THE IMAGE AND KERNEL OF ATKIN'S U_p OPERATOR MODULO p

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ABSTRACT. We compute the image of Atkin's U_p operator on reduced modular forms. If $A \geq 1$ and $2 \leq B \leq p+1$, then $U_p : \widetilde{M}_{Ap+B}(\Gamma_1(N)) \rightarrow \widetilde{M}_{A+B}(\Gamma_1(N))$ is a surjection. In particular, the dimension of $\ker U_p$ is known for weights at least $p+2$.

1. INTRODUCTION

Modular forms modulo p were introduced by Serre [8] and Swinnerton-Dyer [9] (level $N = 1$) and Katz [5] (general level). Fix a level $N \neq 2, 3$, and a prime $p \geq 5$ not dividing N . Set $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$. Suppose that $f = \sum a(n)q^n \in M_k(\Gamma)$ is a weight k , holomorphic modular form on Γ with p -integral coefficients. Let $\widetilde{M}_k(\Gamma)$ denote the space of reductions of forms modulo p . Atkin's U_p operator acts on q -expansions via

$$U_p f = \sum a(pn)q^n.$$

Serre [8] showed that U_p is contractionary: if $k \geq p+2$, then U_p sends $\widetilde{M}_k(1)$ to $\widetilde{M}_{k'}(1)$, where $k' < k$. We sharpen this result by determining k' and generalizing to other levels.

Theorem 1. *Let $p \geq 5$ be prime and $p \nmid N \neq 2, 3$. Fix Γ to be either $\Gamma_1(N)$ or $\Gamma_0(N)$. If $2 \leq B \leq p+1$ and $A \geq 1$, then*

$$U_p : \widetilde{M}_{Ap+B}(\Gamma) \twoheadrightarrow \widetilde{M}_{A+B}(\Gamma)$$

is a surjection.

In particular, the dimension of $\ker U_p$ is now known for weights $Ap+B \geq p+2$; namely,

$$(1.1) \quad \dim \ker U_p = \dim \widetilde{M}_{Ap+B}(\Gamma) - \dim \widetilde{M}_{A+B}(\Gamma).$$

This is in contrast to the low-weight situation in which the action of U_p is still mysterious. Usually, $\ker U_p$ is trivial for small weights, but there are sometimes

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non-ordinary forms (also called *supersingular* forms) of weight less than $p+1$ which are annihilated by U_p . To prepare for a couple of examples, set

$$X_k := \ker \left(U_p|_{\widetilde{M}_k(\Gamma)} \right).$$

Take $\Gamma = \Gamma_1(1)$ and consider the eigenform

$$\Delta = \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_1(1)).$$

In this case, $\dim X_{12} = 1$ if and only if $\tau(p) \equiv 0 \pmod{p}$. The only known primes for which this is true are $p = 2, 3, 5, 7, 2411$, and 7758337633 [7]. It is unknown if there are infinitely many supersingular primes for which $\tau(p) \equiv 0 \pmod{p}$.

Continuing with $\Gamma_1(1)$, consider the prime $p = 59$. The sequence of dimensions $\{\dim X_{2k}\}_{k \geq 2}$ begins with a sea of zeros separating two small blips:

$$\dim X_{2k} = \begin{cases} 0 & \text{when } 4 \leq 2k \leq 60 \text{ and } 2k \neq 16, 46, \\ 1 & \text{when } 2k = 16, 46. \end{cases}$$

However, starting from $2k = 62$, the sequence is totally predictable and regular. In the range $62 \leq 2k \leq 118$ we get

$$4, 5, 5, 5, 4, 6, 4, 5, 5, 5, 4, 6, 4, 5, 5, 5, 4, 6, \dots,$$

and in the range $120 \leq 2k \leq 178$ we get

$$10, 9, 10, 10, 9, 10, 10, 9, 10, 10, 9, 10, \dots,$$

and so on. Of course, the regularity of $\dim X_{2k}$ for $2k \geq 62$ is a consequence of (1.1) and the dimension formulas. The point is that the small glitches at weights 16 and 46 do not produce further complexities later on.

For another example, take $\Gamma = \Gamma_1(4)$. Then for $A \geq 1$ and $2 \leq B \leq p+1$, by (1.1) we have

$$\dim X_{Ap+B} = A \left(\frac{p-1}{2} \right).$$

For $A = 0$ and $2 \leq B \leq p+1$, although $\dim X_B$ is zero for most primes p , there are occasional quirks. For example, the primes p with $5 \leq p \leq 17$ exhibit the typical behaviour: for all $0 \leq k \leq p+1$, we have $\dim X_k = 0$. However when $p = 19$ we have

$$\dim X_k = \begin{cases} 0 & \text{when } 0 \leq k \leq p+1 \text{ and } k \neq 6, 16, \\ 1 & \text{when } k = 6, 16. \end{cases}$$

The space X_6 is generated by the unique normalized weight 6 cusp form of level 4,

$$\begin{aligned} f &= q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} - 418q^{13} - 648q^{15} + 594q^{17} \\ &\quad + 836q^{19} + 1056q^{21} + \dots \\ &\equiv q + 7q^3 + 16q^5 + 7q^7 + 15q^9 + 8q^{11} + 17q^{15} + 5q^{17} + 11q^{21} + \dots \pmod{19}. \end{aligned}$$

Writing $f = \sum a(n)q^n$, the space X_{16} is generated by a form $g \equiv \sum n^{14}a(n)q^n \pmod{19}$. In the terminology of Section 2, f and g are low points of a Tate cycle.

The point of the preceding examples is that $\dim X_k$ is erratic for small k . However, Theorem 1 shows that this unpredictability disappears for $k \geq p+2$.

Section 2 recalls background on the structure of $\widetilde{M}_k(\Gamma)$, various operators, and the Tate cycle. Section 3 contains the key to the proof: a decomposition of $\widetilde{M}_k(\Gamma)$ into many subspaces with clear relationships to $\ker U_p$. Finally, Section 4 ties everything together and proves Theorem 1.

2. PRELIMINARIES

We recall the notion of reduced modular forms due to Serre [8], Swinnerton-Dyer [9], and Katz [5]. Fix a positive integer $N \neq 2$ or 3 and fix $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$. Let $p \geq 5$ be a prime not dividing N . Set $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} : p \nmid n\}$. Let $M_k(\Gamma) \cap \mathbb{Z}_{(p)}[[q]]$ denote the space of holomorphic, weight $k \in \mathbb{Z}_{\geq 0}$ modular forms on Γ with p -integral coefficients. The coefficients of the forms in this space may be reduced modulo p . Let $M_k = \widetilde{M}_k(\Gamma)$ denote the space of such reductions. Henceforth, all forms and spaces are reduced modulo p and equality of forms indicates congruence modulo p . The Kummer congruences imply that the weight $p-1$, normalized Eisenstein series $E_{p-1} = 1$. Hence multiplication by E_{p-1} gives an injection

$$E_{p-1} : M_k \hookrightarrow M_{k+p-1}.$$

This leads to the definition of a filtration on M_k in terms of the “least weight” of a form. For $f \in M_k$, define

$$\omega(f) := \inf \{k' : f \in M_{k'}\}.$$

Define the operators

$$\begin{aligned} V_p : M_k &\hookrightarrow M_{pk}, \\ U_p : M_k &\rightarrow M_k, \\ \Theta : M_k &\rightarrow M_{k+p+1} \end{aligned}$$

via

$$\begin{aligned} V_p \left(\sum a(n)q^n \right) &= \sum a(n)q^{pn}, \\ U_p \left(\sum a(n)q^n \right) &= \sum a(pn)q^n, \\ \Theta \left(\sum a(n)q^n \right) &= \sum a(n)nq^n. \end{aligned}$$

The existence of these operators is detailed in [5, 8, 9]. It will be convenient to have some notation for the kernels:

$$(2.1) \quad X_k := \ker (U_p : M_k \rightarrow M_k),$$

$$(2.2) \quad K_k := \ker (\Theta : M_k \rightarrow M_{k+p+1}).$$

Observe that

$$\begin{aligned} V_p U_p &= 1 - \Theta^{p-1}, \\ X_k \cap K_k &= 0, \\ V_p(M_k) &\subseteq K_{pk}, \\ (2.3) \quad \Theta(M_k) &\subseteq X_{k+p+1}. \end{aligned}$$

Swinnerton-Dyer [9, Lemma 5] and Katz [6] prove the following lemma. (See also Gross [2, Proposition 4.10] for a statement of the following lemma in terms of $\Gamma_1(N)$. The fact that the $\langle d \rangle$ operators commute with $T_p = U_p$ implies that the proposition also holds for $\Gamma_0(N)$.)

Lemma 2. *If $\omega(f) \not\equiv 0 \pmod{p}$, then $\omega(\Theta f) = \omega(f) + p + 1$. If $\omega(f) \equiv 0 \pmod{p}$, then either $\Theta f = 0$ or $\omega(\Theta f) = \omega(f) + p + 1 - s(p - 1)$ for some $s \geq 1$.*

Fermat's little theorem implies that for any $f \in M_k$, we have $\Theta^p f = \Theta f$. Thus repeated iterations of Θ yield the so-called *Tate cycle*

$$\Theta f, \Theta^2 f, \dots, \Theta^{p-1} f.$$

The situation $\Theta^{p-1} f = f$, in which f is in its Tate cycle, is equivalent to $f \in X_k$. By Lemma 2, the filtration usually rises by $p + 1$ at each step of the Tate cycle. Occasionally, the filtration will fall. If i is such that $\omega(\Theta^{i+1} f) < \omega(\Theta^i f) + p + 1$, then call $\Theta^i f$ a *high point* and $\Theta^{i+1} f$ a *low point* of the Tate cycle. An analysis as in Jochowitz [4, Section 7] or the author's [1, Lemma 3.1] gives the following characterization of the rise-and-fall pattern of the filtration in the Tate cycle.

Lemma 3. *Suppose $A \geq 0$ and $1 \leq B \leq p$. If $f \in X_{Ap+B}$ and $\omega(f) = Ap + B$, then:*

- (1) $\Theta f \neq 0$,
- (2) $B \neq 1$,
- (3) $\omega(\Theta f) \neq Ap + B + 2$,
- (4) *either there is one low point in the Tate cycle or there are two low points in the Tate cycle,*
- (5) *the Tate cycle has a single low point if and only if some term in the cycle (which will be the low point) has filtration congruent to 2 modulo p ,*
- (6) *if f is a low point of its Tate cycle and if there are two low points, then the first high point has filtration*

$$\omega(\Theta^{p-B} f) = (A - B + p + 1)p,$$

the other low point has filtration

$$\omega(\Theta^{p-B+1} f) = Ap + (p + 3 - B),$$

and the last high point has filtration

$$\omega(\Theta^{p-2} f) = (A + B - 2)p.$$

3. MAIN DECOMPOSITION

We begin by defining key subspaces of M_k .

Definition 4. Recall (2.1) and consider the following cases:

- (1) For $0 \leq k \leq 2p$, pick once and for all complementary subspaces Y_k and W_k so that

$$M_k = X_k \oplus W_k = Y_k \oplus E_{p-1} M_{k-p+1}.$$

- (2) For $k \geq 2p + 1$ and $k \equiv 1 \pmod{p}$, pick once and for all a complementary subspace Y_k so that

$$M_k = Y_k \oplus E_{p-1} M_{k-p+1}.$$

- (3) For $k \geq 2p + 1$ and $k \not\equiv 1 \pmod{p}$, define

$$Y_k := \Theta(Y_{k-p-1}).$$

The purpose of the Y_k is elucidated by the following lemma.

Lemma 5. *Let $k \geq 2p + 1$. Then $M_k = Y_k \oplus E_{p-1}M_{k-p+1}$, where*

$$Y_k \cap X_k = 0 \iff k \equiv 1 \pmod{p},$$

$$Y_k \subseteq X_k \iff k \not\equiv 1 \pmod{p}.$$

In particular, a nonzero $f \in Y_k$ has filtration $\omega(f) = k$.

Proof. When $k \equiv 1 \pmod{p}$ the conclusion follows from Definition 4 and Lemma 3 (2). On the other hand, when $k \not\equiv 1 \pmod{p}$, by (2.3) and Definition 4 we have $Y_k \subseteq X_k$. If $k \equiv 2 \pmod{p}$, then by Definition 4 we have

$$(3.1) \quad M_{k-p-1} = Y_{k-p-1} \oplus E_{p-1}M_{k-2p}.$$

If $k \not\equiv 2 \pmod{p}$, then take (3.1) as an inductive hypothesis. Consider the diagram below:

$$\begin{array}{ccc} M_{k-2p} & \xrightarrow{E_{p-1}} & M_{k-p-1} = Y_{k-p-1} \oplus E_{p-1}M_{k-2p} \\ \Theta \downarrow & & \downarrow \Theta \\ M_{k-p+1} & \xrightarrow{E_{p-1}} & M_k \end{array}$$

Suppose $0 \neq f \in Y_{k-p-1}$. Since $\omega(f) = k-p-1 \not\equiv 0 \pmod{p}$, Lemma 2 implies that $\omega(\Theta f) = k$. Thus $\Theta f \notin E_{p-1}M_{k-p+1}$. We conclude that $E_{p-1}M_{k-p+1} \cap \Theta Y_{k-p-1} = 0$, and hence that $E_{p-1}M_{k-p+1} \oplus \Theta Y_{k-p-1} \subseteq M_k$. Since both $\Theta|_{Y_{k-p-1}}$ and E_{p-1} are injective, we have

$$\begin{aligned} \dim(E_{p-1}M_{k-p+1} \oplus \Theta Y_{k-p-1}) &= \dim M_{k-p+1} + \dim Y_{k-p-1} \\ &= \dim M_{k-p+1} + (\dim M_{k-p-1} - \dim M_{k-2p}) \\ &= \dim M_k, \end{aligned}$$

where the last equality follows from a calculation (see, for example, [3, Formula 2.6]) with dimension formulas. $\square$

For any $k \geq p + 2$, write $k = E(p-1) + F$, where $p + 2 \leq F \leq 2p$. Iterating the inclusion $M_k = Y_k \oplus M_{k-p+1}$ yields the decomposition:

$$(3.2) \quad \begin{aligned} M_k &= \bigoplus_{i=0}^{E-1} Y_{k-i(p-1)} \oplus W_F \oplus X_F \\ &= \underbrace{\bigoplus_{\substack{0 \leq i \leq E-1 \\ k-i(p-1) \equiv 1 \pmod{p}}} Y_{k-i(p-1)} \oplus W_F}_{\cap X_k = 0} \oplus \underbrace{\bigoplus_{\substack{0 \leq i \leq E-1 \\ k-i(p-1) \not\equiv 1 \pmod{p}}} Y_{k-i(p-1)} \oplus X_F}_{\subseteq X_k}. \end{aligned}$$

By Lemma 5 and an argument with filtrations, the first collection of subspaces has trivial intersection with X_k , and the second collection of subspaces is contained in X_k . Hence

$$(3.3) \quad X_k = \bigoplus_{\substack{0 \leq i \leq E-1 \\ k-i(p-1) \not\equiv 1 \pmod{p}}} Y_{k-i(p-1)} \oplus X_F.$$

Rather than studying the image of U_p directly, we study the image of the composition $V_p U_p = (1 - \Theta^{p-1})$. Since $X_k = \ker(1 - \Theta^{p-1})$, by (3.3) it suffices to determine the images $(1 - \Theta^{p-1})W_F$ and $(1 - \Theta^{p-1})Y_{A_{p+1}}$ for $A \geq 2$.

Lemma 6. *Let $p+2 \leq F \leq 2p$ and set $D = F-p+1$. Then $(1 - \Theta^{p-1})W_F \subseteq K_{Dp}$ and so $\dim W_F \leq \dim K_{Dp}$.*

Proof. Suppose $0 \neq f \in W_F$. Then $\Theta f \in X_{p+F+1}$. If Θf is not a low point or is zero, then there exists $g \in X_F$ such that $\Theta g = \Theta f$. But then $\Theta^{p-1}f = \Theta^{p-1}g = g$. Hence $f - \Theta^{p-1}f \in K_F \subseteq K_{Dp}$. Therefore, assume that $\Theta f \neq 0$ is a low point of its Tate cycle. Since $\omega(\Theta f) = F+p+1$, $F+2$, or $D+2$, by Lemma 3(6) we deduce that $\omega(\Theta^{p-1}f) = Dp$. Hence $(1 - \Theta^{p-1})f \in K_{Dp}$. $\square$

Lemma 7. *Let $A \geq 2$. Then $(1 - \Theta^{p-1})$ sends Y_{Ap+1} into $K_{(A+p)p}$. Moreover, the map*

$$(1 - \Theta^{p-1}) : Y_{Ap+1} \hookrightarrow \frac{K_{(A+p)p}}{K_{(A+1)p}}$$

is injective. Hence $\dim Y_{Ap+1} \leq \dim K_{(A+p)p} - \dim K_{(A+1)p}$.

Proof. Suppose $0 \neq f \in Y_{Ap+1}$. Then by Lemma 2, $\omega(f - \Theta^{p-1}f) = Ap + p^2$. Hence the map $1 - \Theta^{p-1}$ injects Y_{Ap+1} into K_{Ap+p^2}/K_{Ap+p} . $\square$

4. PROOF OF MAIN THEOREM

Let $A \geq 3$ and write $Ap = E(p-1) + F$, where $p+2 \leq F \leq 2p$. Set $D = F-p+1$. By (3.2), we have

$$M_{Ap} = X_{Ap} \oplus Y_{(A-1)p+1} \oplus \left(\bigoplus_{t=1}^{\lfloor \frac{A-3}{p-1} \rfloor} Y_{(A-1)p+1-t(p-1)} \right) \oplus W_F.$$

Lemma 8. *Let $A \geq 3$ and write $Ap = E(p-1) + F$, where $p+2 \leq F \leq 2p$. Then $(X_{Ap} \oplus Y_{(A-1)p+1}) \cap K_{Ap} = 0$. In particular,*

$$\dim K_{Ap} \leq \dim W_F + \sum_{t=1}^{\lfloor \frac{A-3}{p-1} \rfloor} \dim Y_{(A-1)p+1-t(p-1)}.$$

Proof. Suppose, to the contrary, that there exist $f \in X_{Ap}$, $g \in Y_{(A-1)p+1}$ and $h \in K_{Ap}$ such that $f + g = h \neq 0$. Then $-\infty = \omega(\Theta h) = \omega(\Theta f + \Theta g)$ and so $\omega(\Theta f) = \omega(\Theta g) = Ap + 2$. If $\omega(f) = Ap$, then by Lemma 3(3), $\omega(\Theta f) \neq Ap + 2$. By Lemma 3(2), it is impossible for $\omega(f) = Ap - (p-1) \equiv 1 \pmod{p}$. If $\omega(f) \leq Ap - 2(p-1)$, then $\omega(\Theta f) \leq Ap + 2 - (p-1) < Ap + 2$. We have reached a contradiction in that $\omega(\Theta f)$ cannot equal $\omega(\Theta g)$. $\square$

Lemmas 6, 7, and 8 imply that

$$\begin{aligned} \dim K_{Ap} &\leq \dim W_F + \sum_{t=1}^{\lfloor \frac{A-3}{p-1} \rfloor} \dim Y_{(A-1)p+1-t(p-1)} \\ (4.1) \quad &\leq \dim K_{pD} + \sum_{t=1}^{\lfloor \frac{A-3}{p-1} \rfloor} (\dim K_{(A-(t-1)(p-1))p} - \dim K_{(A-t(p-1))p}) \\ &= \dim K_{Ap}, \end{aligned}$$

where the last line follows because the sum telescopes. Therefore there must be equality throughout (4.1). That is, for $3 \leq A \leq p+1$, we have

$$(4.2) \quad K_{Ap} = (1 - \Theta^{p-1}) W_{A+p-1},$$

and for all $A \geq p+2$, we have

$$(4.3) \quad \begin{aligned} K_{Ap} &= (1 - \Theta^{p-1}) \left(\bigoplus_{t=1}^{\lfloor \frac{A-3}{p-1} \rfloor} Y_{(A-1)p+1-tp(p-1)} \oplus W_F \right) \\ &= (1 - \Theta^{p-1}) Y_{(A-p)p+1} \oplus (1 - \Theta^{p-1}) \left(\bigoplus_{t=2}^{\lfloor \frac{A-3}{p-1} \rfloor} Y_{(A-1)p+1-tp(p-1)} \oplus W_F \right) \\ &= (1 - \Theta^{p-1}) Y_{(A-p)p+1} \oplus K_{(A-p+1)p}. \end{aligned}$$

To finish the proof of Theorem 1, we consider the action of U_p on the space M_{Ap+B} , where $A \geq 1$ and $2 \leq B \leq p+1$, by breaking into four cases:

Case 1. Suppose $B = p+1$ and $A \geq 1$. Write $(A+1)p+1 = E'(p-1) + F'$, where $p+2 \leq F' \leq 2p$. Now the decomposition

$$M_{(A+1)p+1} = X_{(A+1)p+1} \oplus \bigoplus_{t=0}^{\lfloor \frac{A-1}{p-1} \rfloor} Y_{(A+1)p+1-tp(p-1)} \oplus W_{F'}$$

and (4.3) show that

$$(1 - \Theta^{p-1}) M_{(A+1)p+1} = K_{(A+1+p)p}.$$

In particular, since V_p is injective and $1 - \Theta^{p-1} = V_p U_p$ is surjective in the diagram

$$\begin{array}{ccc} M_{(A+1)p+1} & \xrightarrow{1-\Theta^{p-1}} & K_{(A+1+p)p} \\ & \searrow U_p \quad \swarrow V_p & \\ & M_{A+1+p} & \end{array}$$

we deduce that both V_p and U_p are surjections. Hence $V_p(M_{A+1+p}) = K_{(A+1+p)p}$ and $U_p(M_{(A+1)p+1}) = M_{A+1+p}$, as desired.

Case 2. Suppose $2 \leq B \leq p$ and $A = 1$. In accordance with our earlier notation, set $F := p+B$ and $D = F-p+1$. Analogously to Case 1, (4.2) implies that $(1 - \Theta^{p-1}) M_F = K_{Dp}$. Since V_p is injective and $1 - \Theta^{p-1} = V_p U_p$ is surjective in the diagram

$$\begin{array}{ccc} M_F & \xrightarrow{1-\Theta^{p-1}} & K_{Dp} \\ & \searrow U_p \quad \swarrow V_p & \\ & M_D & \end{array}$$

we deduce that both V_p and U_p are surjections. Hence $V_p(M_D) = K_{Dp}$ and $U_p(M_F) = M_D$.

Case 3. Suppose $2 \leq B \leq p$ and $A \geq p - B + 2$. In the decomposition

$$M_{Ap+B} = \underbrace{Y_{Ap+B} \oplus Y_{(A-1)p+B+1} \oplus \cdots \oplus Y_{(A+B-p+1)p}}_{\subseteq X_{Ap+B}} \oplus M_{(A+B-p)p+1},$$

the Y spaces are contained in $X_{Ap+B} = \ker U_p$. Hence $U_p(M_{Ap+B}) = U_p(M_{(A+B-p)p+1}) = M_{A+B}$.

Case 4. Suppose $2 \leq B \leq p$ and $1 \leq A \leq p - B + 1$. We have

$$M_{Ap+B} = \underbrace{Y_{Ap+B} \oplus Y_{(A-1)p+B+1} \oplus \cdots \oplus Y_{2p+A+B-2}}_{\subseteq X_{Ap+B}} \oplus M_{p+A+B-1},$$

and so $U_p(M_{Ap+B}) = U_p(M_{p+A+B-1}) = M_{A+B}$. $\square$

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