SHARP WEAK TYPE INEQUALITIES FOR THE HAAR SYSTEM AND RELATED ESTIMATES FOR NONSYMMETRIC MARTINGALE TRANSFORMS

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ABSTRACT. For any $1 \leq p < \infty$, we determine the optimal constant C_p such that the following holds. If $(h_k)_{k \geq 0}$ is the Haar system, then for any vectors a_k from a separable Hilbert space $\mathcal{H}$ and $\theta_k \in \{0, 1\}$, $k = 0, 1, 2, \dots$, we have

$$\left\| \sum_{k=0}^n \theta_k a_k h_k \right\|_{p, \infty} \leq C_p \left\| \sum_{k=0}^n a_k h_k \right\|_p.$$

This is generalized to the weak type inequality

$$\|g\|_{p, \infty} \leq C_p \|f\|_p,$$

where f is an $\mathcal{H}$ -valued martingale and g is its transform by a predictable sequence taking values in $[0, 1]$. We extend this further to the estimate

$$\|Y\|_{p, \infty} \leq C_p \|X\|_p,$$

valid for any two $\mathcal{H}$ -valued continuous-time martingales X, Y , such that $([Y, X - Y]_t)$ is nondecreasing and nonnegative as a function of t .

1. INTRODUCTION

Let $1 < p < \infty$ and let $(h_k)_{k \geq 0}$ be the Haar system in L^p . In his classical result, Marcinkiewicz [9] proved that there is a universal finite constant c_p such that

$$(1.1) \quad c_p^{-1} \left\| \sum_{k=0}^n a_k h_k \right\|_p \leq \left\| \sum_{k=0}^n \varepsilon_k a_k h_k \right\|_p \leq c_p \left\| \sum_{k=0}^n a_k h_k \right\|_p$$

for any n and any $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \dots, n$. This result was extended by Burkholder [1] to the martingale setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_k)_{k \geq 0}$, a nondecreasing family of sub- σ -fields of $\mathcal{F}$. Let $f = (f_k)_{k \geq 0}$ be a real-valued martingale with the difference sequence $(df_k)_{k \geq 0}$ given by $df_0 = f_0$ and $df_k = f_k - f_{k-1}$ for $k \geq 1$. Let g be a transform of f by a real predictable sequence $v = (v_k)_{k \geq 0}$ bounded in absolute value by 1: that is, $dg_k = v_k df_k$ for all $k \geq 0$. By predictability we mean that each term v_k is measurable with respect to

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$\mathcal{F}_{(k-1) \vee 0}$. Then (cf. [1]) for $1 < p < \infty$ there is an absolute constant c'_p for which

$$(1.2) \quad \|g\|_p \leq c'_p \|f\|_p.$$

Here we have used the notation $\|f\|_p = \sup_n \|f_n\|_p$. Let $c_p(1.1)$, $c'_p(1.2)$ denote the optimal constants in (1.1) and (1.2), respectively. The Haar system is a martingale difference sequence with respect to its natural filtration (with the probability space being the Lebesgue unit interval) and hence so is $(a_k h_k)_{k \geq 0}$, for given fixed real numbers $a_0, a_1, a_2, \dots$. Therefore, $c_p(1.1) \leq c'_p(1.2)$ for all $1 < p < \infty$. It follows from the results of Burkholder [2] and Maurey [10] that in fact the constants coincide: $c_p(1.1) = c'_p(1.2)$ for all $1 < p < \infty$. The question about the precise value of $c_p(1.1)$ was answered by Burkholder in [3]: $c_p(1.1) = p^* - 1$ (where $p^* = \max\{p, p/(p-1)\}$) for $1 < p < \infty$. Furthermore, the constant does not change if we allow the martingales and the terms a_k to take values in a separable Hilbert space $\mathcal{H}$. This determines the complex unconditional basis constant of the Haar system:

$$\sup \left\{ \left\| \sum_{k=0}^n \varepsilon_k a_k h_k \right\|_p \right\} = p^* - 1, \quad 1 < p < \infty,$$

where the supremum is taken over all n , all sequences $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ of signs and all complex numbers $a_0, a_1, a_2, \dots$ satisfying $\|\sum_{k=0}^n a_k h_k\|_p = 1$ (cf. [5]).

For $p = 1$ the inequalities (1.1) and (1.2) do not hold with any finite constant, but one can establish a corresponding weak type estimate. Burkholder [3] proved the following sharp bound, for a wider range of parameters: if $1 \leq p \leq 2$, then

$$(1.3) \quad \|g\|_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)} \right)^{1/p} \|f\|_p,$$

where $\|g\|_{p,\infty} = \sup_{\lambda > 0} \lambda (\mathbb{P}(\sup_n |g_n| \geq \lambda))^{1/p}$. For $p > 2$, Suh [12] showed that

$$(1.4) \quad \|g\|_{p,\infty} \leq (p^{p-1}/2)^{1/p} \|f\|_p.$$

Both (1.3), (1.4) remain sharp for the Haar system, even for $\mathcal{H}$ -valued coefficients. In fact, all the martingale inequalities above are valid under the less restrictive assumption of differential subordination and can be extended to the continuous-time setting. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and equip it with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. Let X, Y be two adapted cadlag martingales taking values in $\mathcal{H}$; with no loss of generality we assume, from now on, that $\mathcal{H} = \ell^2$. Following [13], we say that Y is *differentially subordinate* to X , if the process $([X, X]_t - [Y, Y]_t)_{t \geq 0}$ is nondecreasing and nonnegative as a function of t . Here $[X, Y] = \sum_{j=0}^{\infty} [X^j, Y^j]$, where X^j, Y^j stand for the j -th coordinates of X and Y , respectively, and $[X^j, Y^j]$ is the quadratic covariance process of X^j and Y^j (see e.g. Dellacherie and Meyer [7]). If we treat the discrete-time martingales $f = (f_k)_{k=0}^{\infty}$, $g = (g_k)_{k=0}^{\infty}$ as continuous-time processes (by $X_t = f_{[t]}$ and $Y_t = g_{[t]}$ for $t \geq 0$), then the above condition reads

$$|dg_k| \leq |df_k| \quad \text{for } k \geq 0,$$

which is the original definition of the differential subordination due to Burkholder [3]. Clearly, this condition is satisfied by the martingale transforms studied above. Thus the following theorem (cf. [12], [13]) generalizes the previous inequalities (1.2), (1.3) and (1.4). We use the notation $\|X\|_p = \sup_t \|X_t\|_p$ and $\|X\|_{p,\infty} = \sup_{\lambda > 0} \lambda (\mathbb{P}(\sup_t |X_t| \geq \lambda))^{1/p}$, analogous to that of the discrete-time setting.

Theorem 1.1. *If Y is differentially subordinate to X , then*

$$(1.5) \quad \begin{aligned} \|Y\|_p &\leq c_p \|X\|_p, \quad 1 < p < \infty, \\ \|Y\|_{p,\infty} &\leq \left(\frac{2}{\Gamma(p+1)} \right)^{1/p} \|X\|_p, \quad 1 \leq p \leq 2, \\ \|Y\|_{p,\infty} &\leq \left(\frac{p^{p-1}}{2} \right)^{1/p} \|X\|_p, \quad 2 \leq p < \infty, \end{aligned}$$

and the inequalities are sharp.

Let us now turn to the nonsymmetric case, a setting we will be particularly interested in. An alternative way of defining the unconditional basis constant is

$$\sup \left\{ \left\| \sum_{k \in J} a_k h_k \right\|_p \right\},$$

where the supremum is taken over all n , all subsets $J \subseteq \{0, 1, 2, \dots, n\}$ and all complex numbers $a_0, a_1, a_2, \dots, a_n$ satisfying $\|\sum_{k=0}^n a_k h_k\|_p = 1$ (see e.g. [8]). This leads to another natural transformation of Haar series: to throw out some of its terms instead of changing their signs. In other words, it suggests that we consider in (1.1) the case when each ε_k takes values in $\{0, 1\}$. Under this nonsymmetric assumption, Choi [6] found the best constant c_p'' in (1.1) for real coefficients $a_0, a_1, a_2, \dots$. He also showed that if a martingale f is real and each v_k takes values in $[0, 1]$, then (1.2) holds with the same constant c_p'' . Since the description of the constant is quite complicated, we do not present it here and refer the interested reader to [6].

There is a natural question about the best constants in the corresponding weak type estimates for the Haar system and the extension of these bounds to continuous-time martingales. We will study this problem in the general case when the coefficients $a_0, a_1, a_2, \dots$ as well as the processes take values in a Hilbert space $\mathcal{H}$. The role of “non-symmetric differential subordination” is played by the condition

$$(1.6) \quad ([X, Y]_t - [Y, Y]_t) \text{ is nonnegative and nondecreasing as a function of } t.$$

This generalizes nonsymmetric martingale transforms: assume that f is a martingale and g is its transform by a predictable sequence v . If we treat these as continuous-time processes, we see that the condition (1.6) reads $(v_k - v_k^2)df_k^2 \geq 0$ for all k , and hence it is satisfied if the variables v_k take values in $[0, 1]$.

We turn to the formulation of our main result. Let

$$C_p = \begin{cases} 1 & \text{if } 1 \leq p \leq 2, \\ \frac{1}{2} \left[\frac{(2c+p-1)^{p-1}}{c+1} \right]^{1/p} & \text{if } p > 2, \end{cases}$$

where $c = c(p) > 1$ is the unique positive number satisfying

$$(1.7) \quad c^{p-1} = 2c + 1.$$

Theorem 1.2. *Let X, Y be two Hilbert-space-valued martingales satisfying (1.6). Then for any $1 \leq p < \infty$ we have*

$$(1.8) \quad \|Y\|_{p,\infty} \leq C_p \|X\|_p$$

and the constant C_p is the best possible. It is already the best possible in the following one-sided bound for the Haar system:

$$(1.9) \quad \left| \left\{ r \in [0, 1] : \sum_{k=0}^n \theta_k a_k h_k(r) \geq 1 \right\} \right| \leq C_p^p \left\| \sum_{k=0}^n a_k h_k \right\|_p^p$$

for all n , $a_k \in \mathbb{R}$ and $\theta_k \in \{0, 1\}$, $k = 0, 1, 2, \dots, n$.

A few words about the proof and the organization of the paper. Our approach is based on Burkholder's technique, which exploits special functions which have certain convex type properties. To be more precise, the inequality (1.8) reduces to $\mathbb{E}V_p(X_t, Y_t) \leq 0$ for some appropriate function $V_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and all $t \geq 0$. The key to study this inequality is to find a majorant U_p of V_p such that $(U_p(X_t, Y_t))_{t \geq 0}$ is an $(\mathcal{F}_t)$ -supermartingale satisfying $U_p(X_0, Y_0) \leq 0$. This is the way we prove Theorem 1.2 for $1 \leq p \leq 2$; see Section 2 below. For $p > 2$ our argument turns out to be substantially different and uses an "integration trick" developed by the author in [11]. First we show that $\mathbb{E}U_\infty(X_t, Y_t) \leq 0$ for some simple $U_\infty : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and all $t \geq 0$, and then complicate the function by integrating it against a certain positive kernel, thus obtaining the appropriate majorant; see Section 3. The final part of the paper contains the proof of a technical fact needed in the earlier considerations.

2. THE CASE $1 \leq p \leq 2$

The main object in this section is the function $U_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ given by

$$U_p(x, y) = \begin{cases} py \cdot (y - x) & \text{if } |x| + |2y - x| < 2, \\ p - p|x| & \text{if } |x| + |2y - x| \geq 2. \end{cases}$$

Here and below, the dot $\cdot$ denotes the scalar product in $\mathcal{H}$ and $|x|$ stands for the norm of $x \in \mathcal{H}$. Let $V_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ be defined by

$$V_p(x, y) = 1_{\{|y| \geq 1\}} - |x|^p.$$

We have the following majorization.

Lemma 2.1. *For all $x, y \in \mathcal{H}$ we have*

$$(2.1) \quad U_p(x, y) \geq V_p(x, y).$$

Proof. If $|x| + |2y - x| < 2$, then $|y| \leq |x/2| + |y - x/2| < 1$ and, consequently,

$$1_{\{|y| \geq 1\}} - |x|^p = -|x|^p \leq -\frac{p|x|^2}{4} \leq p|y|(|y| - |x|) \leq py \cdot (y - x).$$

On the other hand, if $|x| + |2y - x| \geq 2$, then (2.1) follows immediately from the estimate $p - ps \geq 1 - s^p$, valid for all $s \geq 0$, by virtue of the mean-value theorem. $\square$

Lemma 2.2. *Suppose that martingales X, Y satisfy the condition (1.6). Then for any $t \geq 0$ we have*

$$(2.2) \quad \mathbb{E}U_p(X_t, Y_t) \leq 0.$$

Proof. Let $\mathcal{U} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ be given by

$$\mathcal{U}(x, y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| < 1, \\ 1 - 2|x| & \text{if } |x| + |y| \geq 1. \end{cases}$$

This is Burkholder's special function corresponding to the weak type inequality (1.3) for $p = 1$ (cf. [3]). As shown by Wang (see the proof of Theorem 3 in [13]), if $\zeta = (\zeta_t)$ is differentially subordinate to $\xi = (\xi_t)$, then for any $t \geq 0$ we have

$$\mathbb{E}\mathcal{U}(\zeta_t, \xi_t) \leq 0.$$

We apply this to the martingales $\xi = X/2$ and $\zeta = -X/2 + Y$; the differential subordination follows from the identity

$$[X/2, X/2]_t - [-X/2 + Y, -X/2 + Y]_t = [X, Y]_t - [Y, Y]_t.$$

The proof is completed by noting that $U_p(x, y) = p\mathcal{U}(x/2, -x/2 + y)$ for all x, y . $\square$

Now we turn to the proof of Theorem 1.2 in the case $1 \leq p \leq 2$.

Proof of Theorem 1.2. Obviously, the constant $C_p = 1$ is the best possible in (1.9): take $a_0 = \theta_0 = 1$ and $a_k = \theta_k = 0$ for $k \geq 1$. Therefore all we need is to establish the estimate (1.8). Note that we may assume that X is bounded in L^p ; otherwise there is nothing to prove. By homogeneity, we will be done if we show that

$$(2.3) \quad \mathbb{P}(Y^* \geq 1) \leq \|X\|_p^p,$$

where $Y^* = \sup_{t \geq 0} |Y_t|$ is the maximal function of Y . Observe that by virtue of (2.1) and (2.2) we have

$$(2.4) \quad \mathbb{P}(|Y_t| \geq 1) \leq \mathbb{E}|X_t|^p \quad \text{for } t \geq 0.$$

Now take $\varepsilon \in (0, 1)$ and introduce the stopping time $\tau = \inf\{s \geq 0 : |Y_s| \geq 1 - \varepsilon\}$. We have that

$$\{Y^* \geq 1\} \subset \{|Y_t| \geq 1 - \varepsilon \text{ for some } t\} = \{|Y_{\tau \wedge t}| \geq 1 - \varepsilon \text{ for some } t\}.$$

Since the family $(\{|Y_{\tau \wedge s}| \geq 1 - \varepsilon\})_s$ is nondecreasing and

$$\{|Y_{\tau \wedge t}| \geq 1 - \varepsilon \text{ for some } t\} = \bigcup_{t \geq 0} \{|Y_{\tau \wedge t}| \geq 1 - \varepsilon\},$$

we get $\mathbb{P}(Y^* \geq 1) \leq \lim_{t \rightarrow \infty} \mathbb{P}(Y_{\tau \wedge t} \geq 1 - \varepsilon)$. Now it is easy to see that the pair $(X_{\tau \wedge t}/(1 - \varepsilon), (Y_{\tau \wedge t}/(1 - \varepsilon)))$ satisfies (1.6). Applying (2.4) to this pair gives

$$\lim_{t \rightarrow \infty} \mathbb{P}(Y_{\tau \wedge t} \geq 1 - \varepsilon) \leq (1 - \varepsilon)^{-p} \mathbb{E}|X_{\tau \wedge t}|^p \leq (1 - \varepsilon)^{-p} \|X\|_p^p.$$

Thus (2.3) follows, since ε was arbitrary. $\square$

3. THE CASE $p > 2$

This is more involved. Define an auxiliary function $U_\infty : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ by

$$U_\infty(x, y) = \begin{cases} 0 & \text{if } |x| + |2y - x| < 1, \\ (|2y - x| - 1)^2 - |x|^2 & \text{if } |x| + |2y - x| \geq 1. \end{cases}$$

Later on, we will need the following properties of this function.

Lemma 3.1. (i) *There is an absolute constant $A > 0$ such that for all $x, y \in \mathcal{H}$,*

$$(3.1) \quad U_\infty(x, y) \leq A(|x|^2 + |y|^2 + 1).$$

(ii) *For all $x, y \in \mathcal{H}$ we have*

$$(3.2) \quad U_\infty(x, y) \leq (|2y - x| - 1)^2 - |x|^2.$$

(iii) *If $x, y, h, k \in \mathcal{H}$ satisfy*

$$(3.3) \quad |x| + |2y - x| \leq 1, \quad |x + h| + |2(y + k) - (x + h)| \geq 1$$

and

$$(3.4) \quad |2k - h| \leq |h|,$$

then $U_\infty(x + h, y + k) \leq 0$.

(iv) If $x, y \in \mathcal{H}$ satisfy $x \cdot y - |y|^2 \geq 0$, then $U_\infty(x, y) \leq 0$.

Proof. (i), (ii) These inequalities are evident from the very definition of U_∞ .

(iii) The desired inequality can be written in the form

$$-|x + h| \leq |2(y + k) - (x + h)| - 1 \leq |x + h|.$$

The left inequality is precisely the second condition in (3.3). To get the right one, note that by a triangle inequality, (3.3) and (3.4),

$$|2(y + k) - (x + h)| - 1 \leq |2y - x| + |2k - h| - 1 \leq |2y - x| - 1 + |h| \leq -|x| + |h| \leq |x + h|.$$

(iv) The estimate is trivial if $|x| + |2y - x| \leq 1$. If the reverse holds, note that $x \cdot y - |y|^2 \geq 0$ is equivalent to $|x|^2 \geq |2y - x|^2$ and hence

$$U_\infty(x, y) = (|2y - x| + |x| - 1)(|2y - x| - |x| - 1) \leq 0. \quad \square$$

The next result is a dual version of Lemma 2.2.

Lemma 3.2. *Suppose that the martingales X, Y are bounded in L^2 and satisfy the condition (1.6). Then for any $t \geq 0$,*

$$(3.5) \quad \mathbb{E}U_\infty(X_t, Y_t) \leq 0.$$

Proof. First note that by (3.1), the random variable $U_\infty(X_t, Y_t)$ is integrable. Let $\tau = \inf\{s \geq 0 : |X_s| + |2Y_s - X_s| > 1\}$. We will show the following three statements:

$$(3.6) \quad \mathbb{E}U_\infty(X_t, Y_t)1_{\{|X_0| + |2Y_0 - X_0| > 1\}} \leq \mathbb{E}U_\infty(X_0, Y_0)1_{\{|X_0| + |2Y_0 - X_0| > 1\}},$$

$$(3.7) \quad U_\infty(X_t, Y_t) = U_\infty(X_0, Y_0) = 0 \quad \text{on } \{|X_0| + |2Y_0 - X_0| \leq 1, \tau > t\}$$

and

$$(3.8) \quad \mathbb{E}U_\infty(X_t, Y_t)1_{\{|X_0| + |2Y_0 - X_0| \leq 1, \tau \leq t\}} \leq \mathbb{E}U_\infty(X_0, Y_0)1_{\{|X_0| + |2Y_0 - X_0| \leq 1, \tau \leq t\}}.$$

These three facts yield the claim: indeed, they give $\mathbb{E}U(X_t, Y_t) \leq \mathbb{E}U(X_0, Y_0)$, and it suffices to note that $U(X_0, Y_0) \leq 0$ in view of (1.6) and part (iv) of Lemma 3.1.

To prove (3.6), use (3.2) to get

$$\mathbb{E}[U_\infty(X_t, Y_t)|\mathcal{F}_0] \leq 4\mathbb{E}[Y_t \cdot (Y_t - X_t)|\mathcal{F}_0] - 2\mathbb{E}(|2Y_t - X_t||\mathcal{F}_0) + 1.$$

Clearly, $\mathbb{E}(|2Y_t - X_t||\mathcal{F}_0) \geq |2Y_0 - X_0|$. Moreover, by (1.6), we have that

$$\mathbb{E}[Y_t(Y_t - X_t) - Y_0(Y_0 - X_0)|\mathcal{F}_0] = -\mathbb{E}[(X, Y)_t - (Y, Y)_t - (X, Y)_0 - (Y, Y)_0|\mathcal{F}_0]$$

is nonpositive. Consequently, on the set $\{|X_0| + |2Y_0 - X_0| > 1\}$,

$$\mathbb{E}[U_\infty(X_t, Y_t)|\mathcal{F}_0] \leq 4Y_0 \cdot (Y_0 - X_0) - 2|2Y_0 - X_0| + 1 = U_\infty(X_0, Y_0)$$

and (3.6) follows. The condition (3.7) is obvious, by the definition of U_∞ and τ . To get (3.8), we proceed as previously: by (3.2) and (1.6) we have, on the set $\{|X_0| + |2Y_0 - X_0| \leq 1, \tau \leq t\}$,

$$\begin{aligned} \mathbb{E}[U_\infty(X_t, Y_t)|\mathcal{F}_\tau] &= 4\mathbb{E}[Y_t \cdot (Y_t - X_t)|\mathcal{F}_\tau] - 2\mathbb{E}(|2Y_t - X_t||\mathcal{F}_\tau) + 1 \\ &\leq 4Y_\tau \cdot (Y_\tau - X_\tau) - 2|2Y_\tau - X_\tau| + 1 \\ &= U_\infty(X_\tau, Y_\tau). \end{aligned}$$

Now use part (iii) of Lemma 3.1 with $x = X_{\tau-}$, $y = Y_{\tau-}$, $h = \Delta X_{\tau}$ and $k = \Delta Y_{\tau}$: the condition (3.3) follows from the definition of τ , while (3.4) is a consequence of (1.6). Thus, $U_{\infty}(X_{\tau}, Y_{\tau}) \leq 0 = U_{\infty}(X_0, Y_0)$ and the proof is complete. $\square$

We are ready to introduce the special function $U_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, corresponding to the weak type estimate for $p > 2$. Recall $c = c(p)$ given by (1.7) and let

$$(3.9) \quad b = b(p) = \frac{2(p-1)}{2c+p-1}.$$

Set

$$(3.10) \quad U_p(x, y) = \int_0^b t^{p-1} U_{\infty}(x/t, y/t) dt.$$

Some lengthy but straightforward calculations give that

$$U_p(x, y) = \frac{2}{p(p-1)(p-2)} (|x| + |2y - x|)^{p-1} (|2y - x| - (p-1)|x|)$$

if $|x| + |2y - x| \leq b$, and

$$U_p(x, y) = b^{p-2} \left[\frac{|2y - x|^2 - |x|^2}{p-2} - \frac{2b|2y - x|}{p-1} + \frac{b^2}{p} \right]$$

for $|x| + |2y - x| > b$. We will also need the function $V_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, given by

$$V_p(x, y) = \alpha_p (C_p^{-p} 1_{\{|y| \geq 1\}} - |x|^p),$$

where

$$(3.11) \quad \alpha_p = \frac{2(p-1)^{p-2}}{p(p-2)}.$$

We have the following majorization.

Lemma 3.3. *For all $x, y \in \mathcal{H}$, we have*

$$(3.12) \quad U_p(x, y) \geq V_p(x, y).$$

The justification of this estimate is quite involved, so for the sake of clarity we postpone it to Section 4 and proceed with the proof of Theorem 1.2.

Proof of (1.8). We may assume that X is bounded in L^p . Then so is Y : since $-X/2 + Y$ is differentially subordinate to $X/2$ (see the proof of Lemma 2.2 above), the inequality (1.5) implies that

$$\|Y\|_p \leq \| -X/2 + Y \|_p + \|X/2\|_p \leq p^* \|X/2\|_p < \infty.$$

As in the case $1 \leq p \leq 2$, we reduce the desired estimate to

$$\mathbb{P}(|Y_t| \geq 1) \leq C_p^p |X_t|^p, \quad t \geq 0.$$

By (3.12), this will be done if we show that $\mathbb{E}U_p(X_t, Y_t) \leq 0$. This follows immediately from (3.5), the definition of U_p and Fubini's theorem. To see that the latter is applicable, note that by (3.1),

$$\mathbb{E}|U_p(X_t, Y_t)| \leq A \mathbb{E} \left[\frac{b^{p-2}(|X_t|^2 + |Y_t|^2)}{p-2} + \frac{b^p}{p} \right] < \infty,$$

since X, Y are bounded in L^p . $\square$

We turn to the sharpness of (1.9), which is the most technical element of the paper. We will need the following fact, which relates the validity of a given inequality for the Haar system to a certain boundary value problem (for similar results, see e.g. Section 11 in [3] or Section 7 in [4]).

Lemma 3.4. *Let $V : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given Borel function, locally bounded from below. Assume that*

$$(3.13) \quad \int_0^1 V \left(\sum_{k=0}^n a_k h_k(r), \sum_{k=0}^n a_k \theta_k h_k(r) \right) dr \leq 0$$

for all n and all $a_k \in \mathbb{R}$, $\theta_k \in \{0, 1\}$, $k = 0, 1, 2, \dots, n$. Then there is a function $W : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following properties.

- (a) We have $W(x, x) \leq 0$ for any $x \in \mathbb{R}$.
- (b) For all $x, y \in \mathbb{R}$ we have $W(x, y) \geq V(x, y)$.
- (c) The function W is concave along any line of slope 0 or 1.

Proof. Define $W : \mathbb{R} \times \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$(3.14) \quad W(x, y) = \sup \left\{ \int_0^1 V \left(x + \sum_{k=1}^n a_k h_k(r), y + \sum_{k=1}^n a_k \theta_k h_k(r) \right) dr \right\},$$

where the supremum is taken over all n and all $a_k \in \mathbb{R}$, $\theta_k \in \{0, 1\}$, $k = 1, 2, \dots, n$. Then the property (a) is a consequence of (3.13), while (b) follows from the definition of W by considering the sequence $a_1 = a_2 = \dots = 0$. To get (c), we use Burkholder's "splicing" argument: take any line L of slope 1 and any point (x, y) lying on it. Take $d > 0$, an integer N , sequences $a_1^+, a_2^+, \dots, a_N^+$, $a_1^-, a_2^-, \dots, a_N^-$ of real numbers and $\theta_1^+, \theta_2^+, \dots, \theta_N^+$, $\theta_1^-, \theta_2^-, \dots, \theta_N^-$ with $\theta_k^\pm \in \{0, 1\}$. Let

$$Z_{x,y}^\pm(r) = \left(x \pm d + \sum_{k=1}^N a_k^\pm h_k(r), y \pm d + \sum_{k=1}^N a_k^\pm \theta_k^\pm h_k(r) \right)$$

and splice the functions Z^+ and Z^- together in the following way:

$$Z_{x,y}(r) = \begin{cases} Z_{x,y}^-(2r) & \text{if } 0 \leq r \leq 1/2, \\ Z_{x,y}^+(2r-1) & \text{if } 1/2 < r \leq 1. \end{cases}$$

Then it is easily seen that

$$Z_{x,y}(r) = \left(x + \sum_{k=1}^{2N} a_k h_k(r), y + \sum_{k=1}^{2N} a_k \theta_k h_k(r) \right),$$

where for any $1 \leq k \leq 2N$, there is $\ell \leq N$ such that $(a_k, \theta_k) = (a'_\ell, \theta'_\ell)$ or $(a_k, \theta_k) = (a''_\ell, \theta''_\ell)$. Thus

$$W(x, y) \geq \int_0^1 V(Z_{x,y}(r)) dr = \int_0^{1/2} V(Z_{x,y}^-(2r)) dr + \int_{1/2}^1 V(Z_{x,y}^+(2r-1)) dr$$

and taking the supremum over all N , $a_k^\pm$ and $\theta_k^\pm$ as above yields

$$W(x, y) \geq (W(x-d, y-d) + W(x+d, y+d))/2.$$

Since x , y , and d were arbitrary, W is midpoint concave along L . Analogous arguments lead to the midpoint concavity along the lines of slope 0. This yields the

finiteness of W : indeed, for any $x, y \in \mathbb{R}$ we have, by (a) and (b),

$$0 \geq W(y, y) \geq \frac{1}{2}W(x, y) + \frac{1}{2}W(2y - x, y) \geq \frac{1}{2}W(x, y) + \frac{1}{2}V(2y - x, y),$$

so $W(x, y) \leq -V(2y - x, y)$. Finally, W is locally bounded from below by virtue of (b) and the fact that V also has this property. This, combined with the midpoint concavity of W along the lines of slope 0 or 1, yields (c). $\square$

We are ready to study (1.9). Let $p > 2$, $0 < \gamma_p < C_p$ and assume that

$$(3.15) \quad \left| \left\{ r \in [0, 1] : \sum_{k=0}^n a_k \theta_k h_k(r) \geq 1 \right\} \right| \leq \gamma_p^p \int_0^1 \left| \sum_{k=0}^n a_k h_k(r) \right|^p dr,$$

for all n and all $a_k \in \mathbb{R}$, $\theta_k \in \{0, 1\}$, $k = 0, 1, 2, \dots, n$. Take $\beta_p \in (\gamma_p, C_p)$ and

$$V_p^{\beta_p}(x, y) = 1_{\{y \geq 1\}} - \beta_p^p |x|^p.$$

Let W_p be given by (3.14), with $V = V_p^{\beta_p}$. For clarity, we split the remaining part of the proof into a few steps. Recall b and c , given by (3.9) and (1.7), respectively.

Step 1. The starting point is the equation

$$(3.16) \quad W_p(0, y) = 0 \quad \text{for sufficiently small } y > 0.$$

To see this, note first that if $y < 1$, then, by (b), $W_p(0, y) \geq 0$. On the other hand, let y be a positive number satisfying $\beta_p(1 - y) \geq \gamma_p$. Take an integer n , numbers $a_1, a_2, \dots, a_n$ belonging to $\mathbb{R}$ and $\theta_1, \theta_2, \dots, \theta_n$ belonging to $\{0, 1\}$. We have

$$\begin{aligned} & \left| \left\{ r \in [0, 1] : y + \sum_{k=1}^n a_k \theta_k h_k(r) \geq 1 \right\} \right| - \beta_p^p \int_0^1 \left| \sum_{k=1}^n a_k h_k(r) \right|^p dr \\ &= \left| \left\{ r \in [0, 1] : \sum_{k=1}^n \frac{a_k}{1-y} \theta_k h_k(r) \geq 1 \right\} \right| - \beta_p^p (1-y)^p \int_0^1 \left| \sum_{k=1}^n \frac{a_k}{1-y} h_k(r) \right|^p dr \\ &\leq \left| \left\{ r \in [0, 1] : \sum_{k=1}^n \frac{a_k}{1-y} \theta_k h_k(r) \geq 1 \right\} \right| - \gamma_p^p \int_0^1 \left| \sum_{k=1}^n \frac{a_k}{1-y} h_k(r) \right|^p dr \leq 0, \end{aligned}$$

where the latter estimate follows from (3.15). Since n and the numbers a_k and θ_k were arbitrary, this gives $W_p(0, y) \leq 0$ and we are done.

Step 2. Note that the points

$$P_1 = (1 - b/2, 1), \quad P_2 = (0, b/2), \quad P_3 = (-b/(p-1), b(p-3)/(2(p-1)))$$

lie, in this order, on a certain line of slope 1. Moreover, by (b),

$$W_p(P_1) \geq 1 - \beta_p^p \left(1 - \frac{b}{2}\right)^p = 1 - \beta_p^p c^p \left(\frac{b}{p-1}\right)^p$$

and

$$W_p(P_3) \geq -\beta_p^p \left(\frac{b}{p-1}\right)^p.$$

Thus, combining this with (c),

$$\begin{aligned} (3.17) \quad W_p(P_2) &\geq \frac{|P_3 - P_2|}{|P_3 - P_1|} W_p(P_1) + \frac{|P_2 - P_1|}{|P_3 - P_1|} W_p(P_3) \\ &\geq \frac{1}{c+1} - \frac{c}{c+1} \beta_p^p \left(\frac{b}{p-1}\right)^p (1 + c^{p-1}). \end{aligned}$$

Step 3. Fix positive numbers y and δ . Using (c) and then (b), we get

$$\begin{aligned} W_p(0, y) &\geq \frac{\delta(p-1)}{\delta(p-1)+2} W_p\left(\frac{2y}{p-1}, y\right) + \frac{2}{\delta(p-1)+2} W_p(-\delta y, y) \\ &\geq -\frac{\delta(p-1)}{\delta(p-1)+2} \beta_p^p \left(\frac{2y}{p-1}\right)^p + \frac{2}{\delta(p-1)+2} W_p(-\delta y, y). \end{aligned}$$

Similarly, we get

$$W_p(-\delta y, y) \geq -\frac{\delta(p-1)}{2(1+\delta)} \beta_p^p \left(\frac{2y(1+\delta)}{p-1}\right)^p + \frac{2(1+\delta) - \delta(p-1)}{2(1+\delta)} W_p(0, y(1+\delta)),$$

and plugging this into the preceding estimate gives

$$(3.18) \quad W_p(0, y) \geq A(\delta) \beta_p^p y^p + B(\delta) W_p(0, y(1+\delta)),$$

where

$$A(\delta) = -\left(\frac{2}{p-1}\right)^p \frac{\delta(p-1)}{\delta(p-1)+2} (1 + (1+\delta)^{p-1})$$

and

$$B(\delta) = \frac{2(1+\delta) - \delta(p-1)}{(1+\delta)(\delta(p-1)+2)}.$$

One easily verifies that we have the following asymptotics:

$$(3.19) \quad \lim_{\delta \rightarrow 0} \frac{A(\delta)}{\delta} = -2 \left(\frac{2}{p-1}\right)^{p-1},$$

$$(3.20) \quad \lim_{\delta \rightarrow 0} \frac{B(\delta)(1+\delta)^{p-1} - 1}{\delta^2} = 0.$$

By induction, (3.18) leads to

$$(3.21) \quad W_p(0, y) \geq \beta_p^p y^p A(\delta) \frac{[B(\delta)(1+\delta)^p]^N - 1}{B(\delta)(1+\delta)^p - 1} + B(\delta)^N W_p(0, y(1+\delta)^N)$$

for any positive integer N .

Step 4. This is the final part. Take $\delta = N^{-1/2}$, $y = b(1+\delta)^{-N}/2$, divide both sides of (3.21) by y^{p-1} and let $N \rightarrow \infty$. We obtain the estimate

$$(3.22) \quad 0 \geq -b \beta_p^p \left(\frac{2}{p-1}\right)^{p-1} + \left(\frac{2}{b}\right)^{p-1} W_p\left(0, \frac{b}{2}\right).$$

To see this, note that $\delta \rightarrow 0$, $y \rightarrow 0$ as $N \rightarrow \infty$, and therefore, by (3.16), the left-hand side of (3.21), divided by y^{p-1} , converges to 0. To deal with the right-hand side, observe that, by (3.19) and (3.20),

$$\lim_{\delta \rightarrow 0} \frac{A(\delta)}{B(\delta)(1+\delta)^p - 1} = \lim_{\delta \rightarrow 0} \frac{A(\delta)}{\delta B(\delta)(1+\delta)^{p-1}} = -2 \left(\frac{2}{p-1}\right)^{p-1}.$$

Furthermore, again by (3.20), we have

$$y([B(\delta)(1+\delta)^p]^N - 1) = \frac{b}{2} [B(\delta)(1+\delta)^{p-1}]^{1/\delta^2} \rightarrow \frac{b}{2}$$

and, similarly,

$$\frac{B(\delta)^N}{y^{p-1}} = \left(\frac{2}{b}\right)^{p-1} (B(\delta)(1+\delta)^{p-1})^{1/\delta^2} \rightarrow \left(\frac{2}{b}\right)^{p-1}.$$

The above three limits yield (3.22). Combining this estimate with (3.17) gives

$$\beta_p^p \left[b \left(\frac{2}{p-1} \right)^{p-1} + \left(\frac{2}{b} \right)^{p-1} \frac{c}{c+1} \left(\frac{b}{p-1} \right)^p (1+c^{p-1}) \right] \geq \left(\frac{2}{b} \right)^{p-1} \frac{1}{c+1}.$$

After some cancellations and manipulations, it can be written in the form

$$\beta_p^p \geq \frac{(p-1)^p}{b^p(c+1)} \left[p-1 + \frac{c}{c+1} (1+c^{p-1}) \right]^{-1} = C_p^p,$$

where the equality follows from (1.7). This contradicts the initial assumption $\beta_p \in (\gamma_p, C_p)$ and completes the proof.

4. PROOF OF LEMMA 3.3

We start with a reduction step: it suffices to establish the majorization in the real case and for x, y satisfying $0 \leq x \leq 2y$. To see this, let us (for a moment) write $U_p^{\mathcal{H}}, V_p^{\mathcal{H}}$ instead of U_p, V_p , to indicate the Hilbert space we are working with. For $x, y \in \mathcal{H}$, take $x' = |x|$ and $y' = |x/2| + |x/2 - y|$. Then $0 \leq x' \leq 2y'$, $2y' - x' = |2y - x|$ and $y' \geq |y|$, so

$$U_p^{\mathcal{H}}(x, y) - V_p^{\mathcal{H}}(x, y) \geq U_p^{\mathbb{R}}(x', y') - V_p^{\mathbb{R}}(x', y').$$

This justifies the reduction. We consider the cases $y \leq b/2$, $y \in (b/2, 1)$ and $y \geq 1$ separately in the three lemmas below.

Lemma 4.1. *We have*

$$(4.1) \quad \frac{2}{p(p-1)(p-2)} (2y)^{p-1} (2y - px) \geq -\frac{2(p-1)^{p-2}}{p(p-2)} x^p.$$

This yields the majorization (3.12) for $y \leq b/2$.

Proof. The estimate is clear for $x = 0$. If $x > 0$ and we divide both sides by x^p , the inequality takes the form $F_0(2y/x) \geq 0$, where

$$F_0(s) := \frac{2}{p(p-1)(p-2)} s^{p-1} (s-p) + \frac{2(p-1)^{p-2}}{p(p-2)}$$

for $s > 0$. It suffices to note that F_0 is convex and $F_0(p-1) = F'_0(p-1) = 0$. $\square$

Lemma 4.2. (i) *For all $s \geq 0$,*

$$(4.2) \quad b^{p-2} \left(-\frac{s}{p-2} + \frac{b^2}{p(p-1)^2} \right) + \frac{2(p-1)^{p-2}}{p(p-2)} s^{p/2} \geq 0.$$

(ii) *We have*

$$(4.3) \quad b^{p-2} \left[\frac{4y^2 - 4xy}{p-2} - \frac{2b(2y-x)}{p-1} + \frac{b^2}{p} \right] + \frac{2(p-1)^{p-2}}{p(p-2)} x^p \geq 0.$$

This yields the majorization (3.12) for $y \in (b/2, 1)$.

Proof. (i) Denote the left-hand side of (4.2) by $F_1(s)$. It is evident that the function F_1 is convex on $\mathbb{R}$. In addition, it is straightforward to check that

$$(4.4) \quad F_1(b^2/(p-1)^2) = F'_1(b^2/(p-1)^2) = 0.$$

The claim follows.

(ii) The partial derivative of the left-hand side of (4.3) with respect to y equals

$$\frac{4b^{p-2}}{p-2} \left(2y - x - \frac{b(p-2)}{p-1} \right).$$

Thus it suffices to verify the estimate for $2y = x + b(p-2)/(p-1)$. Plug this into (4.3) to get the inequality $F_1(x^2) \geq 0$, which has already been proved in (i). $\square$

Lemma 4.3. *If $y \geq 1$, then*

$$(4.5) \quad b^{p-2} \left[\frac{4y^2 - 4xy}{p-2} - \frac{2b(2y-x)}{p-1} + \frac{b^2}{p} \right] - \frac{2(p-1)^{p-2}}{p(p-2)} \left(\frac{1}{C_p^p} - x^p \right) \geq 0.$$

This yields the majorization (3.12) for $y \geq 1$.

Proof. We divide the proof into three parts.

Step 1. A reduction. Denoting the left-hand side of (4.5) by $F_2(x, y)$, we derive that its partial derivative with respect to y is given by

$$F_{2y}(x, y) = \frac{4b^{p-2}}{p-2} \left(2y - x - \frac{b(p-2)}{p-1} \right).$$

Consequently, it suffices to establish the estimate on the line segment

$$H_1 = \left\{ (x, y) : y = 1, x \geq 0, 2y - x - \frac{b(p-2)}{p-1} \geq 0 \right\}$$

and the half-line

$$H_2 = \left\{ (x, y) : y > 1, 2y - x - \frac{b(p-2)}{p-1} = 0 \right\}.$$

Step 2. The segment H_1 . It is obvious that $x \mapsto F_2(x, 1)$ is convex on the interval $[0, 2 - b(p-2)/(p-1)]$. After some lengthy but easy calculations we verify that

$$1 - \frac{b}{2} < 2 - \frac{b(p-2)}{p-1} \quad \text{and} \quad F_2 \left(1 - \frac{b}{2}, 1 \right) = F_2' \left(1 - \frac{b}{2}, 1 \right) = 0,$$

which yields the estimate on H_1 .

Step 3. The half-line H_2 . Plugging $2y = x + b(p-2)/(p-1)$ into the estimate transforms it into

$$(4.6) \quad b^{p-2} \left(-\frac{x^2}{p-2} + \frac{b^2}{p(p-1)^2} \right) + \frac{2(p-1)^{p-2}}{p(p-2)} x^p \geq \frac{2(p-1)^{p-2}}{p(p-2)C_p^p}.$$

The left-hand side is equal to $F_1(x^2)$, where F_1 was defined in the proof of Lemma 4.2. Note that we have

$$x \geq 2 - \frac{b(p-2)}{p-1} \geq \frac{b}{p-1},$$

the latter being equivalent to $b \leq 2$, which is obvious. Thus, by the convexity of F_1 and (4.4), we get that the left-hand side of (4.6) attains its minimum at $x = 2 - b(p-2)/(p-1)$. However, then the estimate reads $F_2(2 - b(p-2)/(p-1)) \geq 0$, and we have already showed this in the preceding step. The proof is complete. $\square$

In the lemma below we show (3.12) for $y \leq -1$.

Lemma 4.4. *Let $p > 2$ and $y \leq -1$.*

(i) *We have*

$$(4.7) \quad b^{p-1} \left[-\frac{2(p-1)y}{b} + p - 2 \right] - (2(p-1))^{p-1} |y|^{p-1} \leq 0.$$

(ii) *For any $(x, y) \in \mathbb{R}^2$ satisfying $x \leq 2y$ we have*

$$(4.8) \quad b^{p-2} \left[\frac{4y^2 - 4xy}{p-2} - \frac{2b(2y-x)}{p-1} + \frac{b^2}{p} \right] - \frac{2(p-1)^{p-2}}{p(p-2)} \left(\frac{1}{C_p^p} - |x|^p \right) \geq 0.$$

Proof. (i) Denote the left-hand side by $F_3(y)$. We have that

$$F'_3(y) = (2(p-1))^{p-1} \left[-\left(\frac{b}{2(p-1)} \right)^{p-2} + |y|^{p-2}(p-1) \right],$$

and since $b/(2(p-1)) = (2c+p-1)^{-1} < 1$ and $y \leq -1$, we see that the expression in the square brackets is positive. Thus F_3 is increasing and it suffices to show that $F_3(-1) \leq 0$. We have

$$\begin{aligned} F_3(-1) &= b^{p-1} \left[\frac{2(p-1)}{b} + p - 2 \right] - (2(p-1))^{p-1} \\ &= b^{p-1} [2c + 2p - 3 - (2c + p - 1)^{p-1}]. \end{aligned}$$

It follows from a standard analysis that the function $x \mapsto x + 2p - 3 - (x + p - 1)^{p-1}$ is decreasing on $[0, \infty)$. Using this and the trivial estimate $c \geq 1$ yields

$$2c + 2p - 3 - (2c + p - 1)^{p-1} \leq 2p - 1 - (p + 1)^{p-1}.$$

It is easy to check that the right-hand side, as a function of $p \geq 2$, is decreasing and equal to 0 at 2. This completes the proof of (4.7).

(ii) Let $y \leq -1$ be fixed and denote the left-hand side by $F_4(x)$. Clearly, the function F_4 is convex on $(-\infty, 2y]$ and satisfies $F_4(2y) \leq 0$, by means of Lemma 4.3 (since $F_4(2y) = F_2(-2y, 1)$). Therefore, we will be done if we show that $F'_4(2y-) \leq 0$. However, this inequality is equivalent to (4.7), which we have already established in (i). $\square$

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