THE STRUCTURE OF FREE AUTOMORPHIC MOUFANG LOOPS

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ABSTRACT. We describe the structure of a free loop of rank n in the variety of automorphic Moufang loops as a subdirect product of a free group and a free commutative Moufang loop, both of rank n . In particular, the variety of automorphic Moufang loops is the join of the variety of groups and the variety of commutative Moufang loops.

1. INTRODUCTION

The variety of Moufang loops is defined by the identity

$$(xy)(zx) = [x(yz)]x.$$

In an arbitrary loop one defines commutators and associators by

$$(ba)[a, b] = ab, \quad (a(bc))(a, b, c) = (ab)c.$$

Since in a Moufang loop every subloop generated by two elements is a group, one has

$$[a, b] = (ba)^{-1}ab, \quad (a, b, c) = (a(bc))^{-1}((ab)c).$$

We use the notation

$$[Q, Q] = \langle [a, b] \mid a, b \in Q \rangle$$

for the commutator subloop and

$$A(Q) = \langle (a, b, c) \mid a, b, c \in Q \rangle$$

for the associator subloop of a loop Q .

In an arbitrary loop Q consider the translations

$$L_a : x \mapsto ax, \quad R_b : x \mapsto xb.$$

The group $\mathfrak{J}(Q)$ generated by the mappings

$$(1.1) \quad l_{a,b} = L_a L_b L_{ba}^{-1}, \quad r_{a,b} = R_a R_b R_{ab}^{-1}, \quad T_a = R_a L_a^{-1}$$

is called the *inner mapping group* of Q . A subloop H of Q is normal if H is invariant under $\mathfrak{J}(Q)$ (see [3, p. 60]).

In 1956 Bruck and Paige began to study loops, in which all inner mappings are automorphisms, under the name “ A -loops”. These loops are now known as *automorphic loops*. In an automorphic loop Q all characteristic subloops are normal, in particular, the commutator subloop $[Q, Q]$ and the associator subloop $A(Q)$ are normal subloops of Q .

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Automorphic Moufang loops form a variety which contains all groups and all commutative Moufang loops (CMLs). In [7] and [9] it is shown that every F -quasigroup is isotopic to an automorphic Moufang loop which is a product of a group and a CML.

In our paper we show how a free automorphic Moufang loop of rank n is composed of a free group of rank n and a free CML of rank n . In particular, the variety of automorphic Moufang loops is the join of the variety of all groups and the variety of all CMLs.

2. AUTOMORPHIC MOUFANG LOOPS

The following result is well known (see [6, Theorem 4.16], p. 91).

Theorem 2.1. *A Moufang loop M is an automorphic loop if and only if the quotient loop $M/\text{Nuc}(M)$ is a commutative Moufang loop of exponent 3.* $\square$

Lemma 2.2. *In an automorphic Moufang loop M the following identities hold:*

$$(2.1) \quad (x, y, [z, t]) = 1,$$

$$(2.2) \quad [(x, y, z), t] = 1,$$

$$(2.3) \quad (x, y, z)^3 = 1.$$

Proof. By Theorem 2.1 the commutator $[M, M]$ is contained in the nucleus of M . Hence (2.1) holds.

For $t \in M$ the mapping $x \mapsto x^t = t^{-1}xt$ is an automorphism of M . In an arbitrary loop L satisfying $\text{Nuc}(L) \triangleleft L$ the identities

$$(2.4) \quad (xu, y, z) = (x, yu, z) = (x, y, zu) = (x, y, z)$$

hold for $x, y, z \in L, u \in \text{Nuc}(L)$ (see [8, Lemma 2]). For $x_1, x_2, x_3, t \in M$ it follows from Theorem 2.1 that $u_i = [x_i, t] = x_i^{-1}x_i^t \in \text{Nuc}(M)$. Using equation (2.4) one obtains

$$(x_1, x_2, x_3)^t = (x_1^t, x_2^t, x_3^t) = (x_1u_1, x_2u_2, x_3u_3) = (x_1, x_2, x_3).$$

Thus (2.2) is shown.

Identity (2.2) implies identity (i) in [3, Lemma 5.5, p. 125]. Hence $(x^3, y, z) = (x, y, z)^3$ for all $x, y, z \in M$. But by Theorem 2.1 we have $x^3 \in \text{Nuc}(M)$. Hence (2.3) holds. $\square$

For an arbitrary loop Q we put $D(Q) = [Q, Q] \cap A(Q)$.

Lemma 2.3. *Let M be an automorphic Moufang loop. Then:*

- (i) $D(M)$ is an elementary abelian 3-group contained in the center of M .
- (ii) $D(M/D(M)) = 1$.
- (iii) If $D(M) = 1$, then M is a subdirect product of the commutative Moufang loop $C(M) = M/[M, M]$ and the group $G(M) = M/A(M)$.

Proof. By Lemma 2.2 the identities $([x, y], z, t) = 1$ and $[(x, y, z), t] = 1$ hold in M . Hence (i) is proved, whereas (ii) and (iii) are obvious. $\square$

3. FREE AUTOMORPHIC MOUFANG LOOPS

Let $\mathcal{F}_n$ be the free group generated by $X = \{x_1, \dots, x_n\}$ and let $\mathcal{C}_n$ be the free CML generated by $Y = \{y_1, \dots, y_n\}$. Put

$$z_i = (x_i, y_i) \in \mathcal{F}_n \times \mathcal{C}_n$$

for $1 \leq i \leq n$ and denote by $\mathcal{A}_n$ the subloop of $\mathcal{F}_n \times \mathcal{C}_n$ generated by $Z = \{z_1, \dots, z_n\}$. Obviously one has

Proposition 3.1. *For the subloop $\mathcal{A}_n$ of $\mathcal{F}_n \times \mathcal{C}_n$ the following statements hold:*

- (i) $\mathcal{F}_n \times \mathcal{C}_n = \mathcal{A}_n \mathcal{C}_n = \mathcal{F}_n \mathcal{A}_n$,
- (ii) $[\mathcal{A}_n, \mathcal{A}_n] = [\mathcal{F}_n, \mathcal{F}_n] = [\mathcal{F}_n \times \mathcal{C}_n, \mathcal{F}_n \times \mathcal{C}_n]$,
- (iii) $A(\mathcal{A}_n) = A(\mathcal{C}_n) = A(\mathcal{F}_n \times \mathcal{C}_n)$,
- (iv) $D(\mathcal{A}_n) = 1$. □

Lemma 3.2. *Let $\mathcal{M}_n$ be a free automorphic Moufang loop with a base $\{b_1, \dots, b_n\}$ and let $\varphi : \mathcal{M}_n \rightarrow \mathcal{A}_n$ be a surjective homomorphism such that $\varphi(b_i) = z_i$ for $1 \leq i \leq n$. Then $\text{Ker } \varphi = D(\mathcal{M}_n)$.*

Proof. Since $D(\mathcal{A}_n) = 1$ it follows that $D(\mathcal{M}_n) \leq \text{Ker } \varphi$. Conversely, we consider the restrictions

$$\varphi_c : [\mathcal{M}_n, \mathcal{M}_n] \rightarrow [\mathcal{A}_n, \mathcal{A}_n], \quad \varphi_a : A(\mathcal{M}_n) \rightarrow A(\mathcal{A}_n)$$

and the surjective homomorphisms

$$\varphi^c : \mathcal{M}_n / [\mathcal{M}_n, \mathcal{M}_n] \rightarrow \mathcal{A}_n / [\mathcal{A}_n, \mathcal{A}_n], \quad \varphi^a : \mathcal{M}_n / A(\mathcal{M}_n) \rightarrow \mathcal{A}_n / A(\mathcal{A}_n)$$

induced by φ .

Since commutators and associators form verbal subloops, the homomorphisms φ_c and φ_a are surjective. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & [\mathcal{M}_n, \mathcal{M}_n] & \xrightarrow{\varphi_c} & [\mathcal{A}_n, \mathcal{A}_n] & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \text{Ker } \varphi & \longrightarrow & \mathcal{M}_n & \xrightarrow{\varphi} & \mathcal{A}_n \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{M}_n / [\mathcal{M}_n, \mathcal{M}_n] & \xrightarrow{\varphi^c} & \mathcal{A}_n / [\mathcal{A}_n, \mathcal{A}_n] & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
 \end{array}$$

Both $\mathcal{A}_n / [\mathcal{A}_n, \mathcal{A}_n]$ and $\mathcal{M}_n / [\mathcal{M}_n, \mathcal{M}_n]$ are free CMLs of rank n . Hence φ^c is an isomorphism. Assume that there is $w \in \text{Ker } \varphi$ such that $w \notin [\mathcal{M}_n, \mathcal{M}_n]$. Considering monomorphisms as inclusions and writing surjective homomorphisms as coset projections, we obtain

$$1 = \varphi(w) \mathcal{A}_n / [\mathcal{A}_n, \mathcal{A}_n] = \varphi^c(w[\mathcal{M}_n, \mathcal{M}_n]) \neq 1,$$

a contradiction. It follows that $\text{Ker } \varphi \leq [\mathcal{M}_n, \mathcal{M}_n]$. The same argument applied to φ_a gives us $\text{Ker } \varphi \leq A(\mathcal{M}_n)$. Hence

$$\text{Ker } \varphi \leq [\mathcal{M}_n, \mathcal{M}_n] \cap A(\mathcal{M}_n) = D(\mathcal{M}_n);$$

the lemma is proved. □

Theorem 3.3. *The loop $\mathcal{A}_n$ is a free automorphic Moufang loop of rank n .*

Proof. Let $\mathcal{M}_n$ a free automorphic Moufang loop with basis $\{b_1, \dots, b_n\}$. Then there is a surjective homomorphism $\varphi : \mathcal{M}_n \rightarrow \mathcal{A}_n$ such that $\varphi(b_i) = z_i$ for $1 \leq i \leq n$. We have to show that φ is an isomorphism. In view of Lemma 3.2 this means that we have to show that $D(\mathcal{M}_n) = 1$.

1) We first show the validity of the theorem for the case $n = 3$. Let x, y, z be a free generating system of $\mathcal{M}_3$ and let a be the associator (x, y, z) . Then $a^3 = 1$ and $\mathbb{Z}_3 \cong \langle a \rangle = A(\mathcal{M}_3)$. Assume that $D(\mathcal{M}_3) \neq 1$. Then $D(\mathcal{M}_3) = A(\mathcal{M}_3)$. Thus

$$\varphi(\mathcal{M}_3) \cong \mathcal{M}_3/D(\mathcal{M}_3) \cong \mathcal{A}_3$$

is a group, a contradiction. It follows that $D(\mathcal{M}_3) = 1$ and $\mathcal{M}_3 \cong \mathcal{A}_3$.

It is a consequence of this isomorphism that any commutator identity in at most three variables holds in all automorphic Moufang loops if and only if it holds in all groups.

In particular, the Witt identity

$$(3.1) \quad [[x, y^{-1}], z]^y [[y, z^{-1}], x]^z [[z, x^{-1}], y]^x = 1$$

holds in every automorphic Moufang loop. Let A be an automorphic Moufang loop such that $[A, A]$ is commutative. Then it follows from (3.1) that in A the Jacobi identity

$$(3.2) \quad [[x, y], z][[y, z], x][[z, x], y] = 1$$

is satisfied (see [10, Chap. II, §6, equation (17), p. 63]).

2) We have $\varphi[\mathcal{M}_n, \mathcal{M}_n] = [\mathcal{A}_n, \mathcal{A}_n] = [\mathcal{F}_n, \mathcal{F}_n]$ (Proposition 3.1(ii)). But $[\mathcal{F}_n, \mathcal{F}_n]$ is a free group and $[\mathcal{M}_n, \mathcal{M}_n]$ is a group. By Lemma 3.2 one has $\text{Ker } \varphi = D(\mathcal{M}_n) \leq [\mathcal{M}_n, \mathcal{M}_n]$. Hence $[\mathcal{M}_n, \mathcal{M}_n]$ is the semi-direct product of $D(\mathcal{M}_n)$ and a subgroup $M_c \cong [\mathcal{F}_n, \mathcal{F}_n]$. Since $D(\mathcal{M}_n)$ is contained in the centre of $\mathcal{M}_n$ it follows that $[\mathcal{M}_n, \mathcal{M}_n]$ is even the direct product of M_c and $D(\mathcal{M}_n)$.

3) By 2) we have $[[\mathcal{M}_n, \mathcal{M}_n], [\mathcal{M}_n, \mathcal{M}_n]] = [M_c, M_c]$ since $D(\mathcal{M}_n)$ lies in the center of $\mathcal{M}_n$. In particular, $[M_c, M_c]$ is a normal subloop of $\mathcal{M}_n$. For the quotient loop $\overline{\mathcal{M}_n} = \mathcal{M}_n/[M_c, M_c]$ and the canonical projection $\rho : \mathcal{M}_n \rightarrow \overline{\mathcal{M}_n}$, the following assertions hold:

$$(3.1) \quad D(\overline{\mathcal{M}_n}) = \overline{D(\mathcal{M}_n)} \cong D(\mathcal{M}_n).$$

$$(3.2) \quad \text{The abelian group } G = [\overline{\mathcal{M}_n}, \overline{\mathcal{M}_n}] \text{ is the direct product of the free abelian group } V = \overline{M_c} \text{ and the elementary abelian 3-group } D = D(\overline{\mathcal{M}_n}).$$

$$(3.3) \quad \varphi : \mathcal{M}_n \rightarrow \mathcal{A}_n \text{ induces a surjective homomorphism}$$

$$\overline{\varphi} : G \rightarrow \mathcal{F}_n/[[\mathcal{F}_n, \mathcal{F}_n], [\mathcal{F}_n, \mathcal{F}_n]]$$

$$\text{and } \text{Ker } (\overline{\varphi}) = D(\overline{\mathcal{M}_n}).$$

The action under conjugation of $\overline{\mathcal{M}_n}$ on G leads to a homomorphism

$$\gamma : \overline{\mathcal{M}_n} \rightarrow \text{Aut } G.$$

The abelian group $\Gamma = \gamma(\overline{\mathcal{M}_n})$ acts on G such that for all $d \in D, \alpha \in \Gamma$ one has $d^\alpha = d$. Observe that with the notation $\gamma_i = \gamma(\overline{b_i})$ we see that Γ is a free abelian group generated by the set $\{\gamma_1, \dots, \gamma_n\}$.

4) We consider G as a Γ -module and write G additively for this purpose. Consider the generators $\overline{b_i}$, $1 \leq i \leq n$ of $\overline{\mathcal{M}_n}$ and the elements $c_{i,j} = [\overline{b_i}, \overline{b_j}] \in G$. Obviously one has $c_{i,i} = 0$, $c_{j,i} = -c_{i,j}$. Furthermore, $c_{i,j}$, $1 \leq i < j \leq n$, forms a generating system of the Γ -module G .

By the Witt identity (3.2) we have for any i, j, k :

$$(3.3) \quad c_{i,j}^{1-\gamma_k} + c_{j,k}^{1-\gamma_i} + c_{k,i}^{1-\gamma_j} = 0.$$

5) Now we compare the Γ -module G with the standard relation module of the free abelian group $\Gamma \cong \mathbb{Z}^n$ (see [1], [2], [5]) and we obtain the abelian group $U = [\mathcal{F}_n, \mathcal{F}_n]/[[\mathcal{F}_n, \mathcal{F}_n], [\mathcal{F}_n, \mathcal{F}_n]]$ on which Γ acts by conjugation, too. For the basis $x_1, \dots, x_n$ of $\mathcal{F}_n$ and for $g_i = x_i[[\mathcal{F}_n, \mathcal{F}_n], [\mathcal{F}_n, \mathcal{F}_n]]$, $1 \leq i \leq n$, the elements $[g_i, g_j]$, $1 \leq i < j \leq n$, form a system of generators of U as a Γ -module. Furthermore, $\overline{\varphi}: G \rightarrow U$ is an epimorphism of Γ -modules satisfying $\overline{\varphi}(c_{i,j}) = [g_i, g_j]$.

By Bachmuth's Theorem [1] (see also [2]) the relations

$$(3.4) \quad [g_i, g_j]^{1-\gamma_k} + [g_j, g_k]^{1-\gamma_i} + [g_k, g_i]^{1-\gamma_j} = 0$$

form a complete system of relations for U .

It follows from (3.3) and (3.4) that there exists a Γ -homomorphism $\tau: U \rightarrow G$ satisfying $\tau(g_{i,j}) = c_{i,j}$. Hence $\overline{\varphi}\tau = \text{id}_U$. By 4) we know that $G = \langle c_{i,j} \mid 1 \leq i < j \leq n \rangle^\Gamma$. Hence τ is a surjective and $\overline{\varphi}: G \rightarrow U$ is an isomorphism. Thus $D(\mathcal{M}_n) \cong D(\overline{\mathcal{M}_n}) = \ker(\overline{\varphi}) = 1$. $\square$

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