{C}[\lambda]$. Then G acts on the relative de Rham cohomology groups $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])$. For an irreducible representation χ of G , let $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])^\chi$ denote the χ -isotypic component of $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])$; i.e., $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])^\chi$ is the sum of all G -submodules of $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])$ that are isomorphic to χ . The $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])^\chi$ are $\mathcal{D}$ -modules via the Gauss-Manin connection. We say that a $\mathcal{D}$ -module $\mathcal{M}$ *comes from geometry* if it is isomorphic to $H_{\text{DR}}^i(X/\mathbb{C}[\lambda])^\chi$ for some $(X/\mathbb{C}[\lambda], G, \chi)$.

The main idea of this paper is to show that the $\mathcal{M}_\alpha$ are isomorphic as $\mathcal{D}$ -modules to certain cohomology groups that arise in algebraic geometry. Let $R' = \mathbb{C}[\lambda][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the coordinate ring of the n -torus $\mathbb{T}^n$ over $\mathbb{C}[\lambda]$. Put

$$(1.1) \quad f = \sum_{j=1}^N \lambda_j x^{a^{(j)}} \in R'$$

and let $\Omega_{R'/\mathbb{C}[\lambda]}^k$ denote the module of relative k -forms. We use the set

$$\left\{ \frac{dx_{i_1}}{x_{i_1}} \wedge \dots \wedge \frac{dx_{i_k}}{x_{i_k}} \mid 1 \leq i_1 < \dots < i_k \leq n \right\}$$

as a basis for $\Omega_{R'/\mathbb{C}[\lambda]}^k$ as an R' -module. The map $\nabla_\alpha : \Omega_{R'/\mathbb{C}[\lambda]}^k \rightarrow \Omega_{R'/\mathbb{C}[\lambda]}^{k+1}$ given by

$$\nabla_\alpha(\omega) = d\omega + \sum_{k=1}^n \alpha_i \frac{dx_i}{x_i} \wedge \omega + df \wedge \omega$$

defines a complex $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$. In terms of the above basis, we have for $\xi \in R'$

$$(1.2) \quad \nabla_\alpha \left(\xi \frac{dx_{i_1}}{x_{i_1}} \wedge \dots \wedge \frac{dx_{i_k}}{x_{i_k}} \right) = \left(\sum_{i=1}^n D_{i,\alpha}(\xi) \frac{dx_i}{x_i} \right) \wedge \frac{dx_{i_1}}{x_{i_1}} \wedge \dots \wedge \frac{dx_{i_k}}{x_{i_k}},$$

where

$$(1.3) \quad D_{i,\alpha} = x_i \frac{\partial}{\partial x_i} + \alpha_i + x_i \frac{\partial f}{\partial x_i}.$$

Formally,

$$\nabla_\alpha = \frac{1}{x^\alpha \exp f} \circ d \circ x^\alpha \exp f,$$

so for any derivation $\partial \in \text{Der}_{\mathbb{C}}(\mathbb{C}[\lambda])$ the operator

$$D_\partial = \frac{1}{x^\alpha \exp f} \circ \partial \circ x^\alpha \exp f = \partial + f^\partial$$

on the $\Omega_{R'/\mathbb{C}[\lambda]}^k$ commutes with ∇_α (where f^∂ denotes the polynomial obtained from f by applying ∂ to its coefficients). This defines an action of $\text{Der}_{\mathbb{C}}(\mathbb{C}[\lambda])$ on $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$. In particular, ∂_j acts as $D_j = \partial_j + x^{a^{(j)}}$. This action extends to an action of $\mathcal{D}$ on $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$, which makes the $H^i(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$ into $\mathcal{D}$ -modules.

Let $C(A) \subseteq \mathbb{R}^n$ be the real cone generated by A and let $\ell_1, \dots, \ell_s \in \mathbb{Z}[u_1, \dots, u_n]$ be homogeneous linear forms defining the codimension-one faces of $C(A)$, normalized so that the coefficients of each ℓ_i are relatively prime and so that $\ell_i \geq 0$ on $C(A)$ for each i . We say that α is *nonresonant for A* if $\ell_i(\alpha) \notin \mathbb{Z}$ for all i . Note that α is nonresonant for A if and only if $\alpha + u$ is nonresonant for A for all $u \in \mathbb{Z}^n$. (If $\alpha \in \mathbb{R}^n$, this is equivalent to saying that no proper face of $C(A)$ contains a point of $\alpha + \mathbb{Z}^n$.)

Theorem 1.4. *If α is nonresonant for A , then $\mathcal{M}_\alpha \cong H^n(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$ as $\mathcal{D}$ -modules.*

Remark 1. One can show that the complex $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$ is isomorphic to the Euler-Koszul complex on the localization denoted $S_A[\partial_A^{-1}]$ in Schulze-Walther [14]. Theorem 1.4 then follows from results in [14, Sections 2 and 3]. We believe the approach we follow here to be more convenient for explaining the relationship with the relative de Rham cohomology of algebraic varieties. By [14, Sections 2 and 3], this Euler-Koszul complex is in fact a resolution of $\mathcal{M}_\alpha$ for α nonresonant, so $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$ is also a resolution of $\mathcal{M}_\alpha$.

Remark 2. It is straightforward to check that for $u \in \mathbb{Z}^n$ multiplication by x^u defines an isomorphism of complexes of $\mathcal{D}$ -modules

$$x^u : (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_{\alpha+u}) \rightarrow (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$$

(its inverse is multiplication by x^{-u}). Thus when α is nonresonant the $\mathcal{M}_{\alpha+u}$ for $u \in \mathbb{Z}^n$ are all isomorphic as $\mathcal{D}$ -modules.

Consider the special case $f = x_n g$, where $g \in \mathbb{C}[\lambda][x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. Let $\mathbb{T}^{n-1}$ be the $(n-1)$ -torus over $\mathbb{C}[\lambda]$ with coordinates $x_1, \dots, x_{n-1}$. Let $U \subseteq \mathbb{T}^{n-1}$ be the open set where g is nonvanishing and let $\Omega_{U/\mathbb{C}[\lambda]}^k$ be the module of relative k -forms over the ring of regular functions on U . The map $\tilde{\nabla}_\alpha : \Omega_{U/\mathbb{C}[\lambda]}^k \rightarrow \Omega_{U/\mathbb{C}[\lambda]}^{k+1}$ defined by

$$\tilde{\nabla}_\alpha(\omega) = d\omega + \sum_{i=1}^{n-1} \alpha_i \frac{dx_i}{x_i} \wedge \omega - \alpha_n \frac{dg}{g} \wedge \omega$$

defines a complex $(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$. Formally we have

$$\tilde{\nabla}_\alpha = \frac{g^{\alpha_n}}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \circ d \circ \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{g^{\alpha_n}},$$

so if we define for $\partial \in \text{Der}_{\mathbb{C}}(\mathbb{C}[\lambda])$

$$\tilde{D}_\partial = \frac{g^{\alpha_n}}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \circ \partial \circ \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{g^{\alpha_n}} = \partial - \alpha_n \frac{g^\partial}{g},$$

we get an action of $\text{Der}_{\mathbb{C}}(\mathbb{C}[\lambda])$ on this complex. The action extends to an action of $\mathcal{D}$, making $(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$ into a complex of $\mathcal{D}$ -modules. Note that for $u \in \mathbb{Z}^n$, multiplication by $x_1^{u_1} \cdots x_{n-1}^{u_{n-1}} / g^{u_n}$ defines an isomorphism of complexes of $\mathcal{D}$ -modules

$$\frac{x_1^{u_1} \cdots x_{n-1}^{u_{n-1}}}{g^{u_n}} : (\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_{\alpha+u}) \rightarrow (\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha).$$

Theorem 1.5. *Suppose $f = x_n g(x_1, \dots, x_{n-1})$ and α is nonresonant for A . For all i there are $\mathcal{D}$ -module isomorphisms*

$$H^i(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) \cong H^{i-1}(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha).$$

Remark 3. In view of Remark 1, we have $H^i(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) = 0$ for $i \neq n$, so both sides of the isomorphism of Theorem 1.5 vanish for $i \neq n$.

When $\alpha \in \mathbb{Q}^n$, it is well known that the $H^i(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$ come from geometry. (We sketch the proof of this fact in Section 4.) From Theorems 1.4 and 1.5, we then get the following result.

Corollary 1.6. *Suppose $f = x_n g(x_1, \dots, x_{n-1})$ and α is nonresonant for A . There is an isomorphism of $\mathcal{D}$ -modules $\mathcal{M}_\alpha \cong H^{n-1}(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$. If in addition $\alpha \in \mathbb{Q}^n$, then $\mathcal{M}_\alpha$ comes from geometry.*

The proofs of Theorems 1.4 and 1.5 are based on ideas from [1] and [3]. Those papers in turn are related to earlier work of Dwork, Dwork-Loeser, and N. Katz. We refer the reader to the introductions of [1] and [3] for more details on the connections with that earlier work.

2. PROOF OF THEOREM 1.4

It is straightforward to check that the $Z_{i,\alpha}$ commute with one another and that

$$\square_l \circ Z_{i,\alpha} = Z_{i,\beta} \circ \square_l,$$

where $\beta = \alpha + \sum_{l_j > 0} l_j a^{(j)}$ ($= \alpha - \sum_{l_j < 0} l_j a^{(j)}$ since $l \in L$). It follows that right multiplication by $Z_{i,\alpha}$ maps the left ideal $\sum_{l \in L} \mathcal{D} \square_l$ into itself. If we put $\mathcal{P} = \mathcal{D} / \sum_{l \in L} \mathcal{D} \square_l$, then right multiplication by the $Z_{i,\alpha}$ is a family of commuting endomorphisms of $\mathcal{P}$ as a left $\mathcal{D}$ -module. Let $\mathcal{C}^\bullet$ be the cohomological Koszul complex on $\mathcal{P}$ defined by the $Z_{i,\alpha}$. Concretely,

$$\mathcal{C}^k = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \mathcal{P} e_{i_1} \wedge \dots \wedge e_{i_k},$$

where the e_i are formal symbols satisfying $e_i \wedge e_j = -e_j \wedge e_i$ and the boundary operator $\delta_\alpha : \mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$ is defined by additivity and the formula (for $\sigma \in \mathcal{P}$)

$$\delta_\alpha(\sigma e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{i=1}^n \sigma Z_{i,\alpha} e_i \wedge e_{i_1} \wedge \dots \wedge e_{i_k}.$$

One obtains a complex of left $\mathcal{D}$ -modules $(\mathcal{C}^\bullet, \delta_\alpha)$ for which

$$(2.1) \quad H^n(\mathcal{C}^\bullet, \delta_\alpha) = \mathcal{M}_\alpha.$$

Let $R = \mathbb{C}[\lambda][x^{a^{(1)}}, \dots, x^{a^{(N)}}]$, a subring of R' which is also a $\mathcal{D}$ -submodule of R' , and let

$$\Omega_R^k \langle \log \rangle = \left\{ \sum_{1 \leq i_1 < \dots < i_k \leq n} \xi_{i_1 \dots i_k} \frac{dx_{i_1}}{x_{i_1}} \wedge \dots \wedge \frac{dx_{i_k}}{x_{i_k}} \mid \xi_{i_1 \dots i_k} \in R \right\} \subseteq \Omega_{R'/\mathbb{C}[\lambda]}^k.$$

By (1.2) and (1.3), one has $\nabla_\alpha(\Omega_R^k \langle \log \rangle) \subseteq \Omega_R^{k+1} \langle \log \rangle$, so this defines a subcomplex $(\Omega_R^\bullet \langle \log \rangle, \nabla_\alpha)$ of $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$.

The $\mathbb{C}[\lambda]$ -module homomorphism $\phi : \mathcal{P} \rightarrow R$ defined by

$$\phi(\partial_1^{b_1} \dots \partial_N^{b_N}) = x^{\sum_{j=1}^N b_j a^{(j)}}$$

is an isomorphism of $\mathcal{D}$ -modules by [1, Theorem 4.4]. It extends to a map $\phi : \mathcal{C}^k \rightarrow \Omega_R^k \langle \log \rangle$ by additivity and the formula

$$\phi(\sigma e_{i_1} \wedge \cdots \wedge e_{i_k}) = \phi(\sigma) \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}.$$

By [2, Corollary 2.4], this is an isomorphism of complexes of $\mathcal{D}$ -modules:

$$(2.2) \quad \phi : (\mathcal{C}^\bullet, \delta_\alpha) \xrightarrow{\cong} (\Omega_R^\bullet \langle \log \rangle, \nabla_\alpha).$$

By (2.1) and (2.2), Theorem 1.4 is a consequence of the following result.

Proposition 2.3. *If α is nonresonant for A , then the inclusion map*

$$(\Omega_R^\bullet \langle \log \rangle, \nabla_\alpha) \hookrightarrow (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$$

is a quasi-isomorphism of complexes of $\mathcal{D}$ -modules.

We state and prove a generalization of Proposition 2.3. Let $U \subseteq \mathbb{Z}^n$ be a nonempty subset satisfying the following condition:

$$(2.4) \quad \text{if } u \in U, \text{ then } u + a^{(j)} \in U \text{ for } j = 1, \dots, N.$$

If we denote by M_U the free $\mathbb{C}[\lambda]$ -module with basis $\{x^u \mid u \in U\}$, then (2.4) implies that M_U is both an R -submodule and a $\mathcal{D}$ -submodule of R' . Note that $R' = M_{\mathbb{Z}^n}$ and that $R = M_{U_0}$, where $U_0 = \{\sum_{j=1}^N c_j a^{(j)} \mid c_j \in \mathbb{Z}_{\geq 0}\}$. Let $\Omega_{M_U}^k \langle \log \rangle \subseteq \Omega_{R'/\mathbb{C}[\lambda]}^k$ be the subset

$$\Omega_{M_U}^k \langle \log \rangle = \left\{ \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1 \dots i_k} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \mid \xi_{i_1 \dots i_k} \in M_U \right\}.$$

By (1.2) and (1.3), we have $\nabla_\alpha(\Omega_{M_U}^k \langle \log \rangle) \subseteq \Omega_{M_U}^{k+1} \langle \log \rangle$, so we get a subcomplex $(\Omega_{M_U}^\bullet \langle \log \rangle, \nabla_\alpha)$ of $(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$. Proposition 2.3 is the special case $U = U_0$ of the following more general result.

Proposition 2.5. *If α is nonresonant for A and U satisfies (2.4), then the inclusion*

$$(\Omega_{M_U}^\bullet \langle \log \rangle, \nabla_\alpha) \hookrightarrow (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$$

is a quasi-isomorphism of complexes of $\mathcal{D}$ -modules.

As in Section 1, let $\ell_1, \dots, \ell_s \in \mathbb{Z}[u_1, \dots, u_n]$ be the normalized homogeneous linear forms defining the codimension-one faces of $C(A)$ and put $S = \{1, \dots, s\}$. For $v = (v_1, \dots, v_s) \in \mathbb{Z}^s$ and a subset $T \subseteq S$, put

$$W(v, T) = \{u \in \mathbb{Z}^n \mid \ell_i(u) \geq v_i \text{ for all } i \in T\}.$$

Since $\ell_i(a^{(j)}) \geq 0$ for all i, j , the set $W(v, T)$ satisfies (2.4). By [1, Lemma 3.12], there exists v such that $W(v, S) \subseteq U_0$. It follows that if U satisfies (2.4), then there exists v such that $W(v, S) \subseteq U$.

Lemma 2.6. *If α is nonresonant for A , U satisfies (2.4), and $W(v, T) \subseteq U$, then the inclusion*

$$(2.7) \quad (\Omega_{M_{W(v, T)}}^\bullet \langle \log \rangle, \nabla_\alpha) \hookrightarrow (\Omega_{M_U}^\bullet \langle \log \rangle, \nabla_\alpha)$$

is a quasi-isomorphism of complexes of $\mathcal{D}$ -modules.

Since $W(v, T) \subseteq \mathbb{Z}^n$ for all v, T , Lemma 2.6 implies that the inclusion

$$(2.8) \quad (\Omega_{M_{W(v, T)}}^\bullet \langle \log \rangle, \nabla_\alpha) \hookrightarrow (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$$

is a quasi-isomorphism for all v, T . The quasi-isomorphisms (2.7) and (2.8) imply Proposition 2.5.

Proof of Lemma 2.6. For $T' \subseteq T$, let

$$U(v, T') = \{u \in U \mid \ell_i(u) \geq v_i \text{ for all } i \in T'\}.$$

Note that $U(v, \emptyset) = U$ and that $U(v, T) = W(v, T)$ (since $W(v, T) \subseteq U$). By induction, it thus suffices to show that if $T' \subseteq T$ and $T'' = T' \cup \{t\}$ with $t \in T \setminus T'$, then the inclusion

$$(2.9) \quad (\Omega_{M_{U(v, T'')}}^\bullet \langle \log \rangle, \nabla_\alpha) \hookrightarrow (\Omega_{M_{U(v, T')}}^\bullet \langle \log \rangle, \nabla_\alpha)$$

is a quasi-isomorphism. Let $Q^\bullet$ be the quotient complex

$$Q^\bullet = \Omega_{M_{U(v, T')}}^\bullet \langle \log \rangle / \Omega_{M_{U(v, T'')}}^\bullet \langle \log \rangle.$$

We show that (2.9) is a quasi-isomorphism by showing that $H^k(Q^\bullet) = 0$ for all k .

We define a filtration $\{F_p\}_{p \leq v_t}$ on the complex $\Omega_{M_{U(v, T')}}^\bullet \langle \log \rangle$. For $p \leq v_t$, let $F_p \Omega_{M_{U(v, T')}}^k \langle \log \rangle$ be the $\mathbb{C}[\lambda]$ -submodule spanned by differential forms

$$(2.10) \quad x^u \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}$$

satisfying $\ell_t(u) \geq p$. By (1.2) and (1.3), ∇_α respects this filtration. We also denote by F_p the induced filtrations on the subcomplex $\Omega_{M_{U(v, T'')}}^\bullet \langle \log \rangle$ and the quotient complex $Q^\bullet$. Note that since

$$F_{v_t} \Omega_{M_{U(v, T')}}^\bullet \langle \log \rangle = \Omega_{M_{U(v, T'')}}^\bullet \langle \log \rangle$$

we have $F_{v_t} Q^\bullet = 0$. To show that $H^k(Q^\bullet) = 0$ for all k , it suffices to show that $H^k(\text{gr}_p Q^\bullet) = 0$ for all k and p , where $\text{gr}_p Q^\bullet$ denotes the p -th graded piece of the associated graded of the filtration F_p .

Write $\ell_t(u) = \sum_{i=1}^n c_i u_i \in \mathbb{Z}[u]$. Define $\rho : \Omega_{M_{U(v, T')}}^k \langle \log \rangle \rightarrow \Omega_{M_{U(v, T'')}}^{k-1} \langle \log \rangle$ to be the $\mathbb{C}[\lambda]$ -module homomorphism satisfying

$$\rho \left(x^u \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \right) = x^u \sum_{j=1}^k (-1)^{j-1} c_{i_j} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \widehat{\frac{dx_{i_j}}{x_{i_j}}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}.$$

It is straightforward to check that

$$(\nabla_\alpha \circ \rho + \rho \circ \nabla_\alpha) \left(x^u \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \right) = \left(\sum_{i=1}^n c_i D_{i, \alpha}(x^u) \right) \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}},$$

and a calculation using (1.1) and (1.3) shows that

$$(2.11) \quad \sum_{i=1}^n c_i D_{i, \alpha}(x^u) = \ell_t(\alpha + u)x^u + \sum_{j=1}^N \lambda_j \ell_t(a^{(j)})x^{u+a^{(j)}}.$$

Suppose that the form (2.10) lies in $F_p \setminus F_{p+1}$. Then $\ell_t(\alpha + u) = \ell_t(\alpha) + p$, and if $\ell_t(a^{(j)}) \neq 0$, then $\ell_t(u + a^{(j)}) > p$. It follows that the second term on the right-hand side of (2.11) lies in F_{p+1} , so on the associated graded complex the induced map

$$(\nabla_\alpha \circ \rho + \rho \circ \nabla_\alpha) : \text{gr}_p Q^k \rightarrow \text{gr}_p Q^k$$

is just multiplication by $\ell_t(\alpha) + p$. Since α is nonresonant for A , $\ell_t(\alpha) + p \neq 0$. Thus multiplication by a nonzero constant is homotopic to the zero map, which implies that $H^k(\mathrm{gr}_p Q^\bullet) = 0$ for all k . $\square$

3. PROOF OF THEOREM 1.5

In this section we assume that $f = x_n g(x_1, \dots, x_{n-1})$. By the remark following Theorem 1.4, we may assume that if $\alpha_n \in \mathbb{Z}$, then $\alpha_n \geq 1$. Let $R_+ = \mathbb{C}[\lambda][x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}, x_n]$. By Proposition 2.5, the inclusion

$$(\Omega_{R_+}^\bullet \langle \log \rangle, \nabla_\alpha) \hookrightarrow (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha)$$

is a quasi-isomorphism of complexes of $\mathcal{D}$ -modules. Theorem 1.5 is then a consequence of the following result.

Proposition 3.1. *If $\alpha_n \notin \mathbb{Z}_{\leq 0}$, then there is a quasi-isomorphism of complexes of $\mathcal{D}$ -modules*

$$(\Omega_{R_+}^\bullet \langle \log \rangle, \nabla_\alpha) \rightarrow (\Omega_{U/\mathbb{C}[\lambda]}^\bullet[-1], \tilde{\nabla}_\alpha).$$

Proof. We regard $\Omega_{R_+}^\bullet \langle \log \rangle$ as the total complex associated to a certain two-row double complex: let

$$\Omega^{k,0} = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n-1} R_+ \frac{dx_{i_1}}{x_{i_1}} \wedge \dots \wedge \frac{dx_{i_k}}{x_{i_k}}$$

and let

$$\Omega^{k,1} = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n-1} R_+ \frac{dx_{i_1}}{x_{i_1}} \wedge \dots \wedge \frac{dx_{i_k}}{x_{i_k}} \wedge \frac{dx_n}{x_n}.$$

Let $\partial_h : \Omega^{k,i} \rightarrow \Omega^{k+1,i}$ be the map

$$\partial_h(\omega) = d'\omega + \sum_{i=1}^{n-1} \alpha_i \frac{dx_i}{x_i} \wedge \omega + x_n(d'g \wedge \omega),$$

where d' is an exterior differentiation relative to the variables $x_1, \dots, x_{n-1}$, and let $\partial_v : \Omega^{k,0} \rightarrow \Omega^{k,1}$ be the map

$$\partial_v(\omega) = d''\omega + \alpha_n \frac{dx_n}{x_n} \wedge \omega + g dx_n \wedge \omega,$$

where d'' is an exterior differentiation relative to the variable x_n . Since ∂_v is injective, the canonical projection $\Omega_{R_+}^k \langle \log \rangle \rightarrow \Omega^{k-1,1}$ induces a quasi-isomorphism

$$(3.2) \quad (\Omega_{R_+}^\bullet \langle \log \rangle, \nabla_\alpha) \rightarrow ((\Omega^{\bullet,1}/\partial_v \Omega^{\bullet,0})[-1], \partial_h)$$

(see [13, Appendix B]).

To complete the proof of Proposition 3.1, we define a quasi-isomorphism of complexes between $(\Omega^{\bullet,1}/\partial_v \Omega^{\bullet,0}, \partial_h)$ and $(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$. Define $\gamma : \Omega^{k,1} \rightarrow \Omega_{U/\mathbb{C}[\lambda]}^k$ to be the $\mathbb{C}[\lambda]$ -module homomorphism satisfying

$$\gamma\left(x^u \frac{dx_{i_1}}{x_{i_1}} \dots \frac{dx_{i_k}}{x_{i_k}} \frac{dx_n}{x_n}\right) = \frac{(-1)^{u_n} (\alpha_n)_{u_n} x_1^{u_1} \dots x_{n-1}^{u_{n-1}}}{g^{u_n}} \frac{dx_{i_1}}{x_{i_1}} \dots \frac{dx_{i_k}}{x_{i_k}},$$

where $(\alpha_n)_{u_n} = \alpha_n(\alpha_n + 1) \dots (\alpha_n + u_n - 1)$. It is straightforward to check that γ commutes with boundary operators and hence defines a homomorphism of complexes from $\Omega^{\bullet,1}$ to $\Omega_{U/\mathbb{C}[\lambda]}^\bullet$. The hypothesis that $\alpha_n \notin \mathbb{Z}_{\leq 0}$ implies that γ is

surjective, and it is straightforward to check that $\gamma(\partial_v \Omega^{k,0}) = 0$. It remains only to show that

$$\ker(\gamma : \Omega^{k,1} \rightarrow \Omega_{U/\mathbb{C}[\lambda]}^k) = \partial_v \Omega^{k,0}.$$

Let $\xi \in \Omega^{k,1}$ satisfy $\gamma(\xi) = 0$. Write

$$\xi = \sum_{i=m}^M x_n^i \xi_i \frac{dx_n}{x_n},$$

where $\xi_i \in \Omega^{k,0}$ is in the $\mathbb{C}[\lambda]$ -span of the forms

$$(3.3) \quad x_1^{w_1} \cdots x_{n-1}^{w_{n-1}} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}, \quad 1 \leq i_1 < \cdots < i_k \leq n-1.$$

We prove by induction on $M-m$ that $\xi \in \partial_v \Omega^{k,0}$. We have

$$0 = \gamma(\xi) = \sum_{i=m}^M (-1)^i (\alpha_n)_i \frac{\xi_i}{g^i}.$$

Since $(\alpha_n)_M \neq 0$, this equation implies $\xi_M = g\eta$, where η is in the $\mathbb{C}[\lambda]$ -span of the forms (3.3). It follows that

$$\xi = (-1)^k \partial_v (x_n^{M-1} \eta) - (\alpha_n + M - 1) x_n^{M-1} \eta \frac{dx_n}{x_n} + \sum_{m=1}^{M-1} x_n^i \xi_i \frac{dx_n}{x_n}.$$

By induction we are reduced to the case $\xi = x_i^m \xi_m \frac{dx_n}{x_n}$. However, in this case

$$0 = \gamma(\xi) = (-1)^m (\alpha_n)_m \xi_m / g^m,$$

so $\xi_m = 0$. □

If $\alpha_n \notin \mathbb{Z}$, there is no need to introduce the complex $\Omega_{R_+}^\bullet \langle \log \rangle$.

Proposition 3.4. *If $\alpha_n \notin \mathbb{Z}$, then there is a quasi-isomorphism of complexes of $\mathcal{D}$ -modules*

$$(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) \rightarrow (\Omega_{U/\mathbb{C}[\lambda]}^\bullet[-1], \tilde{\nabla}_\alpha).$$

Sketch of the proof. One proceeds as in the proof of Proposition 3.1 with the complex $\Omega_{R_+}^\bullet \langle \log \rangle$ replaced by $\Omega_{R'/\mathbb{C}[\lambda]}^\bullet$. One defines γ as before with the understanding that for $u_n < 0$

$$(\alpha_n)_{u_n} = ((\alpha_n - 1)(\alpha_n - 2) \cdots (\alpha_n + u_n))^{-1}.$$

The proof then proceeds unchanged. (See [3, Lemma 2.5] for the details in a similar situation.) □

4. APPLICATION OF N. KATZ'S RESULTS

We begin by sketching the proof that the $H^i(\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$ come from geometry. Let $X \subseteq \mathbb{T}_{\mathbb{C}[\lambda]}^n$ be the hypersurface $x_n^D g(x_1^D, \dots, x_{n-1}^D) - 1 = 0$, where $D \in \mathbb{Z}_{>0}$. Since $\Omega_{X/\mathbb{C}[\lambda]}^1$ is a free module with basis $\{dx_i\}_{i=1}^{n-1}$, $\mathcal{D}$ acts on global i -forms by acting on their coefficients relative to exterior powers of this basis. The group $(\mu_D)^n$ acts on X and its relative de Rham complex $(\Omega_{X/\mathbb{C}[\lambda]}^\bullet, d)$. The irreducible representations χ of $(\mu_D)^n$ can be indexed by n -tuples $(a_1, \dots, a_n)$, $0 \leq a_i < D$, so that if χ corresponds to $(a_1, \dots, a_n)$, then there is an isomorphism of complexes of

$\mathcal{D}$ -modules $(\Omega_{X/\mathbb{C}[\lambda]}^\bullet, d)^\chi \cong (\Omega_{U/\mathbb{C}[\lambda]}^\bullet, \tilde{\nabla}_\alpha)$, where $\alpha = (a_1/D, \dots, a_n/D)$.) The first assertion of Corollary 1.6 then implies that

$$(4.1) \quad \mathcal{M}_\alpha \cong H_{\text{DR}}^{n-1}(X/\mathbb{C}[\lambda])^\chi,$$

which establishes the second assertion of Corollary 1.6.

Now suppose that $C = \text{Spec}(A)$ is a smooth connected curve over $\mathbb{C}$ and $\phi : C \rightarrow \mathbb{A}^N = \text{Spec}(\mathbb{C}[\lambda])$ is a morphism. Let X' be the pullback of X to a variety over C , i.e., $X' = A \otimes_{\mathbb{C}[\lambda]} X$. Then X' is the hypersurface in $\mathbb{T}_A^n$ defined by the equation $x_n^D g^\phi(x_1^D, \dots, x_{n-1}^D) - 1 = 0$, where $g^\phi \in A[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ is the Laurent polynomial obtained from g by applying ϕ to its coefficients (by abuse of notation, we also denote by ϕ the homomorphism $\mathbb{C}[\lambda] \rightarrow A$ corresponding to $\phi : C \rightarrow \mathbb{A}^N$). The varieties X and X' are smooth affine schemes whose de Rham cohomology can be computed as the cohomology of the complex of global sections of the de Rham complex. By the right-exactness of tensor products, one has

$$(4.2) \quad \phi^*(H_{\text{DR}}^{n-1}(X/\mathbb{C}[\lambda])) \cong H_{\text{DR}}^{n-1}(X'/A).$$

It follows from [9, Section 14] that $H_{\text{DR}}^{n-1}(X'/A)$ has regular singular points and quasi-unipotent local monodromy at infinity (i.e., at all points of the quotient field of A). Therefore $H_{\text{DR}}^{n-1}(X/\mathbb{C}[\lambda])$ has regular singular points and quasi-unipotent local monodromy at infinity (in the sense of [11, Section VIII]). Equation (4.1) then implies that $\mathcal{M}_\alpha$ has regular singular points and quasi-unipotent local monodromy at infinity.

To apply the results of [10], we observe that the results of this paper are valid when one replaces $\mathbb{C}[\lambda]$ by $\mathbb{C}(\lambda)$. Let $\bar{\mathcal{D}}$ denote the ring of differential operators with coefficients in $\mathbb{C}(\lambda)$ and define

$$\bar{\mathcal{M}}_\alpha = \bar{\mathcal{D}} \left/ \left(\sum_{l \in L} \bar{\mathcal{D}} \square_l + \sum_{i=1}^n \bar{\mathcal{D}} Z_{i,\alpha} \right) \right.$$

Put

$$\bar{R} = \mathbb{C}(\lambda)[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

the coordinate ring of the n -torus $\mathbb{T}_{\mathbb{C}(\lambda)}$. The proof of Theorem 1.4 establishes the following result.

Proposition 4.3. *If α is nonresonant for A , then $\bar{\mathcal{M}}_\alpha \cong H^n(\Omega_{\bar{R}/\mathbb{C}(\lambda)}^\bullet, \nabla_\alpha)$ as $\bar{\mathcal{D}}$ -modules.*

In the situation of Theorem 1.5, let $\bar{U} \subseteq \mathbb{T}_{\mathbb{C}(\lambda)}^n$ be the open set where g is nonvanishing. Then we have the following result.

Proposition 4.4. *Suppose $f = x_n g(x_1, \dots, x_{n-1})$ and α is nonresonant for A . For all i there are $\bar{\mathcal{D}}$ -module isomorphisms*

$$H^i(\Omega_{\bar{R}/\mathbb{C}(\lambda)}^\bullet, \nabla_\alpha) \cong H^{i-1}(\Omega_{\bar{U}/\mathbb{C}(\lambda)}^\bullet, \tilde{\nabla}_\alpha).$$

Combining these propositions gives the following result.

Corollary 4.5. *Suppose $f = x_n g(x_1, \dots, x_{n-1})$ and α is nonresonant for A . There is an isomorphism of $\bar{\mathcal{D}}$ -modules $\bar{\mathcal{M}}_\alpha \cong H^{n-1}(\Omega_{\bar{U}/\mathbb{C}(\lambda)}^\bullet, \tilde{\nabla}_\alpha)$. If in addition $\alpha \in \mathbb{Q}^n$, then $\bar{\mathcal{M}}_\alpha$ comes from geometry.*

Explicitly, letting $\bar{X} \subseteq \mathbb{T}_{\mathbb{C}(\lambda)}^n$ be the hypersurface $x_n^D g(x_1^D, \dots, x_{n-1}^D) - 1 = 0$, we have (corresponding to equation (4.1))

$$(4.6) \quad \bar{\mathcal{M}}_\alpha \cong H_{\text{DR}}^{n-1}(\bar{X}/\mathbb{C}(\lambda))^\chi.$$

By [10, Theorem 5.7] $H_{\text{DR}}^{n-1}(\bar{X}/\mathbb{C}(\lambda))^\chi$ has a full set of polynomial solutions modulo p for almost all primes p if and only if it has a full set of algebraic solutions. Note that the solution sets of $\mathcal{M}_\alpha$ and $\bar{\mathcal{M}}_\alpha$ in the algebraic closure of $\mathbb{C}(\lambda)$ are identical. From equation (4.6), we then get the following result.

Corollary 4.7. *Suppose $f = x_n g(x_1, \dots, x_{n-1})$ and $\alpha \in \mathbb{Q}^n$ is nonresonant for A . The hypergeometric $\mathcal{D}$ -module $\mathcal{M}_\alpha$ has a full set of polynomial solutions modulo p for almost all primes p if and only if it has a full set of algebraic solutions.*

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