CONNECTIVITY OF JULIA SETS FOR WEIERSTRASS ELLIPTIC FUNCTIONS ON SQUARE LATTICES

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ABSTRACT. We show that the Julia set of a Weierstrass elliptic function on a square lattice is connected. The techniques used to prove this result are used to show a similar result for a related family of rational maps obtained from the Laurent series.

1. INTRODUCTION

It is a classical result that the Julia set of $z^2 + c$ is connected if and only if the orbit of 0 is bounded. This could be restated as: “The Julia set of $z^2 + c$ is connected if and only if each Fatou component contains at most one critical value.” It is our goal in this paper to show that the Julia set of a Weierstrass elliptic function on a square lattice is connected by proving that each Fatou component contains at most one critical value. In certain specific cases the Julia set has been shown to be connected. These cases include when $\wp_\Omega$ has a real super-attracting fixed point [12] or a cycle of Siegel disks [14]. We complete the cases when $\wp_\Omega$ has attracting or parabolic periodic points. A result of [5] proves that all of these cases do indeed occur.

Using different techniques to show that critical values lie in distinct Fatou components, the Julia set of a Weierstrass elliptic on so-called triangular lattices has been shown to be connected [14, 13, 15]. Our techniques are necessarily different due to the fact that one critical value is always a pole. Furthermore, the triangular lattice case has three different but symmetric (under rotations of $2\pi/3$) orbits of critical values, whereas the square case has one critical value which is a pole and the other two critical values map to the same value determining one free critical orbit. This results in at most one non-repelling cycle [4]. More generally, a Weierstrass elliptic function can have as many as three different non-repelling cycles of possibly different types.

This paper is setup as follows. In Section 2 we give the relevant technical details of Weierstrass elliptic functions. In particular, we discuss lattices, critical points, critical values, and mapping properties. Section 3 summarizes known results about

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the Fatou and Julia sets of Weierstrass elliptic functions and their application to the square lattice setting. The main theorem is stated and proved in Section 4. The last section applies the technique used in Section 4 to a related family of rational maps obtained from the Laurent series of the Weierstrass elliptic functions.

2. LATTICES AND WEIERSTRASS ELLIPTIC FUNCTIONS

The Weierstrass $\wp$ function is considered to be the most basic of the family of elliptic functions. With that consideration we would like to study it in detail so that we may better understand other elliptic functions. Let $\mathbb{C}$ denote the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere.

Definition 2.1. A *lattice* Ω is an additive subgroup of $\mathbb{C}$ with two generators that is isomorphic to $\mathbb{Z}^2$ such that the generators $\omega_1, \omega_2 \in \mathbb{C}^*$ are $\mathbb{R}$ linearly independent:

$$(2.1) \quad \Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

We denote this by $\Omega = \langle \omega_1, \omega_2 \rangle$.

The *residue class* of a point z with respect to a lattice Ω is the set of values

$$(2.2) \quad z + \Omega = \{z + \omega : \omega \in \Omega\},$$

and a *fundamental region* is a simply connected region in $\mathbb{C}$ that contains a representative of each residue class of each point. The most commonly used example of a fundamental region is a *period parallelogram*, which is defined as

$$(2.3) \quad P_\Omega = \{t\omega_1 + s\omega_2 : 0 \leq t, s < 1\}.$$

The generators of a lattice are by no means unique.

Proposition 2.2 ([8]). *Two lattices Ω and Ω' are equal if and only if*

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Remark 2.3. Given a lattice Ω , one must choose generators ω_1 and ω_2 before a period parallelogram can be defined.

We say that two lattices are *similar* if $\Omega' = \lambda\Omega$ for some $\lambda \in \mathbb{C}^*$.

Theorem 2.4 ([8]). *Every lattice is similar to a lattice of the form $\langle 1, \tau \rangle$ for some unique $\tau \in T$, where*

$$(2.4) \quad T = \{z : \operatorname{Im}(z) > 0, -1/2 \leq \operatorname{Re}(z) < 1/2, |z| \geq 1 \text{ (} |z| > 1 \text{ when } \operatorname{Re}(z) > 0)\}.$$

To specific values of τ we attach names of similarity classes called *shapes*.

Definition 2.5. The following lattice shapes/collections of shape are defined as follows:

- (1) A lattice Ω is *square* if $\tau(\Omega) = i$.
- (2) A lattice Ω is *triangular* if $\tau(\Omega) = e^{2\pi i/3}$.
- (3) A lattice Ω is *rectangular* if $\tau(\Omega)$ is pure imaginary.
- (4) A lattice Ω is *rhombic* if $|\tau(\Omega)| = 1$.

We now define elliptic functions on a given lattice.

Definition 2.6. A meromorphic function, $f : \mathbb{C} \rightarrow \mathbb{C}_\infty$, is elliptic if there is a lattice Ω so that $f(z + \omega) = f(z)$ for all $\omega \in \Omega$.

The elliptic functions of Weierstrass are defined as the following convergent series:

$$(2.5) \quad \wp_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega^*} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

where $\Omega^* = \Omega \setminus \{0\}$.

By differentiating the series defining $\wp_\Omega$ we can obtain a series for $\wp'_\Omega$:

$$(2.6) \quad \wp'_\Omega(z) = -2 \sum_{\omega \in \Omega^*} \frac{1}{(z - \omega)^3}.$$

When scaling the lattice we have the following relations, which we refer to as the *homogeneity equations*. For $k \in \mathbb{C}^*$ we have

$$(2.7) \quad \wp_{k\Omega}(kz) = \frac{1}{k^2} \wp_\Omega(z)$$

or, similarly,

$$(2.8) \quad \wp_{k\Omega}(z) = \frac{1}{k^2} \wp_\Omega\left(\frac{z}{k}\right).$$

These relations follow directly from the definition of $\wp_\Omega$ and give a very useful tool for analyzing the families $\wp_{\lambda\Omega}$ where $\lambda \in \mathbb{C}^*$. The Weierstrass elliptic function $\wp_\Omega$ satisfies the following differential equation [8]:

$$(2.9) \quad (\wp'_\Omega(z))^2 = 4\wp_\Omega(z)^3 - g_2(\Omega)\wp_\Omega(z) - g_3(\Omega),$$

where g_2 and g_3 are invariants of the lattice. The invariants g_2 and g_3 can be written as follows:

$$(2.10) \quad g_2(\Omega) = 60 \sum_{\omega \in \Omega^*} \frac{1}{\omega^4}; \quad g_3(\Omega) = 140 \sum_{\omega \in \Omega^*} \frac{1}{\omega^6},$$

and so these invariants also have homogeneity properties:

$$(2.11) \quad g_2(k\Omega) = \frac{1}{k^4} g_2(\Omega); \quad g_3(k\Omega) = \frac{1}{k^6} g_3(\Omega).$$

These invariants also have the property that given complex numbers g_2 and g_3 with $g_2^3 - 27g_3^2 \neq 0$ there exists a unique lattice Ω so that $g_2 = g_2(\Omega)$ and $g_3 = g_3(\Omega)$ [8].

Critical values of $\wp_\Omega$ correspond to zeros of $4\wp_\Omega(z)^3 - g_2\wp_\Omega(z) - g_3$. Using the Fundamental Theorem of Algebra we can write

$$(2.12) \quad (\wp'_\Omega(z))^2 = 4(\wp_\Omega(z) - e_1)(\wp_\Omega(z) - e_2)(\wp_\Omega(z) - e_3).$$

It is known that the roots $\{e_1, e_2, e_3\}$ are distinct [8] and are $e_1 = \wp_\Omega(\omega_1/2)$, $e_2 = \wp_\Omega(\omega_2/2)$, and $e_3 = \wp_\Omega((\omega_1 + \omega_2)/2)$.

The critical values satisfy important algebraic relations obtained by equating like terms in the factorization and the original expression $(\wp'_\Omega(z))^2 = 4\wp_\Omega(z)^3 - g_2\wp_\Omega(z) - g_3$. We list the ones most relevant:

$$(2.13) \quad e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_1e_3 + e_2e_3 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}.$$

Because of our focus on square lattices we develop in detail the formulas for the critical values in terms of the lattice.

Proposition 2.7 ([8]). *Suppose that Ω is a square lattice. Then $e_3 = 0$ and $e_2 = -e_1$. Furthermore, $g_2 = 4e_1^2$ and $g_3 = 0$.*

Certain lattices have symmetry about the real axis. These lattices are called *real lattices*.

Proposition 2.8. *If Ω is a lattice so that $\overline{\Omega} = \Omega$, where $\overline{\Omega} = \{\overline{\omega} : \omega \in \Omega\}$, then*

$$\wp_{\Omega} : \mathbb{R} \setminus \Omega \rightarrow \mathbb{R}$$

and

$$\wp'_{\Omega} : \mathbb{R} \setminus \Omega \rightarrow \mathbb{R}.$$

Proposition 2.9. *Given a lattice Ω , $\wp_{\Omega}$ is even.*

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$. Then we have the following mapping property.

Proposition 2.10. *If $\overline{\Omega} = \Omega$, then*

$$\wp_{\Omega} : \mathbb{R} \setminus \Omega \rightarrow \overline{\mathbb{R}^+}.$$

An important benchmark quantity is defined as the following:

Definition 2.11. We define $\gamma > 0$ as $\gamma^2 = \wp_{\langle 1, i \rangle}(1/2)$.

This constant is referred to as the *lemniscate constant*; $\gamma \approx 2.62206\dots$, and an exact formula for γ is the following:

$$\gamma = \frac{1}{4} \sqrt{\frac{2}{\pi}} \left(\Gamma\left(\frac{1}{4}\right) \right)^2,$$

where $\Gamma(z)$ is the classical gamma function. Both the decimal approximation and the exact formula can be found in [1].

Using the homogeneity equations and lemniscate constant we can describe the critical values of $\wp_{\Omega}$ when Ω is square. Using (2.7) and the lemniscate constant γ , we have

$$(2.14) \quad e_1 = \wp_{\langle \lambda, \lambda i \rangle} \left(\frac{\lambda}{2} \right) = \frac{1}{\lambda^2} \wp_{\langle 1, i \rangle}(1/2) = \frac{\gamma^2}{\lambda^2}.$$

This gives the following.

Proposition 2.12. *Let $\lambda \in \mathbb{C}^*$. The critical values of $\wp_{\langle \lambda, \lambda i \rangle}$ are*

$$(2.15) \quad \{-e_1, 0, e_1\} = \left\{ -\frac{\gamma^2}{\lambda^2}, 0, \frac{\gamma^2}{\lambda^2} \right\}.$$

3. THE FATOU AND JULIA SETS

The notation for composition in this paper is $f^{\circ n}(z) = f^n(z)$ and does not mean $(f(z))^n$. Since a transcendental meromorphic function has an essential singularity at ∞ that cannot be extended, we need to carefully account for poles and points that arrive at poles in their forward orbits. These points have finite forward orbits that terminate at infinity.

Definition 3.1. The *Fatou set* $F(f)$ of a meromorphic function $f : \mathbb{C} \rightarrow \mathbb{C}_{\infty}$ is defined to be

$$F(f) = \{z : \{f^n\}_{n \in \mathbb{N}} \text{ are defined and form a normal family at } z\}.$$

Definition 3.2. The *Julia set* $J(f)$ of a meromorphic function f is defined to be $J(f) = \mathbb{C}_\infty \setminus F(f)$.

The Fatou and Julia sets are forward and backward invariant [3, 4]. Furthermore, the Fatou set is open, since it is a union of open sets, and so the Julia set is closed.

Theorem 3.3 ([3, Montel's Theorem]). *Suppose that a family of functions $\mathcal{F}$ mapping $U \subset \mathbb{C}$ to $\mathbb{C}_\infty$ has the property that $f(U) \subset \mathbb{C}_\infty \setminus \{a, b, c\}$, where a, b and c are distinct points in $\mathbb{C}_\infty$. Then $\mathcal{F}$ is a normal family on U .*

Using Montel's Theorem, we have the following fact. We say a set X is *completely invariant* if $z \in X$ implies $f(z) \in X$ (unless $f(z)$ is undefined) and $f(w) = z$ implies $w \in X$.

Proposition 3.4 ([4]). *The Julia set is the smallest completely invariant closed set containing at least three points.*

Since all lattice points are in the Julia set, the Julia set of an elliptic function is the closure of all the pre-images of infinity; i.e. $J(f) = \overline{O^-(\infty)}$.

The classification of Fatou components is an important tool for understanding the dynamics of both rational and transcendental meromorphic functions. This classification theorem is mostly due to Cremer [6] and Fatou [9, 10] and was first presented in this form by Baker, Kotus, and Lü [2].

Theorem 3.5 ([4]). *Let f be a meromorphic function on $\mathbb{C}$ and let U be a periodic component of $F(f)$ of period p . Then we have exactly one of the following properties:*

- (1) *U contains an attracting periodic point z_0 of period p . Then $f^{np}(z) \rightarrow z_0$ for $z \in U$ as $n \rightarrow \infty$, and U is called the immediate basin of attraction for z_0 .*
- (2) *∂U contains a periodic point z_0 of period p and $f^{np}(z) \rightarrow z_0$ for $z \in U$ as $n \rightarrow \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. (For $z_0 = \infty$ we have $(g^p)'(0) = 1$, where $g(z) = 1/(f(1/z))$.) In this case, U is a Leau domain (also called a parabolic domain).*
- (3) *There exists an analytic homeomorphism $\phi : U \rightarrow \mathbb{D}$, where $\mathbb{D}$ is the unit disk, such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a Siegel disk.*
- (4) *There exists an analytic homeomorphism $\phi : U \rightarrow \mathbb{A}$ where $\mathbb{A}$ is an annulus, and $\mathbb{A} = \{z : 1 < |z| < r\}$ for some $r > 1$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a Herman ring.*
- (5) *There exists $z_0 \in \partial U$ such that $f^{np}(z) \rightarrow z_0$ for $z \in U$ as $n \rightarrow \infty$, but $f^p(z_0)$ is not defined. In this case, U is called a Baker domain.*

The following result concerning elliptic functions was proved in [13].

Theorem 3.6. *For any lattice Ω , $\wp_\Omega$ has no cycle of Herman rings.*

To rule out the existence of wandering domains and Baker domains we introduce the following family of functions.

Definition 3.7. A function is of *class S* if it has finitely many critical values and finitely many asymptotic values.

Theorem 3.8 ([4]). *A meromorphic function of class S has no wandering domains or Baker domains.*

It is the case that elliptic functions are of class S and therefore have no wandering domains or Baker domains. The following theorem follows from the fact that if a lattice Ω is square, then $\wp_\Omega$ has one free critical orbit.

Theorem 3.9 ([13]). *Let Ω be a square lattice. Then exactly one of the following must occur:*

- (1) $F(\wp_\Omega)$ is empty and $J(\wp_\Omega) = \mathbb{C}_\infty$;
- (2) $F(\wp_\Omega)$ is the basin of an attracting cycle;
- (3) $F(\wp_\Omega)$ is the basin of a parabolic cycle;
- (4) or $F(\wp_\Omega)$ consists of a cycle of Siegel disks and their pre-images.

4. THE MAIN THEOREM

Theorem 4.1. *For each $\lambda \in \mathbb{C}^*$ the Julia set of $\wp_\lambda = \wp_{\langle \lambda, \lambda i \rangle}$ is connected.*

The following theorem, adapted from [16, Theorem 3.1], is proved in [14]. We present a similar proof here. From here on we will use the notation $\wp_\lambda = \wp_{\langle \lambda, \lambda i \rangle}$.

Theorem 4.2 ([14]). *If every Fatou component of $F(\wp_\Omega)$ contains 0 or 1 critical values, then $J(\wp_\Omega)$ is connected.*

Definition 4.3. Fix $\lambda \in \mathbb{C}^*$. Define $L_1 = \{t\lambda^{-2} : t \in \mathbb{R}^+\}$ and $L_2 = -L_1$. Furthermore, define $S_1 = \{t\lambda : t \in \mathbb{R} \setminus \mathbb{Z}\}$. Set $S_2 = iS_1$.

Proposition 4.4. *For each $\lambda \in \mathbb{C}^*$ we have that*

$$\wp_\lambda(S_1) \subset L_1$$

and

$$\wp_\lambda(S_2) \subset L_2.$$

Proof. If $z \in S_1$, then $z = t\lambda$ for some $t \in \mathbb{R} \setminus \mathbb{Z}$. By the homogeneity equations (2.7), we have that

$$\wp_\lambda(t\lambda) = \lambda^{-2}\wp_1(t).$$

An elementary calculus exercise shows that $\wp_1(t) \geq \gamma^2$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$ giving $\lambda^{-2}\wp_1(t) \in L_1$.

If $z \in S_2$, then $z = itk$ for $t \in \mathbb{R} \setminus \mathbb{Z}$. We have that $i\langle \lambda, \lambda i \rangle = \langle \lambda, \lambda i \rangle$. Using the homogeneity equations a second time, we have

$$\wp_\lambda(it\lambda) = (i\lambda)^{-2}\wp_1(t) = -\lambda^{-2}\wp_1(t).$$

Since $\wp_1(t) > 0$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$, we have that $-\lambda^{-2}\wp_1(t) \in L_2$. □

We observe that $e_1 \in L_1$ and $e_2 \in L_2$ by equation (2.14).

Lemma 4.5. *Fix $\lambda \in \mathbb{C}^*$. Let $p \neq 0$. Then*

$$K(p) = \max\left\{\inf_{z \in L_1} d(z, p), \inf_{z \in L_2} d(z, p)\right\} > 0,$$

where d is the Euclidean metric on $\mathbb{C}$.

The distance $K(p)$ depends on λ , but we write $K(\lambda, p) = K(p)$ as in Lemma 4.5.

Lemma 4.6. *Let C be a curve connecting $z_1 \in L_1$ and $z_2 \in L_2$ in $\mathbb{C}$. Let H be a line passing through the origin. Then $C \cap H \neq \emptyset$.*

Proof. Parameterize C on $[0, 1]$ by $\gamma : [0, 1] \rightarrow C$. If C contains the origin, then we are done. Suppose that C does not contain the origin. Let $\gamma(t) = r(t)e^{i\theta(t)}$. Let $\theta_0 = \theta(0)$ and $\theta_1 = \theta(1) = \theta_0 + (2k+1)\pi$, where $k \in \mathbb{Z}$. If $w \in H$, then $\arg(w) = \theta_H$ or $\arg(w) = \theta_H - \pi$ for some $\theta_H \in (0, \pi]$, and H contains all such w 's. We have that θ is a continuous function of t , so the Intermediate Value Theorem gives that $\theta(t_0) = \theta_H$ or $\theta(t_0) = \theta_H - \pi$, for some $t_0 \in [0, 1]$. Thus, there is $t_0 \in [0, 1]$ such that $\gamma(t_0) = r(t_0)e^{i\theta(t_0)} \in H$. $\square$

We now prove the connectivity of $J(\wp_\lambda)$ in the case when $\wp_\lambda$ has an attracting or parabolic cycle. If $\wp_\lambda$ has an attracting or parabolic cycle $\{p_0, \dots, p_{l-1}\}$, then $\wp_\lambda^{ln} \rightarrow p_0$ locally uniformly on the immediate basin of p_0 [4]. Furthermore, the immediate basin of an attracting or parabolic cycle contains a critical value [4].

Theorem 4.7. *Suppose that $\wp_\lambda$ has an attracting or parabolic cycle $\{p_0, \dots, p_{l-1}\}$. Then $J(\wp_\lambda)$ is connected.*

Proof. It is sufficient to show that each Fatou component contains at most one critical value [14]. The critical value 0 is not in the Fatou set, since it is a pole. We proceed by contradiction.

Suppose that a Fatou component U contains both e_1 and $e_2 = -e_1$. It is evident that U must be the immediate basin of the cycle, since the immediate basin must contain a critical value [3, 4] and since e_1 and $-e_1$ are both in U . Each Fatou component is path connected, since it is open and connected. Let C be a curve connecting e_1 and $-e_1$ in U .

All iterates of $\wp_\lambda$ are defined on C since all pre-poles lie in the Julia set. Since C is a compact subset of the immediate basin, $\{\wp_\lambda^{ln}\}_{n \in \mathbb{N}}$ converges uniformly on C to the constant function p_0 in the Euclidean metric [3, 4]. This means that for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $\sup_{z \in C} |\wp_\lambda^{ln}(z) - p_0| < \epsilon$ for $n > N$.

We claim that for each $m \in \mathbb{N}$, $\wp_\lambda^m(C)$ contains a point on each of L_1, L_2, S_1 , and S_2 and is connected. We proceed by induction. We first establish these properties for the $m = 0$ case, C .

By construction C contains a point on L_1 and on L_2 . We establish that $C \cap S_i \neq \emptyset$ for $i = 1, 2$. Fix i . Now $\overline{S_i}$ is a line passing through the origin, so by Lemma 4.6 we have that $C \cap \overline{S_i} \neq \emptyset$. Since C does not contain any poles, we have that $C \cap S_i \neq \emptyset$.

For the induction hypothesis, we assume that $\wp_\lambda^m(C)$ contains a point in L_1, L_2, S_1 , and S_2 and is connected. We can immediately establish that $\wp_\lambda^{m+1}(C)$ is connected, since all iterates are defined on C and $\wp_\lambda$ is continuous off its pre-poles. Furthermore, $\wp_\lambda^{m+1}(C)$ contains a point on L_1 and a point on L_2 by Proposition 4.4. Lastly, we can establish that $\wp_\lambda^{m+1}(C)$ contains a point on S_1 and S_2 by Lemma 4.6.

If $p_0 \notin L_2$, choose a_n in $\wp_\lambda^{ln}(C) \cap L_2$ for each $n \in \mathbb{N}$ (if $p_0 \in L_2$, then choose a_n in L_1). We have that

$$\sup_{z \in C} |\wp_\lambda^{ln}(z) - p_0| \geq |a_n - p_0| > K(p_0) > 0$$

for all $n \in \mathbb{N}$ by Lemma 4.5. This contradicts the uniform convergence of $\{\wp_\lambda^{ln}\}_{n \in \mathbb{N}}$ on C to the constant function p_0 . Thus every Fatou component contains at most one critical value, and so the Julia set is connected. $\square$

We now prove the main result for all square lattices.

Proof of Theorem 4.1. If $F(\wp)$ is empty, then $J(\wp) = \mathbb{C}_\infty$ and so is connected. (For example, this occurs when the lattice is a so-called “rhombic square” lattice [11]. It also occurs when the critical orbit lands on a pole. There are other examples as well; n.b. [12]). If $\wp$ has an attracting cycle or a parabolic cycle, we have that $J(\wp)$ is connected by Theorem 4.7. If the Fatou set consists of pre-periodic Siegel disks, then the Julia set is connected [4, 14]. This exhausts all possible Fatou components. Thus $J(\wp_\lambda)$ is connected for all $\lambda \neq 0$. $\square$

5. RELATED RATIONAL MAPS

In this section we begin exploring connections between higher degree rational maps and Weierstrass elliptic functions on square lattices. The main result of this chapter is Theorem 5.6. This section is motivated by work on singularly perturbed maps by Bob Devaney et al. In [7] B. Devaney, D. Look, and D. Uminsky describe an escape trichotomy similar to the escape dichotomy for the quadratic map. In particular, they prove that if the free critical orbit remains bounded, then the Julia set is connected. Using the techniques developed in the previous section we present a proof of this result for a conformally conjugate family of rational maps.

Proposition 5.1 ([1]). *For the $\{g_2, g_3\}$ invariant parameterization $\wp_{\{g_2, g_3\}}$ of the Weierstrass elliptic functions we have the following Laurent series centered at zero:*

$$(5.1) \quad \wp_{\{g_2, g_3\}}(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} c_k z^{2k-2},$$

where $c_2 = g_2/20$, $c_3 = g_3/28$, and

$$(5.2) \quad c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}$$

for $k > 3$. This series converges on $0 < |z| < R$, where $R = \min_{\omega \in \Omega^*} |\omega|$.

Our family of interest is the similarity class of square lattices. Thus, we have that $g_3 = 0$ and $g_2 = 4\gamma^4/\lambda^4$, where λ generates the lattice $\langle \lambda, \lambda i \rangle$. This gives the following.

Proposition 5.2 ([1]). *For $\wp_\lambda$ we have the following Laurent series centered at zero:*

$$(5.3) \quad \wp_\lambda(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} c_{2k} z^{4k-2} = \frac{1}{z^2} + \frac{\gamma^4}{5\lambda^4} z^2 + \cdots,$$

where $c_2 = \gamma^4 5\lambda^4$ and

$$c_{2k} = \frac{3}{(4k+1)(2k-3)} \sum_{m=1}^{k-1} c_{2m} c_{2(k-m)}$$

for $k > 1$. This series converges on $0 < |z| < |\lambda|$.

By truncating the infinite Laurent series we obtain parameterized families of rational approximations to $\wp_\lambda$. We define the following family:

$$(5.4) \quad P_\lambda(z) = \frac{1}{z^2} + \frac{\gamma^4}{5\lambda^4} z^2,$$

since

$$(5.5) \quad \wp_\lambda(z) = P_\lambda(z) + O\left(\frac{z^6}{\lambda^8}\right)$$

from Proposition 5.2.

Just as with $\wp_\lambda$, P_λ satisfies a homogeneity equation:

$$(5.6) \quad P_\lambda(\lambda z) = \frac{1}{\lambda^2} P_\lambda(z).$$

This family also has similar mapping properties.

Definition 5.3. Fix $\lambda \in \mathbb{C}^*$. Define

$$L_1 = \{t\lambda^{-2} : t \in \mathbb{R}^+\}$$

and

$$L_2 = -L_1.$$

Furthermore, define

$$\tilde{S}_1 = \{t\lambda : t \in \mathbb{R} \setminus \{0\}\}.$$

Set

$$\tilde{S}_2 = i\tilde{S}_1.$$

Proposition 5.4. For $\lambda \in \mathbb{C}^*$ we have that $P_\lambda(\tilde{S}_1) \subset L_1$ and $P_\lambda(\tilde{S}_2) \subset L_2$.

Proof. If $z \in \tilde{S}_1$, then $z = t\lambda$ for some $t \in \mathbb{R} \setminus \{0\}$. By the homogeneity equations (5.6), we have that

$$P_\lambda(t\lambda) = \lambda^{-2} P_1(t).$$

Furthermore, $P_1(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$, giving the fact that $\lambda^{-2} P_1(t) \in L_1$.

If $z \in \tilde{S}_2$, then $z = itk$ for $t \in \mathbb{R} \setminus \{0\}$. We have that $P_{i\lambda} = P_\lambda$. Using the homogeneity equations a second time, we have

$$P_\lambda(it\lambda) = P_{i\lambda}(it\lambda) = (i\lambda)^{-2} P_1(t) = -\lambda^{-2} P_1(t).$$

Since $P_1(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$, we have that $-\lambda^{-2} P_1(t) \in L_2$. $\square$

The critical points of P_λ are $\{z : z^4 = \frac{5\lambda^4}{\gamma^4}\}$ and the point at infinity. The finite critical points can be written as $i^k \frac{\sqrt[4]{5}\lambda}{\gamma}$, where $k = 0, 1, 2$, or 3 . The point at infinity is a super-attracting fixed point. This leaves one free critical orbit, since after two iterations all critical points (except infinity) land on the same point. Thus P_λ can have at most one finite non-repelling periodic cycle [3].

Theorem 5.5. Suppose that P_λ has a finite attracting or parabolic cycle. Then each Fatou component contains at most one critical value.

Proof. This proof proceeds identically to the proof of Theorem 4.7. $\square$

We present the following theorem.

Theorem 5.6. Suppose that P_λ has an attracting cycle, parabolic cycle, cycle of Siegel disks, or a strictly pre-periodic critical value. Then $J(P_\lambda)$ is connected.

Proof. If P_λ has an attracting cycle or parabolic cycle, then Theorem 4.7 gives that each Fatou component contains at most one critical value. If P_λ has a cycle of Siegel disks or a strictly pre-periodic critical value, then the critical values are contained in the Julia set, and so again, each Fatou component contains at most one critical value. By applying [16, Theorem 3.1] we have the result. $\square$

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