A NECESSARY AND SUFFICIENT CONDITION FOR RICCI SHRINKERS TO HAVE POSITIVE AVR

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ABSTRACT. In this short paper we observe that a recent result of C.-W. Chen meshes well with earlier work of H.-D. Cao and D.-T. Zhou, O. Munteanu, J. Carrillo and L. Ni, and S.-J. Zhang. We give a necessary and sufficient condition for complete noncompact shrinking gradient Ricci solitons to have positive asymptotic volume ratio.

Let $(\mathcal{M}^n, g, f)$ denote a complete shrinking gradient Ricci soliton (*shrinker* for short) with $R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} g_{ij} = 0$. Throughout we shall assume that f is the normalized potential function in the sense that $R + |\nabla f|^2 - f = 0$ holds on $\mathcal{M}$.

It was proved by B.-L. Chen [3] that complete ancient solutions to the Ricci flow, and in particular shrinkers, must have nonnegative scalar curvature. As a consequence, the potential function f satisfies the estimate:

$$(1) \quad 0 \leq f(x) \leq \left(\frac{1}{2} r(x) + f(O)^{\frac{1}{2}} \right)^2,$$

where $r(x)$ denotes the distance function to a fixed point O in $\mathcal{M}$. H.-D. Cao and Detang Zhou [1] proved that there exists a positive constant C which depends on the dimension n , $\sup_{y \in B(O,1)} |\nabla f|(y)$, and the minimum of the Ricci curvature Rc_g in the ball $B(O, 1)$ such that f satisfies the lower estimate

$$(2) \quad f(x) \geq \frac{1}{4} (r(x) - C)^2$$

for $x \in \mathcal{M} - B(O, C)$ (see Fang, Man, and Zhang [5] for related estimates). In fact, carefully following the proof of [1] and integrating by parts yield:

$$(3) \quad f(x) \geq \frac{1}{4} \left[\left(r(x) - 4n - 2f(O)^{\frac{1}{2}} + \frac{4}{3} \right)_+ \right]^2,$$

where $c_+ \doteq \max(c, 0)$. Recently Haslhofer and Müller [6] further observed that if the reference point O is chosen to be a global minimum point of f (its existence is ensured by (1) and (2)), then one obtains improved estimates with constants depending only on n :

$$(4) \quad \frac{1}{4} [(r(x) - 5n)_+]^2 \leq f(x) \leq \frac{1}{4} (r(x) + \sqrt{2n})^2.$$

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Define the functions

$$V : \mathbb{R} \rightarrow [0, \infty), \quad R : \mathbb{R} \rightarrow [0, \infty)$$

by

$$V(c) \doteq \int_{\{f < c\}} d\mu, \quad R(c) \doteq \int_{\{f < c\}} R d\mu.$$

In [1], the following ODE relating $V(c)$ and $R(c)$ was established:

$$(5) \quad 0 \leq \frac{n}{2} V(c) - R(c) = c V'(c) - R'(c).$$

Recall that the asymptotic volume ratio (AVR) of a complete noncompact Riemannian manifold $(\mathcal{N}^n, h)$ is defined by

$$(6) \quad \text{AVR}(h) \doteq \lim_{r \rightarrow \infty} \frac{\text{Vol } B(p, r)}{\omega_n r^n}$$

if the limit exists, where $B(p, r)$ denotes the geodesic ball in $\mathcal{N}$ with center p and radius r and where ω_n is the volume of the unit Euclidean n -ball. It is easy to check that the $\text{AVR}(h)$ is independent of the choice of p . Moreover, if h has nonnegative Ricci curvature, then this limit exists by the Bishop–Gromov volume comparison theorem.

H.-D. Cao and Detang Zhou [1] proved the following using (5) and aided by an observation of Munteanu [7] (see [6] for an improvement).

Theorem 1. *Any complete noncompact shrinking gradient Ricci soliton must have at most Euclidean volume growth; i.e., $\limsup_{r \rightarrow \infty} \frac{\text{Vol } B(O, r)}{\omega_n r^n}$ is finite.*

Note that an earlier result by Carrillo and Ni [2] states that any nonflat shrinker with nonnegative Ricci curvature must have zero AVR. Based on Cao and Zhou's work, Zhang [8] proved a sharp upper bound on the volume growth of shrinkers under the assumption that $R \geq \delta$ for some constant $\delta > 0$. More recently, C.-W. Chen [4] proved that the AVR of a shrinker is bounded from below by some $c > 0$ if the average scalar curvature satisfies $\frac{1}{\text{Vol } B(O, r)} \int_{B(O, r)} R d\mu \leq r^\alpha$, where α is a negative constant (see also [1] for a similar result in the case where $\alpha = 0$).

We observe that the results in [2], [1], [7], [8], and [4] lead to a necessary and sufficient condition for noncompact shrinkers to have positive AVR.

Theorem 2. *Let $(\mathcal{M}^n, g, f)$ be a complete noncompact shrinking gradient Ricci soliton. Then $\text{AVR}(g)$ exists and is finite (by [6], it is bounded by a constant depending only on n). Moreover, $\text{AVR}(g) > 0$ if and only if $\int_{n+2}^\infty \frac{R(c)}{cV(c)} dc < \infty$.*

Proof. Let $P(c) \doteq \frac{V(c)}{c^{\frac{n}{2}}} - \frac{R(c)}{c^{\frac{n}{2}+1}}$ and $N(c) \doteq \frac{R(c)}{cV(c)}$. Note that $\frac{R(c)}{V(c)}$ is the average scalar curvature over the set $\{f < c\}$. The ODE (5) implies that

$$(7) \quad P'(c) = - \left(1 - \frac{n+2}{2c} \right) \frac{R(c)}{c^{\frac{n}{2}+1}} = - \frac{(1 - \frac{n+2}{2c}) N(c)}{1 - N(c)} P(c).$$

Since $0 \leq R(c) \leq \frac{n}{2} V(c)$ by (5), we have

$$(8) \quad \left(1 - \frac{n}{2c} \right) \frac{V(c)}{c^{\frac{n}{2}}} \leq P(c) \leq \frac{V(c)}{c^{\frac{n}{2}}}.$$

Hence, by the bounds (1) and (2) for f ,

$$2^n \omega_n \text{AVR}(g) = \lim_{c \rightarrow \infty} \frac{V(c)}{c^{n/2}} = \lim_{c \rightarrow \infty} P(c),$$

which exists by (7).

Integrating (7) yields

$$(9) \quad P(c) = P(n+2) e^{-\int_{n+2}^c \frac{\left(1 - \frac{n+2}{2c}\right) N(c)}{1 - N(c)} dc}$$

for $c \geq n+2$. From $\frac{R(c)}{V(c)} \leq \frac{n}{2}$ it is easy to see that for any $c \in [n+2, \infty)$ we have

$$(10) \quad \frac{1}{2} \int_{n+2}^c N(c) dc \leq \int_{n+2}^c \left(1 - \frac{n+2}{2c}\right) \frac{N(c)}{1 - N(c)} dc \leq 2 \int_{n+2}^c N(c) dc.$$

If $\int_{n+2}^{\infty} N(c) dc = \infty$, then by (9) we have $\text{AVR}(g) = \frac{1}{2^n \omega_n} \lim_{c \rightarrow \infty} P(c) = 0$.

If $\int_{n+2}^{\infty} N(c) dc < \infty$, then by (9) and (10), we have

$$P(c) \geq P(n+2) e^{-2 \int_{n+2}^{\infty} N(c) dc} > 0.$$

Hence $\text{AVR}(g) > 0$. □

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