IDEAL GAMES AND RAMSEY SETS

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ABSTRACT. It is shown that Matet’s characterization of the Ramsey property relative to a selective co-ideal $\mathcal{H}$, in terms of games of Kastanas, still holds if we consider semiselectivity instead of selectivity. Moreover, we prove that a co-ideal $\mathcal{H}$ is semiselective if and only if Matet’s game-theoretic characterization of the $\mathcal{H}$ -Ramsey property holds. This lifts Kastanas’s characterization of the classical Ramsey property to its optimal setting, from the point of view of the local Ramsey theory, and gives a game-theoretic counterpart to a theorem of Farah, asserting that a co-ideal $\mathcal{H}$ is semiselective if and only if the family of $\mathcal{H}$ -Ramsey subsets of $\mathbb{N}^{[\infty]}$ coincides with the family of those sets having the abstract $\mathcal{H}$ -Baire property. Finally, we show that under suitable assumptions, for every semiselective co-ideal $\mathcal{H}$ all sets of real numbers are $\mathcal{H}$ -Ramsey.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of nonnegative integers. Given an infinite set $A \subseteq \mathbb{N}$, the symbol $A^{[\infty]}$ (resp. $A^{(<\infty]}$) represents the collection of the infinite (resp. finite) subsets of A . Let $A^{[n]}$ denote the set of all the subsets of A with n elements. If $a \in \mathbb{N}^{(<\infty]}$ is an **initial segment** of $A \in \mathbb{N}^{[\infty]}$, then we write $a \sqsubset A$. Also, let $A/a := \{n \in A : \max(a) < n\}$ and write A/n to mean $A/\{n\}$. For $a \in \mathbb{N}^{(<\infty]}$ and $A \in \mathbb{N}^{[\infty]}$ let

$$[a, A] := \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A\}.$$

The family $\{[a, A] : (a, A) \in \mathbb{N}^{(<\infty]} \times \mathbb{N}^{[\infty]}\}$ is a basis for **Ellentuck’s topology**, also known as **exponential topology**. In [2], Ellentuck gave a characterization of the Ramsey property in terms of the Baire property relative to this topology (see Theorem 2.1 below).

Let $(P, \leq)$ be a poset. A subset $D \subseteq P$ is **dense** in P if for every $p \in P$, there is $q \in D$ with $q \leq p$. The subset $D \subseteq P$ is **open** if $p \in D$ and $q \leq p$ imply $q \in D$. We say P is **σ -distributive** if the intersection of countably many dense open subsets of P is dense. P is **σ -closed** if every decreasing sequence of elements of P has a lower bound.

Definition 1.1. A family $\mathcal{H} \subset \wp(\mathbb{N})$ is a **co-ideal** if it satisfies the following:

- (i) $A \subseteq B$ and $A \in \mathcal{H}$ implies $B \in \mathcal{H}$ and
- (ii) $A \cup B \in \mathcal{H}$ implies $A \in \mathcal{H}$ or $B \in \mathcal{H}$.

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The complement $\mathcal{I} = \wp(\mathbb{N}) \setminus \mathcal{H}$ is the **dual ideal** of $\mathcal{H}$. We will suppose that co-ideals differ from $\wp(\mathbb{N})$. Also, we say that a nonempty family $\mathcal{F} \subseteq \mathcal{H}$ is **$\mathcal{H}$ -disjoint** if for every $A, B \in \mathcal{F}$, $A \cap B \notin \mathcal{H}$. We say that $\mathcal{F}$ is a **maximal $\mathcal{H}$ -disjoint family** if it is $\mathcal{H}$ -disjoint and it is not properly contained in any other $\mathcal{H}$ -disjoint family.

A subset $\mathcal{X}$ of $\mathbb{N}^{[\infty]}$ is **Ramsey** if for every $[a, A] \neq \emptyset$ with $A \in \mathbb{N}^{[\infty]}$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. Some authors have used the term “completely Ramsey” to express this property, reserving the term “Ramsey” for a weaker property. Galvin and Prikry [4] showed that all Borel subsets of $\mathbb{N}^{[\infty]}$ are Ramsey, and Silver [12] extended this to all analytic sets. Mathias in [10] showed that if the existence of an inaccessible cardinal is consistent with ZFC , then it is consistent, with $ZF + DC$, that every subset of $\mathbb{N}^{[\infty]}$ is Ramsey. Mathias introduced the concept of a selective co-ideal (or a happy family), which has turned out to be of wide interest. Ellentuck [2] characterized the Ramsey sets as those having the Baire property with respect to the exponential topology of $\mathbb{N}^{[\infty]}$.

A game-theoretic characterization of the Ramsey property was given by Kastanas in [6], using games in the style of Banach-Mazur with respect to Ellentuck’s topology.

In this work we will deal with a game-theoretic characterization of the following property:

Definition 1.2. Let $\mathcal{H} \subset \mathbb{N}^{[\infty]}$ be a co-ideal. $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is **$\mathcal{H}$ -Ramsey** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. We say $\mathcal{X}$ is **$\mathcal{H}$ -Ramsey null** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

Mathias considered sets that are $\mathcal{H}$ -Ramsey with respect to a selective co-ideal $\mathcal{H}$ and generalized Silver’s result to this context. Matet [9] used games to characterize sets which are Ramsey with respect to a selective co-ideal $\mathcal{H}$. These games coincide with the games of Kastanas if $\mathcal{H}$ is $\mathbb{N}^{[\infty]}$ and with the games of Louveau [8] if $\mathcal{H}$ is a Ramsey ultrafilter.

Given a co-ideal $\mathcal{H} \subset \mathbb{N}^{[\infty]}$, the collection $\{[a, A] : (a, A) \in \mathbb{N}^{(<\infty)} \times \mathcal{H}\}$ is not, in general, a basis for a topology on $\mathbb{N}^{[\infty]}$, but the following abstract version of the Baire property and related concepts will be useful.

Definition 1.3. Let $\mathcal{H} \subset \mathbb{N}^{[\infty]}$ be a co-ideal. $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ has the abstract **$\mathcal{H}$ -Baire property** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $[b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \cap \mathcal{X} = \emptyset$. We say $\mathcal{X}$ is **$\mathcal{H}$ -nowhere dense** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $[b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that $[b, B] \cap \mathcal{X} = \emptyset$. Also, $\mathcal{X}$ is **$\mathcal{H}$ -meager** if it is the union of countably many $\mathcal{H}$ -nowhere dense sets.

Given a decreasing sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ of infinite subsets of $\mathbb{N}$, a set B is a **diagonalization** of the sequence (or B **diagonalizes** the sequence) if and only if $B/n \subseteq A_n$ for each $n \in B$. A co-ideal $\mathcal{H}$ is **selective** if and only if every decreasing sequence in $\mathcal{H}$ has a diagonalization in $\mathcal{H}$.

A co-ideal $\mathcal{H}$ has the **Q^+ -property** if for every $A \in \mathcal{H}$ and every partition $(F_n)_n$ of A into finite sets, there is $S \in \mathcal{H}$ such that $S \subseteq A$ and $|S \cap F_n| \leq 1$ for every $n \in \mathbb{N}$.

Proposition 1.4 ([10]). *A co-ideal $\mathcal{H}$ is selective if and only if the poset $(\mathcal{H}, \subseteq^*)$ is σ -closed and $\mathcal{H}$ has the Q^+ -property.*

Given a co-ideal $\mathcal{H}$, recall that a set $\mathcal{D} \subseteq \mathcal{H}$ is **dense open** in the ordering $(\mathcal{H}, \subseteq)$ if (a) for every $A \in \mathcal{H}$ there exists $B \in \mathcal{D}$ such that $B \subseteq A$ and (b) for every $A, B \in \mathcal{H}$, if $B \subseteq A$ and $A \in \mathcal{D}$, then $B \in \mathcal{D}$. Please notice that we will also consider the ordering $(\mathcal{H}, \subseteq^*)$, where $A \subseteq^* B$ if and only if $A \setminus B$ is a finite set, but any reference to “dense open” in this paper will be *only* with respect to the ordering $(\mathcal{H}, \subseteq)$.

Given a sequence $\{D_n\}_{n \in \mathbb{N}}$ of dense open sets in $(\mathcal{H}, \subseteq)$, a set B is a **diagonalization** of $\{D_n\}_{n \in \mathbb{N}}$ if and only if $B/n \in D_n$ for every $n \in B$. A co-ideal $\mathcal{H}$ is **semiselective** if for every sequence $\{D_n\}_{n \in \mathbb{N}}$ of dense open subsets of $\mathcal{H}$, the family of its diagonalizations is dense in $(\mathcal{H}, \subseteq)$.

Proposition 1.5 ([3]). *A co-ideal $\mathcal{H}$ is semiselective if and only if the poset $(\mathcal{H}, \subseteq^*)$ is σ -distributive and $\mathcal{H}$ has the Q^+ -property.*

Since σ -closedness implies σ -distributivity, then semiselectivity follows from selectivity, but the converse does not hold (see [3] for an example).

In Section 2 we list results of Ellentuck, Mathias and Farah that characterize topologically the Ramsey property and the local Ramsey property. In Section 3 we define a family of games and present the main result, which states that a co-ideal $\mathcal{H}$ is semiselective if and only if the $\mathcal{H}$ -Ramsey sets are exactly those for which the associated games are determined. This generalizes results of Kastanas [6] and Matet [9]. The proof is given in Section 4. In Section 5 we show that in Solovay’s model, for every semiselective co-ideal $\mathcal{H}$ all sets of real numbers from $L(\mathbb{R})$ are $\mathcal{H}$ -Ramsey.

2. TOPOLOGICAL CHARACTERIZATION OF THE RAMSEY PROPERTY

The following are the main results concerning the characterization of the Ramsey property and the local Ramsey property in topological terms.

Theorem 2.1 (Ellentuck). *Let $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ be given.*

- (i) $\mathcal{X}$ is Ramsey if and only if $\mathcal{X}$ has the Baire property with respect to Ellentuck’s topology.
- (ii) $\mathcal{X}$ is Ramsey null if and only if $\mathcal{X}$ is meager with respect to Ellentuck’s topology.

Theorem 2.2 (Mathias). *Let $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ and let a selective co-ideal $\mathcal{H}$ be given.*

- (i) $\mathcal{X}$ is $\mathcal{H}$ -Ramsey if and only if $\mathcal{X}$ has the abstract $\mathcal{H}$ -Baire property.
- (ii) $\mathcal{X}$ is $\mathcal{H}$ -Ramsey null if and only if $\mathcal{X}$ is $\mathcal{H}$ -meager.

Theorem 2.3 (Farah, Todorćević). *Let $\mathcal{H}$ be a co-ideal. The following are equivalent:*

- (i) $\mathcal{H}$ is semiselective.
- (ii) The $\mathcal{H}$ -Ramsey subsets of $\mathbb{N}^{[\infty]}$ are exactly those sets having the abstract $\mathcal{H}$ -Baire property, and the following three families of subsets of $\mathbb{N}^{[\infty]}$ coincide and are σ -ideals:
 - (a) $\mathcal{H}$ -Ramsey null sets,
 - (b) $\mathcal{H}$ -nowhere dense sets, and
 - (c) $\mathcal{H}$ -meager sets.

In the next section we state results by Kastanas [6] and Matet [9] (Theorems 3.1 and 3.2 below) which are the game-theoretic counterparts of Theorems 2.1 and 2.2, respectively, and we also present our main result (Theorem 3.3 below), which is the game-theoretic counterpart of Theorem 2.3.

3. CHARACTERIZING THE RAMSEY PROPERTY WITH GAMES

The following is a relativized version of a game due to Kastanas [6], employed to obtain a characterization of the family of completely Ramsey sets (i.e. $\mathcal{H}$ -Ramsey for $\mathcal{H} = \mathbb{N}^{[\infty]}$). The same game was used by Matet in [9] to obtain the analog result when $\mathcal{H}$ is selective.

Let $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$ be a fixed co-ideal. For each $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ we define a two-player game $G_{\mathcal{H}}(a, A, \mathcal{X})$ as follows: player I chooses an element $A_0 \in \mathcal{H} \upharpoonright A$; II answers by playing $n_0 \in A_0$ such that $a \subseteq n_0$, and $B_0 \in \mathcal{H} \cap (A_0/n_0)^{[\infty]}$; then I chooses $A_1 \in \mathcal{H} \cap B_0^{[\infty]}$; II answers by playing $n_1 \in A_1$ and $B_1 \in \mathcal{H} \cap (A_1/n_1)^{[\infty]}$; and so on. Player I wins if and only if $a \cup \{n_j : j \in \mathbb{N}\} \in \mathcal{X}$:

$$\begin{array}{ccccccc} I & A_0 & & A_1 & & \cdots & A_k & & \cdots \\ II & n_0, B_0 & & n_1, B_1 & & \cdots & n_k, B_k & & \cdots \end{array}$$

A **strategy** for a player is a rule that tells him (or her) what to play based on the previous moves. A strategy is a **winning strategy for player I** if player I wins the game whenever he (or she) follows the strategy, no matter what player II plays. Analogously, it can be defined what a winning strategy for player II is. The precise definitions of strategy for two-player games can be found in [7, 11].

Let $s = \{s_0, \dots, s_k\}$ be a nonempty finite subset of $\mathbb{N}$, written in its increasing order, and let $\vec{B} = \{B_0, \dots, B_k\}$ be a sequence of elements of $\mathcal{H}$. We say that the pair $(s, \vec{B})$ is a **legal position for player II** if $(s_0, B_0), \dots, (s_k, B_k)$ is a sequence of possible consecutive moves of II in the game $G_{\mathcal{H}}(a, A, \mathcal{X})$, respecting the rules. In this case, if σ is a winning strategy for player I in the game, we say that $\sigma(s, \vec{B})$ is a **realizable move of player I** according to σ . Notice that if $r \in B_k/s_k$ and $C \in \mathcal{H} \upharpoonright B_k/s_k$, then $(s_0, B_0), \dots, (s_k, B_k), (r, C)$ is also a sequence of possible consecutive moves of II in the game. We will sometimes use the notation $(s, \vec{B}, r, C)$ and say that $(s, \vec{B}, r, C)$ is a legal position for player II and $\sigma(s, \vec{B}, r, C)$ is a realizable move of player I according to σ .

We say that the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is **determined** if one of the players has a winning strategy.

Theorem 3.1 (Kastanas). *$\mathcal{X}$ is Ramsey if and only if for every $A \in \mathbb{N}^{[\infty]}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathbb{N}^{[\infty]}}(a, A, \mathcal{X})$ is determined.*

Theorem 3.2 (Matet). *Let $\mathcal{H}$ be a selective co-ideal. $\mathcal{X}$ is $\mathcal{H}$ -Ramsey if and only if for every $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined.*

Now we state our main result:

Theorem 3.3. *Let $\mathcal{H}$ be a co-ideal. The following are equivalent:*

- (1) $\mathcal{H}$ is semiselective.
- (2) $\forall \mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $\mathcal{X}$ is $\mathcal{H}$ -Ramsey if and only if for every $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined.

So Theorem 3.3 is a game-theoretic counterpart to Theorem 2.3 in the previous section, in the sense that it gives us a game-theoretic characterization of semiselectivity. Obviously, it also gives us a characterization of the $\mathcal{H}$ -Ramsey property, for semiselective $\mathcal{H}$, which generalizes the main results of Kastanas in [6] and Matet in [9] (Theorems 3.1 and 3.2 above).

It is known that every analytic set is $\mathcal{H}$ -Ramsey for $\mathcal{H}$ semiselective (see Theorem 2.2 in [3] or Lemma 7.18 in [14]). Assuming $AD_{\mathbb{R}}$, i.e., the axiom of determinacy for games over the reals (see [7] or [11]), we obtain the following from Theorem 3.3:

Corollary 3.4. *Assume $AD_{\mathbb{R}}$. If $\mathcal{H}$ is a semiselective co-ideal, then every subset of $\mathbb{N}^{[\infty]}$ is $\mathcal{H}$ -Ramsey.*

4. PROOF OF THE MAIN RESULT

Throughout the rest of this section, fix a semiselective co-ideal $\mathcal{H}$. Before proving Theorem 3.3, in Propositions 4.1 and 4.7 below we will deal with winning strategies of players in a game $G_{\mathcal{H}}(a, A, \mathcal{X})$.

Proposition 4.1. *For every $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{<\infty}$, I has a winning strategy in $G_{\mathcal{H}}(a, A, \mathcal{X})$ if and only if there exists $E \in \mathcal{H} \restriction A$ such that $[a, E] \subseteq \mathcal{X}$.*

Proof. Suppose σ is a winning strategy for I. We will suppose that $a = \emptyset$ and $A = \mathbb{N}$ without loss of generality.

Let $A_0 = \sigma(\emptyset)$ be the first move of I using σ . We will define a tree T of finite subsets of A_0 , and for each $s \in T$ we will also define a family $M_s \subseteq A_0^{[\infty]}$ and a family $N_s \subseteq (A_0^{[\infty]})^{|s|}$, where $|s|$ is the length of s . Put $\{p\} \in T$ for each $p \in A_0$ and let

$$M_{\{p\}} \subseteq \{\sigma(p, B) : B \in \mathcal{H} \restriction A_0\}$$

be a maximal $\mathcal{H}$ -disjoint family (see paragraph after Definition 1.1), and set

$$N_{\{p\}} = \{\{B\} : \sigma(p, B) \in M_{\{p\}}\}.$$

Suppose we have defined $T \cap A_0^{[n]}$ and we have chosen a maximal $\mathcal{H}$ -disjoint family M_s of realizable moves of player I of the form $\sigma(s, \vec{B})$ for every $s \in T \cap A_0^{[n]}$. Let

$$N_s = \{\vec{B} : \sigma(s, \vec{B}) \in M_s\}.$$

Given $s \in T \cap A_0^{[n]}$, $\vec{B} \in N_s$ and $r \in \sigma(s, \vec{B})/s$, we put $s \cup \{r\} \in T$. Then choose a maximal $\mathcal{H}$ -disjoint family

$$M_{s \cup \{r\}} \subseteq \{\sigma(s, \vec{B}, r, C) : \vec{B} \in N_s, C \in \mathcal{H} \restriction \sigma(s, \vec{B})/r\}.$$

Put

$$N_{s \cup \{r\}} = \{(\vec{B}, C) : \sigma(s, \vec{B}, r, C) \in M_{s \cup \{r\}}\}.$$

Now, for every $s \in T$, let

$$\mathcal{U}_s = \{E \in \mathcal{H} : (\exists F \in M_s) E \subseteq F\} \text{ and}$$

$$\mathcal{V}_s = \{E \in \mathcal{H} : (\forall F \in M_{s \setminus \{max(s)\}}) max(s) \in F \rightarrow F \cap E \notin \mathcal{H}\}.$$

Claim 4.2. For every $s \in T$, $\mathcal{U}_s \cup \mathcal{V}_s$ is dense open in $(\mathcal{H} \restriction A_0, \subseteq)$.

Proof. Fix $s \in T$ and $A \in \mathcal{H} \upharpoonright A_0$. If $(\forall F \in M_{s \setminus \{max(s)\}}) max(s) \in F \rightarrow F \cap A \notin \mathcal{H}$ holds, then $A \in \mathcal{V}_s$. Otherwise, fix $F \in M_{s \setminus \{max(s)\}}$ such that $max(s) \in F$ and $F \cap A \in \mathcal{H}$. Let $\vec{B} \in N_{s \setminus \{max(s)\}}$ be such that $\sigma(s \setminus \{max(s)\}, \vec{B}) = F$. Notice that since $max(s) \in F$, then

$$(s \setminus \{max(s)\}, \vec{B}, max(s), F \cap A / max(s))$$

is a legal position for player II. Then, using the maximality of M_s , choose $\hat{F} \in M_s$ such that

$$E := \sigma(s \setminus \{max(s)\}, \vec{B}, max(s), F \cap A / max(s)) \cap \hat{F}$$

is in $\mathcal{H}$. So $E \in \mathcal{U}_s$ and $E \subseteq A$. This completes the proof of Claim 4.2. $\square$

Claim 4.3. There exists $E \in \mathcal{H} \upharpoonright A_0$ such that for every $s \in T$ with $s \subset E$, $E/s \in \mathcal{U}_s$.

Proof. For each $n \in \mathbb{N}$, let

$$\mathcal{D}_n = \bigcap_{max(s)=n} \mathcal{U}_s \cup \mathcal{V}_s,$$

$$\mathcal{U}_n = \bigcap_{max(s)=n} \mathcal{U}_s$$

(if there is no $s \in T$ with $max(s) = n$, then we put $\mathcal{D}_n = \mathcal{U}_n = \mathcal{H} \upharpoonright A_0$). By Claim 4.2, every $\mathcal{D}_n$ is dense open in $(\mathcal{H} \upharpoonright A_0, \subseteq)$. Using semiselectivity, choose a diagonalization $\hat{E} \in \mathcal{H} \upharpoonright A_0$ of the sequence $(\mathcal{D}_n)_n$. Let

$$E_0 := \{n \in \hat{E} : \hat{E}/n \in \mathcal{U}_n\} \text{ and } E_1 := \hat{E} \setminus E_0.$$

Let us prove that $E_1 \notin \mathcal{H}$. Suppose $E_1 \in \mathcal{H}$. By the definitions, $(\forall n \in E_1) \hat{E}/n \notin \mathcal{U}_n$. Let $n_0 = \min(E_1)$ and fix $s_0 \subset \hat{E}$ such that $max(s_0) = n_0$, satisfying, in particular, the following:

$$(\forall F \in M_{s_0 \setminus \{n_0\}}) n_0 \in F \rightarrow F \cap E_1 / n_0 \notin \mathcal{H}.$$

Notice that $|s_0| > 1$, by the construction of the M_s 's.

Now, let $m = max(s_0 \setminus \{n_0\})$. Then $m \in E_0$ and therefore $\hat{E}/m \in \mathcal{U}_m \subseteq \mathcal{U}_{s_0 \setminus \{n_0\}}$. So there exists $F \in M_{s_0 \setminus \{n_0\}}$ such that $\hat{E}/m \subseteq F$. Since $m < n_0$, $n_0 \in F$. But $F \cap E_1 / n_0 = E_1 / n_0 \in \mathcal{H}$, a contradiction.

Hence, $E_1 \notin \mathcal{H}$ and therefore $E_0 \in \mathcal{H}$. Then $E := E_0$ is as required. $\square$

Claim 4.4. Let E be as in Claim 4.3 and let $s \cup \{r\} \in T$ with $s \subset E$ and $r \in E/s$. If $E/s \subseteq \sigma(s, \vec{B})$ for some $\vec{B} \in N_s$, then there exists $C \in \mathcal{H} \upharpoonright \sigma(s, \vec{B})/r$ such that $E/r \subseteq \sigma(s, \vec{B}, r, C)$ and $(\vec{B}, C) \in N_{s \cup \{r\}}$.

Proof. Fix s and r as in the hypothesis. Suppose $E/s \subseteq \sigma(s, \vec{B})$ for some $\vec{B} \in N_s$. Since $E/r \in \mathcal{U}_{s \cup \{r\}}$, there exists $(\vec{D}, C) \in N_{s \cup \{r\}}$ such that $E/r \subseteq \sigma(s, \vec{D}, r, C)$. Notice that $E/r \subseteq \sigma(s, \vec{B}) \cap \sigma(s, \vec{D})$. Since M_s is $\mathcal{H}$ -disjoint, then $\sigma(s, \vec{D})$ is necessarily equal to $\sigma(s, \vec{B})$ and therefore $\sigma(s, \vec{B}, r, C) = \sigma(s, \vec{D}, r, C)$. Hence $(\vec{B}, C) \in N_{s \cup \{r\}}$ and $E/r \subseteq \sigma(s, \vec{B}, r, C)$. $\square$

Claim 4.5. Let E be as in Claim 4.3. Then $[\emptyset, E] \subseteq \mathcal{X}$.

Proof. Let $\{k_i\}_{i \geq 0} \subseteq E$ be given. Since $E/k_0 \in \mathcal{U}_{\{k_0\}}$, there exists $B_0 \in N_{\{k_0\}}$ such that $E/k_0 \subseteq \sigma(k_0, B_0)$. Thus, by the choice of E and applying Claim 4.4 iteratively, we prove that $\{k_i\}_{i \geq 0}$ is generated in a run of the game in which player I has used his winning strategy σ . Therefore $\{k_i\}_{i \geq 0} \in \mathcal{X}$. $\square$

The converse is trivial. This completes the proof of Proposition 4.1. $\square$

Now we turn to the case when player II has a winning strategy. The proof of the following is similar to the proof of Proposition 4.3 in [9]. First we show a result we will need in the sequel. It should be compared with Lemma 4.2 in [9].

Lemma 4.6. *Let $B \in \mathcal{H}$, $f : \mathcal{H} \restriction B \rightarrow \mathbb{N}$, and $g : \mathcal{H} \restriction B \rightarrow \mathcal{H} \restriction B$ be given such that $f(A) \in A$ and $g(A) \subseteq A/f(A)$. Then there is $E_{f,g} \in \mathcal{H} \restriction B$ with the property that for each $p \in E_{f,g}$ there exists $A \in \mathcal{H} \restriction B$ such that $f(A) = p$ and $E_{f,g}/p \subseteq g(A)$.*

Proof. For each $n \in \{f(A) : A \in \mathcal{H} \restriction B\}$, let

$$U_n = \{E \in \mathcal{H} \restriction B : (\exists A \in \mathcal{H} \restriction B) (f(A) = n \wedge E \subseteq g(A))\}$$

and

$$V_n = \{E \in \mathcal{H} \restriction B : (\forall A \in \mathcal{H} \restriction B) (f(A) = n \rightarrow |g(A) \setminus E| = \infty)\}.$$

The set $D_n = U_n \cup V_n$ is dense open in $\mathcal{H} \restriction B$. Choose $E \in \mathcal{H} \restriction B$ such that for each $n \in E$, $E/n \in D_n$. Let

$$E_0 = \{n \in E : E/n \in U_n\} \text{ and } E_1 = \{n \in E : E/n \in V_n\}.$$

Now, suppose $E_1 \in \mathcal{H}$. Then, for each $n \in E_1$, $E_1/n \in V_n$. Let $n_1 = f(E_1)$. So $n_1 \in E_1$ by the definition of f . But, by the definition of g , $g(E_1) \subseteq E_1/n_1$ and so $E_1/n_1 \notin V_{n_1}$, a contradiction. Therefore, $E_1 \notin \mathcal{H}$. Hence $E_0 \in \mathcal{H}$, since $\mathcal{H}$ is a co-ideal. The set $E_{f,g} := E_0$ is as required. $\square$

Proposition 4.7. *For every $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$, II has a winning strategy in $G_{\mathcal{H}}(a, A, \mathcal{X})$ if and only if $\forall A' \in \mathcal{H} \restriction A$ there exists $E \in \mathcal{H} \restriction A'$ such that $[a, E] \cap \mathcal{X} = \emptyset$.*

Proof. Let τ be a winning strategy for II in $G_{\mathcal{H}}(a, A, \mathcal{X})$ and let $A' \in \mathcal{H} \restriction A$ be given. We are going to define a winning strategy σ for I, in $G_{\mathcal{H}}(a, A', \mathbb{N}^{[\infty]} \setminus \mathcal{X})$, in such a way that we will get the required result by means of Proposition 4.1. So, in a play of the game $G_{\mathcal{H}}(a, A', \mathbb{N}^{[\infty]} \setminus \mathcal{X})$, with II's successive moves being (n_j, B_j) , $j \in \mathbb{N}$, define $A_j \in \mathcal{H}$ and E_{f_j, g_j} as in Lemma 4.6, for f_j and g_j such that

(1) for all $\hat{A} \in \mathcal{H} \restriction A'$,

$$(f_0(\hat{A}), g_0(\hat{A})) = \tau(\hat{A});$$

(2) for all $\hat{A} \in \mathcal{H} \restriction B_j \cap g_j(A_j)$,

$$(f_{j+1}(\hat{A}), g_{j+1}(\hat{A})) = \tau(A_0, \dots, A_j, \hat{A});$$

(3) $A_0 \subseteq A'$ and $A_{j+1} \subseteq B_j \cap g_j(A_j)$;

(4) $n_j = f_j(A_j)$ and $E_{f_j, g_j}/n_j \subseteq g_j(A_j)$.

Now, let $\sigma(\emptyset) = E_{f_0, g_0}$ and $\sigma((n_0, B_0), \dots, (n_j, B_j)) = E_{f_{j+1}, g_{j+1}}$.

Conversely, let A_0 be the first move of I in the game. Then there exists $E \in \mathcal{H} \restriction A_0$ such that $[a, E] \cap \mathcal{X} = \emptyset$. We define a winning strategy for player II by letting him (or her) play $(\min E, E \setminus \{\min E\})$ at the first turn and arbitrarily from there on. $\square$

We are ready now for the following:

Proof of Theorem 3.3. If $\mathcal{H}$ is semiselective, then part (2) of Theorem 3.3 follows from Propositions 4.1 and 4.7.

Conversely, suppose part (2) holds and $(\mathcal{D}_n)_n$ is a sequence of dense open sets in $(\mathcal{H}, \subseteq)$. For every $a \in \mathbb{N}^{[<\infty]}$, let

$\mathcal{X}_a = \{B \in [a, \mathbb{N}] : B/a \text{ diagonalizes some decreasing } (A_n)_n \text{ such that } (\forall n) A_n \in \mathcal{D}_n\}$ and define

$$\mathcal{X} = \bigcup_{a \in \mathbb{N}^{[<\infty]}} \mathcal{X}_a.$$

Fix $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ with $[a, A] \neq \emptyset$, and define a winning strategy σ for player I in $G_{\mathcal{H}}(a, A, \mathcal{X})$, as follows: let $\sigma(\emptyset)$ be any element of $\mathcal{D}_0$ such that $\sigma(\emptyset) \subseteq A$. At stage k , if II's successive moves in the game are (n_j, B_j) , $j \leq k$, let $\sigma((n_0, B_0), \dots, (n_k, B_k))$ be any element of $\mathcal{D}_{k+1}$ such that $\sigma((n_0, B_0), \dots, (n_k, B_k)) \subseteq B_k$. Notice that $a \cup \{n_0, n_1, n_2, \dots\} \in \mathcal{X}_a$.

So the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined for every $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ with $[a, A] \neq \emptyset$. Then, by our assumptions, $\mathcal{X}$ is $\mathcal{H}$ -Ramsey. So given $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \restriction A$ such that $B^{[\infty]} \subseteq \mathcal{X}$ or $B^{[\infty]} \cap \mathcal{X} = \emptyset$. The second alternative does not hold, so $\mathcal{X} \cap \mathcal{H}$ is dense in $(\mathcal{H}, \subseteq)$. Hence, $\mathcal{H}$ is semiselective. $\square$

5. THE RAMSEY PROPERTY IN SOLOVAY MODELS

Recall that the Mathias forcing notion $\mathbb{M}$ is the collection of all the sets of the form

$$[a, A] := \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A\},$$

ordered by $[a, A] \leq [b, B]$ if and only if $[a, A] \subseteq [b, B]$.

If $\mathcal{H}$ is a co-ideal, then $\mathbb{M}_{\mathcal{H}}$, the Mathias partial order with respect to $\mathcal{H}$, is the collection of all the $[a, A]$ as above but with $A \in \mathcal{H}$, ordered in the same way.

A co-ideal $\mathcal{H}$ has the *Mathias property* if it satisfies the following: if x is $\mathbb{M}_{\mathcal{H}}$ -generic over a model M , then every $y \in x^{[\infty]}$ is $\mathbb{M}_{\mathcal{H}}$ -generic over M . Also, $\mathcal{H}$ has the *Prikry property* if for every $[a, A] \in \mathbb{M}_{\mathcal{H}}$ and every formula φ of the forcing language of $\mathbb{M}_{\mathcal{H}}$, there is $B \in \mathcal{H} \restriction A$ such that $[a, B]$ decides φ .

Theorem 5.1 ([3, Theorem 4.1]). *For a co-ideal $\mathcal{H}$ the following are equivalent:*

- (1) $\mathcal{H}$ is semiselective.
- (2) $\mathbb{M}_{\mathcal{H}}$ has the Prikry property.
- (3) $\mathbb{M}_{\mathcal{H}}$ has the Mathias property.

Suppose M is a model of ZFC and there is an inaccessible cardinal λ in M . The Levy partial order $Col(\omega, < \lambda)$ produces a generic extension $M[G]$ of M where λ becomes $\aleph_1$. Solovay's model (see [13]) is obtained by taking the submodel of $M[G]$ formed by all the sets hereditarily definable in $M[G]$ from a sequence of ordinals (see [10] or [5]).

In [10], Mathias shows that if $V = L$, λ is a Mahlo cardinal and $V[G]$ is a generic extension obtained by forcing with $Col(\omega, < \lambda)$, then every set of real numbers defined in the generic extension from a sequence of ordinals is $\mathcal{H}$ -Ramsey for $\mathcal{H}$ a selective co-ideal. This result can be extended to semiselective co-ideals under a suitable large cardinal hypothesis.

Theorem 5.2. *Suppose λ is a weakly compact cardinal. Let $V[G]$ be a generic extension by $Col(\omega, < \lambda)$. Then, if $\mathcal{H}$ is a semiselective co-ideal in $V[G]$, every set of real numbers in $L(\mathbb{R})$ of $V[G]$ is $\mathcal{H}$ -Ramsey.*

Proof. Let $\mathcal{H}$ be a semiselective co-ideal in $V[G]$. Let $\mathcal{A}$ be a set of reals in $L(\mathbb{R})^{V[G]}$; in particular, $\mathcal{A}$ is defined in $V[G]$ by a formula φ from a sequence of ordinals. Let $[a, A]$ be a condition of the Mathias forcing $\mathbb{M}_{\mathcal{H}}$ with respect to the semiselective co-ideal $\mathcal{H}$. Finally, let $\dot{H}$ be a name for $\mathcal{H}$. Notice that $\dot{H} \subseteq V_{\lambda}$.

Since $V[G]$ satisfies that $\mathcal{H}$ is semiselective, the following statement holds in $V[G]$: For every sequence $D = (D_n : n \in \omega)$ of open dense subsets of $\mathcal{H}$ and for every $x \in \mathcal{H}$ there is $y \in \mathcal{H}$, $y \subseteq x$, such that y diagonalizes the sequence D .

Therefore, there is $p \in G$ such that, in V , the following statement holds:

$$\forall \dot{D} \forall \tau (p \Vdash_{Col(\omega, < \lambda)} (\dot{D} \text{ is a name for a sequence of dense open subsets of } \dot{H} \\ \text{and } \tau \in \dot{H}) \longrightarrow (\exists x (x \in \dot{H}, x \subseteq \tau, x \text{ diagonalizes } \dot{D}))).$$

Notice that every real in $V[G]$ has a name in V_{λ} , and names for subsets of $\mathcal{H}$ or countable sequences of subsets of $\mathcal{H}$ are contained in V_{λ} . Also, the forcing $Col(\omega, < \lambda)$ is a subset of V_{λ} . Therefore the same statement is valid in the structure $(V_{\lambda}, \in, \dot{H}, Col(\omega, < \lambda))$. This statement is Π_1^1 over this structure, and since λ is Π_1^1 -indescribable, there is $\kappa < \lambda$ such that in $(V_{\kappa}, \in, \dot{H} \cap V_{\kappa}, Col(\omega, < \lambda) \cap V_{\kappa})$

$$\forall \dot{D} \forall \tau (p \Vdash_{Col(\omega, < \kappa)} (\dot{D} \text{ is a name for a sequence of dense open subsets of } \dot{H} \cap V_{\kappa} \\ \text{and } \tau \in \dot{H} \cap V_{\kappa}) \longrightarrow (\exists x (x \in \dot{H} \cap V_{\kappa}, x \subseteq \tau, x \text{ diagonalizes } \dot{D}))).$$

We can get κ inaccessible, since there is a Π_1^1 formula expressing that λ is inaccessible. Also, κ is such that p and the names for the real parameters in the definition of $\mathcal{A}$ and for A belong to V_{κ} .

If we let $G_{\kappa} = G \cap Col(\omega, < \kappa)$, then $G_{\kappa} \subseteq Col(\omega, < \kappa)$ and it is generic over V . Also, $p \in G_{\kappa}$. Here $\dot{H} \cap V_{\kappa}$ is a $Col(\omega, < \kappa)$ -name in V which is interpreted by G_{κ} as $\mathcal{H} \cap V[G_{\kappa}]$; thus $\mathcal{H} \cap V[G_{\kappa}] \in V[G_{\kappa}]$. Moreover, since every subset (or sequence of subsets) of $\mathcal{H} \cap V[G_{\kappa}]$ which belongs to $V[G_{\kappa}]$ has a name contained in V_{κ} , we have that, in $V[G_{\kappa}]$, $\mathcal{H} \cap V_{\kappa}$ is semiselective, and in consequence it has both the Prikry and the Mathias properties.

Now the proof can be finished as in [10]. Let $\dot{r}$ be the canonical name of an $\mathbb{M}_{\mathcal{H} \cap V[G_{\kappa}]}$ -generic real and consider the formula $\varphi(\dot{r})$ in the forcing language of $V[G_{\kappa}]$. By the Prikry property of $\mathcal{H} \cap V[G_{\kappa}]$, there is $A' \subseteq A$, $A' \in \mathcal{H} \cap V[G_{\kappa}]$, such that $[a, A']$ decides $\varphi(\dot{r})$. Since $2^{2^{\omega}}$ computed in $V[G_{\kappa}]$ is countable in $V[G]$, there is (in $V[G]$) an $\mathbb{M}_{\mathcal{H} \cap V[G_{\kappa}]}$ -generic real x over $V[G_{\kappa}]$ such that $x \in [a, A']$.

To see that there is such a generic real in $\mathcal{H}$ we argue as in 5.5 of [10] using the semiselectivity of $\mathcal{H}$ and the fact that $\mathcal{H} \cap V[G_\kappa]$ is countable in $V[G]$ to obtain an element of $\mathcal{H}$ which is generic. By the Mathias property of $\mathcal{H} \cap V[G_\kappa]$, every $y \in [a, x \setminus a]$ is also $\mathbb{M}_{\mathcal{H} \cap V[G_\kappa]}$ -generic over $V[G_\kappa]$, and also $y \in [a, A']$. Thus $\varphi(x)$ if and only if $[a, A'] \Vdash \varphi(\dot{r})$, if and only if $\varphi(y)$. Therefore, $[a, x \setminus a]$ is contained in $\mathcal{A}$ or is disjoint from $\mathcal{A}$. $\square$

As in [10], we obtain the following.

Corollary 5.3. *If ZFC is consistent with the existence of a weakly compact cardinal, then so is the statement that for every semiselective co-ideal $\mathcal{H}$ all sets of real numbers from $L(\mathbb{R})$ are $\mathcal{H}$ -Ramsey.*

Eisworth ([1]) showed that the hypothesis of the existence of a Mahlo cardinal in Mathias's result cannot be weakened.

Question. Can the weakly compact cardinal hypothesis in the statement of Theorem 5.2 be weakened? Would a Mahlo cardinal suffice?

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