BORSUK-ULAM TYPE THEOREMS FOR MANIFOLDS

OLEG R. MUSIN

(Communicated by Alexander N. Dranishnikov)

ABSTRACT. This paper establishes a Borsuk-Ulam type theorem for PL-manifolds with a finite group action, depending on the free equivariant cobordism class of a manifold. In particular, necessary and sufficient conditions are considered for a manifold with a free involution to be of Borsuk-Ulam type.

1. INTRODUCTION

The Borsuk-Ulam theorem states that any continuous antipodal $f : \mathbb{S}^n \rightarrow \mathbb{R}^n$ has zeros. (A map f is called *antipodal* if $f(-x) = -f(x)$.) One of the most interesting proofs of this theorem is Bárány’s geometric proof [1] (see also [9, Sec. 2.2]). A very similar proof was given in [10], and [13] has several more references for proofs of this type.

Let $X = \mathbb{S}^n \times [0, 1]$, $X_0 = \mathbb{S}^n \times \{0\}$, and $X_1 = \mathbb{S}^n \times \{1\}$. Let $\tau(x, t) = (-x, t)$, where $(x, t) \in X$, $x \in \mathbb{S}^n$, and $t \in [0, 1]$. Clearly, τ is a free involution on X .

The first step of Bárány’s proof is to show that any continuous antipodal (i.e. $F(\tau(x)) = -F(x)$) map $F : X \rightarrow \mathbb{R}^n$ can be approximated by “sufficiently generic” antipodal maps (see [9, Sec. 2.2]).

Let $f_i : \mathbb{S}^n \rightarrow \mathbb{R}^n$, where $i = 0, 1$, be antipodal generic maps. Let $F(x, t) = tf_1(x) + (1 - t)f_0(x)$. Since F is generic, the set $Z_F := F^{-1}(0)$ is a manifold of dimension one. Then Z_F consists of arcs $\{\gamma_k\}$ with ends in $Z_{f_i} := Z_F \cap X_i = f_i^{-1}(0)$ and cycles which do not intersect X_i . Note that $\tau(Z_F) = Z_F$ and $\tau(\gamma_i) = \gamma_j$ with $i \neq j$. Therefore, (Z_F, Z_{f_0}, Z_{f_1}) is a $\mathbb{Z}_2$ -cobordism. It is not hard to see that Z_{f_0} is $\mathbb{Z}_2$ -cobordant to Z_{f_1} if and only if $|Z_{f_1}| = |Z_{f_0}| \pmod{4}$.

To complete the proof, take f_0 as the standard orthogonal projection of $\mathbb{S}^n$ onto $\mathbb{R}^n$:

$$f_0(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n), \text{ where } x_1^2 + \dots + x_{n+1}^2 = 1.$$

Since $|Z_{f_0}| = 2$, we have $|Z_{f_1}| = 2 \pmod{4}$. This equality shows that for any antipodal generic f_1 the set $Z_{f_1} = f_1^{-1}(0)$ is not empty.

Our analysis of this proof shows that it can be extended for a wide class of manifolds. For instance, consider two-dimensional orientable manifolds $N = M_g^2$ of even genus g and nonorientable manifolds $N = P_m^2$ with even m . Without loss of generality, we can assume that N is “centrally symmetric” embedded to $\mathbb{R}^k$, where $k = 3$ for $N = M_g^2$ and $k = 4$ for $N = P_m^2$. That means $A(N) = N$, where

Received by the editors December 4, 2009 and, in revised form, May 24, 2010; December 18, 2010; January 13, 2011; January 18, 2011; January 23, 2011; and February 24, 2011.

2010 *Mathematics Subject Classification.* Primary 55M35, 55M99, 57R85.

Key words and phrases. Borsuk-Ulam theorem, group action, equivariant cobordism.

This research was supported in part by NSF grant DMS-0807640 and NSA grant MSPF-08G-201.

$A(x) = -x$ for $x \in \mathbb{R}^k$. Then $T := A|_N : N \rightarrow N$ is a free involution. It can be shown that there is a projection of $N \subset \mathbb{R}^k$ into a 2-plane R passing through the origin 0 with $|Z_{f_0}| = 2$. (See the details in Corollary 1.)

Actually, Bárány's proof can be (almost word for word) applied for N . Namely, the following statement holds:

Let $N = M_g^2$ with even g or $N = P_m^2$ with even m . Then for any continuous $h : N \rightarrow \mathbb{R}^2$ there is $x \in N$ such that $h(T(x)) = h(x)$.

Note that N can be represented as a connected sum $M \# M$. This statement can be extended for $N = M \# M$ with any closed manifold M (see Corollary 1).¹ We see that one of the most important steps here is the existence of a generic equivariant $f_0 : N^n \rightarrow \mathbb{R}^n$ with $|Z_{f_0}| = 2 \pmod{4}$.

This approach works for any free group action. Namely, for both technical steps, a “generic approximation lemma” and a “ (Z_F, Z_{f_0}, Z_{f_1}) cobordism lemma” can be extended for free actions of a finite group G (Section 2).

Let G be a finite group acting free on a closed connected PL-manifold M^m and linearly on $\mathbb{R}^n$. In Section 3 we show a Borsuk-Ulam type theorem for M , depending on the free equivariant cobordism class of M (Theorem 1). For the case $m = n$, Theorem 1 shows that if there is a continuous equivariant transversal to zeros $h : M^n \rightarrow \mathbb{R}^n$ with $|Z_h| = |G| \pmod{2|G|}$, then for any continuous equivariant $f : M^n \rightarrow \mathbb{R}^n$, the zero set Z_f is not empty.

In Section 4, this approach is applied to the classical case: manifolds with free involutions. There we give necessary and sufficient conditions for $\mathbb{Z}_2$ -manifolds to have a Borsuk-Ulam type theorem (Theorem 2).

Lemma 4 shows that the zero set Z_f is invariant up to G -cobordisms. Based on this fact, in Section 5 we define homomorphisms μ of free G -cobordism groups which can be considered as obstructions for G -maps. Theorem 3 shows that for a free G -manifold M^m and a given linear action G on $\mathbb{R}^n$, if this invariant is not zero in the free equivariant cobordisms, then for any continuous equivariant $f : M^m \rightarrow \mathbb{R}^n$ the set Z_f is not empty. In this section we also provide some applications of Theorem 3 for the case $G = (\mathbb{Z}_2)^k$.

The main goal of this paper is to show that the geometric proof gives a method for checking whether a G -manifold M is of the Borsuk-Ulam type. Namely, from Theorem 1 and its extension, Corollary 2, it follows that if there exists a generic equivariant $h : M^m \rightarrow \mathbb{R}^n$ with $[Z_h] \neq 0$ in the correspondent group of cobordisms, then for any continuous equivariant $f : M^m \rightarrow \mathbb{R}^n$ the set of zeros $Z_f \neq \emptyset$.

2. GENERIC G -MAPS

If G is a group and X is a set, then a group action of G on X is a binary function $G \times X \rightarrow X$ denoted by $(g, x) \rightarrow g(x)$ which satisfies the following two axioms: (1) $(gh)(x) = g(h(x))$ for all g, h in G and x in X ; (2) $e(x) = x$ for every x in X (where e denotes the identity element of G). The set X is called a G -set. The group G is said to act on X .

Let a finite group G act on a set X . We say that $Y \subset X$ is a G -subset if $g(y) \in Y$ for all $g \in G$ and $y \in Y$, i.e. $G(Y) = Y$. Clearly, for any $x \in X$ the orbit $G(x) := \{g(x), g \in G\}$ is a G -subset. Denote by $\text{Orb}(X, G)$ the set of orbits of G on X , i.e. $\text{Orb}(X, G) = \{G(x)\}$.

¹It is not hard to prove the corollary using the standard techniques developed in [4]. However, we could not find this kind of statement in the Borsuk-Ulam theorem literature.

Let $G_x := \{g \in G \mid g(x) = x\}$ denote the *stabilizer* (or *isotropy subgroup*). Let $X^H = \{x \in X \mid g(x) = x, \forall g \in H\}$ denote the fixed point set of a subgroup $H \subset G$. If H is isomorphic to $G_x \neq e$, then $X^H \neq \emptyset$.

Recall that a group action is called *free* if $G_x = \{e\}$ for all x ; i.e. $g(x) = x$ if and only if $g = e$. Note that for a free action G on X each orbit $G(x)$ consists of $|G|$ points.

Here we consider *piecewise linear* (or *PL*, or *simplicial*) G -manifolds X ; i.e. there is a triangulation Λ of X such that for any $g \in G$, any simplex σ of Λ is mapped bijectively onto the simplex $g(\sigma)$. The triangulation Λ is called *equivariant*. Actually, any smooth G -manifold admits an equivariant triangulation [8].

Every free action of a finite group G on a compact PL-manifold X admits an equivariant triangulation such that for each simplex there are no vertices in the same orbit. Otherwise, we indeed can subdivide these simplices.

A map $f : X \rightarrow Y$ of G -sets X and Y is called *equivariant* (or a G -map) if $f(g(x)) = g(f(x))$ for all $g \in G, x \in X$.

For a PL G -space (in particular, for a PL G -manifold) we say that an equivariant map $f : X \rightarrow \mathbb{R}^n$ is *simplicial* if f is a linear map for each simplex $\sigma \in \Lambda$. For an equivariant triangulation Λ , any simplicial map $f : X \rightarrow \mathbb{R}^n$ is uniquely determined by the set of vertices $V(\Lambda)$. Indeed, for each simplex σ of Λ , f is linear and therefore is determined by the vertices of σ .

Any equivariant continuous map can be approximated by an equivariant simplicial map.

Lemma 1. *Let G be a finite group acting linearly on $\mathbb{R}^n$. Let X be a compact simplicial G -space. Then for any equivariant continuous $f : X \rightarrow \mathbb{R}^n$ and $\varepsilon > 0$, there is an equivariant simplicial map $\bar{f} : X \rightarrow \mathbb{R}^n$ such that $\|f(x) - \bar{f}(x)\| \leq \varepsilon$ for all $x \in X$.*

Proof. It is well known (see, for instance, [7, Section 2.2]) that any continuous map on a compact PL-complex can be approximated by simplicial maps. Therefore there exists a triangulation Λ and a simplicial map $\hat{f}$ such that $\|f(x) - \hat{f}(x)\| \leq \varepsilon$ for all $x \in X$. Clearly, there exists an equivariant subdivision Λ' of Λ with $G(V(\Lambda)) \subset V(\Lambda')$. For $v \in V(\Lambda')$, let $\bar{f}(v) := f(v)$. Then $\bar{f}|_{V(\Lambda')}$ defines a simplicial map with $\|f(x) - \bar{f}(x)\| \leq \varepsilon$. $\square$

Let $\rho : G \rightarrow \text{GL}(n, \mathbb{R})$ be a representation of a group G on $\mathbb{R}^n$. In other words, G is acting linearly on $\mathbb{R}^n$. Then $F := (\mathbb{R}^n)^G$ is a linear subspace of $\mathbb{R}^n$. Denote by L an invariant linear subspace of $\mathbb{R}^n$ which is transverse to F . Then $L^G = \{0\}$. So without loss of generality we can consider only linear actions with $(\mathbb{R}^n)^G = \{0\}$, i.e. linear actions on $\mathbb{R}^n$ such that the fixed point set of these actions is the origin.

Let $f : X \rightarrow \mathbb{R}^n$. Denote by Z_f the set of zeros, i.e.

$$Z_f = \{x \in X : f(x) = 0\}.$$

In this paper we need generic maps with respect to Z_f . Let X be a simplicial m -dimensional G -manifold. Set $O_\varepsilon := \{v \in \mathbb{R}^n : \|v\| < \varepsilon\}$. A continuous equivariant $f : X^m \rightarrow \mathbb{R}^n$ is called *transversal to zeros* if Z_f is a manifold of dimension $m - n$, and there is an $\varepsilon > 0$ such that for any $v \in O_\varepsilon$, the sets $f^{-1}(v)$ and Z_f are homeomorphic.

Definition. Given a finite group G , let X^m be a PL free G -manifold. Let Λ be an equivariant triangulation of X . We say that an equivariant simplicial map

$f : X \rightarrow \mathbb{R}^n$ is *generic* (with respect to zeros) if Z_f does not intersect the $(n-1)$ -skeleton $\Lambda(n-1)$ of Λ . (Recall that the k -skeleton of Λ is the subcomplex $\Lambda(k)$ that consists of all simplices of dimension at most k .)

All simplicial generic maps are transversal to zeros.

Lemma 2. *Let G be a finite group acting linearly on $\mathbb{R}^n$ with $(\mathbb{R}^n)^G = \{0\}$. Let X^m be a PL compact m -dimensional free G -manifold with or without boundary. Let $f : X \rightarrow \mathbb{R}^n$ be an equivariant simplicial generic map. If $n \leq m$, then Z_f is an invariant submanifold of X of dimension $m-n$. Moreover, if the boundary ∂Z_f is not empty, then it lies in ∂X .*

Proof. Note that $Z_f \subset X$ is a locally polyhedral surface consisting of $(m-n)$ -dimensional cells. This is because for each m -simplex σ we have the linear map $f|_\sigma : \sigma \rightarrow \mathbb{R}^n$, and $Z_f \cap \sigma$ is defined by the generic linear equation $f(x) = 0$. Hence, the components of Z_f are PL-manifolds. $\square$

For G -spaces, the Tietze-Gleason theorem states: *Let a compact group G act on X with a closed invariant set A . Let G also act linearly on $\mathbb{R}^n$. Then any equivariant $f : A \rightarrow \mathbb{R}^n$ extends to $f : X \rightarrow \mathbb{R}^n$ (see [3, Theorem 2.3]).*

It is known from results of Bierstone [2] and Field [6] on G -transversality theory that the zero set is a stratified set. If G acts freely on a compact smooth G -manifold X , then this theory implies that the set of nongeneric equivariant smooth $f : X^m \rightarrow \mathbb{R}^n$ has measure zero in the space of all smooth G -maps. Let us extend this result as well as the Tietze-Gleason theorem for the PL case.

Lemma 3. *Let G be a finite group acting linearly on $\mathbb{R}^n$ with $(\mathbb{R}^n)^G = \{0\}$. Let X^m be a PL compact free G -manifold with or without boundary. Let Λ be an equivariant triangulation of X . Let A be an invariant subset of $V(\Lambda)$ or let $A = \emptyset$. Let $h : A \rightarrow \mathbb{R}^n$ be a given equivariant generic map. Then the set of nongeneric simplicial maps $f : X \rightarrow \mathbb{R}^n$ with $f|_A = h$ has measure zero in the space of all possible equivariant maps $f : V(\Lambda) \rightarrow \mathbb{R}^n$ with $f|_A = h$.*

Proof. Let $V' := V(\Lambda) \setminus A$. Let $C \in \text{Orb}(V', G)$ be an orbit of G on V' . Then $|C| = p$, where $p := |G|$. Since for any equivariant $f : V(\Lambda) \rightarrow \mathbb{R}^n$ and $v \in V(\Lambda)$ we have $f(g(v)) = g(f(v))$, the vector $u = f(v) \in \mathbb{R}^n$ yields the map $f : C \rightarrow \mathbb{R}^n$.

Suppose G has k orbits on V' . Take in each orbit $C_i \in \text{Orb}(V', G)$ a vertex x_i . Denote by M_G the space of all equivariant maps $f : V' \rightarrow \mathbb{R}^n$. Then the space M_G is of dimension $N = kn$.

Consider an $(n-1)$ -simplex $\sigma \in \Lambda$ with vertices $v_1, \dots, v_n$ which are not all from A . Note that $f \in M_G$ is not generic on σ if $0 \in f(\sigma)$. In particular, the hyperplane in $\mathbb{R}^n$ which is defined by vectors $f(v_1), \dots, f(v_n)$ passes through the origin 0. In other words, the determinant of n vectors $f(v_1), \dots, f(v_n)$ equals zero.

This constraint gives a proper algebraic subvariety $s(\sigma)$ in M_G . Let $v_1, \dots, v_d$ be vertices of σ which are not in A . Then $s(\sigma)$ is of dimension $N-1$. Indeed, $u_i = f(v_i)$, $i = 1, \dots, d$, are vectors in $\mathbb{R}^n$. Then the equation

$$\det(u_1, \dots, u_d, h(v_{d+1}), \dots, h(v_n)) = 0$$

defines the subvariety $s(\sigma)$ in M_G of dimension $N-1$.

Since the union of all subvarieties $s(\sigma)$ with $\dim \sigma = n-1$ is at most of dimension $N-1$, it has measure zero in M_G . From this it follows that the set of all nongeneric f has measure zero in this space. $\square$

Let M_0 and M_1 be closed m -dimensional simplicial manifolds with free actions of a finite group G . Then an $(m+1)$ -dimensional simplicial free G -manifold W is called a *free G -cobordism* (W, M_0, M_1) if the boundary ∂W consists of M_0 and M_1 , and the action G on W respects actions on M_i .

Lemma 4. *Let G be a finite group acting linearly on $\mathbb{R}^n$ with $(\mathbb{R}^n)^G = \{0\}$. Let (W, M_0, M_1) be a free G -cobordism. Let $f_i : M_i^m \rightarrow \mathbb{R}^n$, $i = 0, 1$, be equivariant simplicial generic maps. Then there is an equivariant simplicial generic $F : W \rightarrow \mathbb{R}^n$ with $F|_{M_i} = f_i$ and such that (Z_F, Z_{f_0}, Z_{f_1}) is a free G -cobordism.*

Proof. The existence of such a generic F follows from Lemma 3. Lemma 2 implies that Z_F is a G -submanifold of W . Clearly, $g(Z_F) = Z_F$ and $g(Z_{f_i}) = Z_{f_i}$ for all $g \in G$. Since $\partial Z_F = Z_{f_0} \sqcup Z_{f_1}$ is an $(m-n)$ -dimensional G -manifold, we have a free G -cobordism (Z_F, Z_{f_0}, Z_{f_1}) . $\square$

3. G -MAPS AND A BORSUK-ULAM TYPE THEOREM

In this section we consider a Borsuk-Ulam type theorem for the case $m = n$.

Definition. Given a finite group G acting freely on a closed PL-manifold M^n and acting linearly on $\mathbb{R}^n$ with $(\mathbb{R}^n)^G = \{0\}$, let $f : M^n \rightarrow \mathbb{R}^n$ be a continuous equivariant transversal to zeros map. Since Z_f is a finite free G -invariant subset of M , we have $|Z_f| = k|G|$, where $k = 0, 1, 2, \dots$. Set $\deg_G(f) := 1$ if k is odd, and $\deg_G(f) := 0$ if k is even.

Lemma 5. *Let G be a finite group acting linearly on $\mathbb{R}^n$ with $(\mathbb{R}^n)^G = \{0\}$. Let (W^{n+1}, M_0^n, M_1^n) be a free G -cobordism. Let $h_i : M_i^n \rightarrow \mathbb{R}^n$, $i = 0, 1$, be continuous equivariant transversal to zeros maps. Then $\deg_G(h_0) = \deg_G(h_1)$.*

Proof. Lemma 1 and Lemma 3 yield that for any $h_i : M_i^n \rightarrow \mathbb{R}^n$ and $\varepsilon > 0$ there is a generic map $a_{i,\varepsilon}$ such that $\|a_{i,\varepsilon}(x) - h_i(x)\| \leq \varepsilon$ for all $x \in M_i$. If $\varepsilon \rightarrow 0$, then $Z_{i,\varepsilon} := a_{i,\varepsilon}^{-1}(0) \rightarrow Z_{h_i}$. This implies that for a sufficiently small ε there is an equivariant bijection between $Z_{i,\varepsilon}$ and Z_{h_i} . Therefore, $\deg_G(a_{i,\varepsilon}) = \deg_G(h_i)$.

Let ε be sufficiently small. Set $f_i := a_{i,\varepsilon}$. From Lemma 4 it follows that there is an equivariant simplicial generic $F : W \rightarrow \mathbb{R}^n$ with $F|_{M_i} = f_i$ such that (Z_F, Z_{f_0}, Z_{f_1}) is a free G -cobordism. We have $m = n$. Then Z_F is a submanifold of W of dimension one. Therefore Z_F consists of arcs $\gamma_1, \dots, \gamma_\ell$ with ends in Z_{f_i} and cycles which do not intersect M_i .

Since any continuous map $s : \gamma_k \rightarrow \gamma_k$ has a fixed point, G cannot act freely on γ_k . Therefore, G acts freely on the set of all arcs $\{\gamma_k\}$. Moreover, the ends of any arc cannot lie in the same orbit of G . From this it follows that $|Z_{f_1}| = |Z_{f_0}| \pmod{2|G|}$, i.e. $\deg_G(f_0) = \deg_G(f_1)$. $\square$

Theorem 1. *Let G be a finite group acting linearly on $\mathbb{R}^n$ with $(\mathbb{R}^n)^G = \{0\}$. Let M^n be a closed connected PL free G -manifold. If there is a closed PL free G -manifold N^n which is free G -cobordant to M^n and a continuous equivariant transversal to zeros $h : N^n \rightarrow \mathbb{R}^n$ with $\deg_G(h) = 1$, then for any continuous equivariant $f : M^n \rightarrow \mathbb{R}^n$ the zero set Z_f is not empty.*

Proof. Let $\deg_G(h) = 1$. Suppose that $Z_f = \emptyset$. Since M is compact, there is an $\varepsilon > 0$ such that $\|f(x)\| \geq \varepsilon$ for all $x \in M$. From Lemma 3 it follows that there exists a generic $\tilde{f}$ such that $\|f(x) - \tilde{f}(x)\| \leq \varepsilon/2$ for all $x \in M$. Then $\|\tilde{f}(x)\| \geq \varepsilon/2$

for all $x \in M$ and $Z_{\tilde{f}} = \emptyset$. Therefore, $\deg_G(\tilde{f}) = 0$. On the other hand, Lemma 5 implies that $0 = \deg_G(\tilde{f}) = \deg_G(h) = 1$, a contradiction. $\square$

Remark. Actually, Lemma 5 immediately implies that the assumption in the theorem: *There is a free G -manifold N^n which is free G -cobordant to M^n and a continuous equivariant transversal to zeros $h : N^n \rightarrow \mathbb{R}^n$ with $\deg_G(h) = 1$* is equivalent to the following statement: *There is a continuous equivariant transversal to zeros map $h : M^n \rightarrow \mathbb{R}^n$ with $\deg_G(h) = 1$.* However, since the assumption in the theorem is more general, it sometimes can be more easily checked for N which are free G -cobordant to M . (For instance, see our proof of Theorem 2.)

4. BUT MANIFOLDS

In this section, we consider the classical case $G = \mathbb{Z}_2$. Let M be a closed PL-manifold with a free simplicial involution $T : M \rightarrow M$, i.e. $T^2(x) = x$ and $T(x) \neq x$ for all $x \in M$. For any $\mathbb{Z}_2$ -manifold (M, T) we say that a map $f : M^n \rightarrow \mathbb{R}^n$ is *antipodal* (or equivariant) if $f(T(x)) = -f(x)$.

Definition. We say that a closed PL free $\mathbb{Z}_2$ -manifold (M^n, T) is a *BUT (Borsuk-Ulam Type) manifold* if for any continuous $g : M^n \rightarrow \mathbb{R}^n$ there is a point $x \in M$ such that $g(T(x)) = g(x)$. Equivalently, if a continuous map $f : M^n \rightarrow \mathbb{R}^n$ is antipodal, then Z_f is not empty.

Let us recall several facts about $\mathbb{Z}_2$ -cobordisms which are related to our main theorem in this section. We write $\mathfrak{N}_n$ for the group of unoriented cobordism classes of n -dimensional manifolds. Thom's cobordism theorem says that the graded ring of cobordism classes $\mathfrak{N}_*$ is $\mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, x_9, \dots]$ with one generator x_k in each degree k not of the form $2^i - 1$. Note that $x_{2k} = [\mathbb{RP}^{2k}]$.

Let $\mathfrak{N}_*(\mathbb{Z}_2)$ denote the unoriented cobordism group of free involutions. Then $\mathfrak{N}_*(\mathbb{Z}_2)$ is a free $\mathfrak{N}_*$ -module with basis $[\mathbb{S}^n, A]$, $n \geq 0$, where $[\mathbb{S}^n, A]$ is the cobordism class of the antipodal involution on the n -sphere [5, Theorem 23.2]. Thus, each $\mathbb{Z}_2$ -manifold can be uniquely represented in $\mathfrak{N}_n(\mathbb{Z}_2)$ in the form

$$[M, T] = \sum_{k=0}^n [V^k][\mathbb{S}^{n-k}, A].$$

Theorem 2. *Let M^n be a closed connected PL-manifold with a free simplicial involution T . Then the following statements are equivalent:*

- (a) *M is a BUT manifold.*
- (b) *M admits an antipodal continuous transversal to zeros map $h : M^n \rightarrow \mathbb{R}^n$ with $\deg_{\mathbb{Z}_2}(h) = 1$.*
- (c) *$[M^n, T] = [\mathbb{S}^n, A] + [V^1][\mathbb{S}^{n-1}, A] + \dots + [V^n][\mathbb{S}^0, A]$ in $\mathfrak{N}_n(\mathbb{Z}_2)$.*

Proof. (1) First we prove that (c) is equivalent to (b). Let

$$[M, T] = [V^0][\mathbb{S}^n, A] + [V^1][\mathbb{S}^{n-1}, A] + \dots + [V^n][\mathbb{S}^0, A].$$

Denote by

$$N := V^1 \times \mathbb{S}^{n-1} \sqcup \dots \sqcup V^n \times \mathbb{S}^0.$$

Note that for any k , where $0 \leq k < n$, the standard antipodal embedding of $\mathbb{S}^k$ into $\mathbb{R}^n$ has no zeros. This implies that there is a generic antipodal $p : N^n \rightarrow \mathbb{R}^n$ with $Z_p = \emptyset$. If $[V^0] = 0$, then M is free $\mathbb{Z}_2$ -cobordant to N . Therefore, Lemma 5

yields that for any generic antipodal $f : M \rightarrow \mathbb{R}^n$ we have $\deg_{\mathbb{Z}_2}(f) = \deg_{\mathbb{Z}_2}(p) = 0$. This contradicts (b), and thus $[V^0] = 1 \in \mathbb{Z}_2$.

On the other hand, if $[V^0] = 1$, then M is free $\mathbb{Z}_2$ -cobordant to $L := \mathbb{S}^n \sqcup N$. Take any generic antipodal $h : M^n \rightarrow \mathbb{R}^n$. Since for any generic antipodal $g : L \rightarrow \mathbb{R}^n$ we have $\deg_{\mathbb{Z}_2}(g) = 1$, it again follows from Lemma 5 that $\deg_{\mathbb{Z}_2}(h) = 1$.

(2) Theorem 1 yields: (b) implies (a).

(3) (a) implies (c). In fact, this follows from [12, Theorem 3].² This theorem yields that if $w_1^n(M/\mathbb{Z}_2) = 0 \in H^n(M/\mathbb{Z}_2, \mathbb{Z}_2)$, then there is an antipodal continuous map $h : M^n \rightarrow \mathbb{S}^{n-1}$, i.e. $Z_h = \emptyset$. Therefore, $w_1^n(M/\mathbb{Z}_2) \neq 0$. This holds if and only if we have (c). $\square$

Remark. Actually, the list of equivalent versions in Theorem 2 can be continued. Tucker's lemma is a discrete version of the Borsuk-Ulam theorem:

Let Λ be any equivariant triangulation of $\mathbb{S}^n$. Let

$$L : V(\Lambda) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$$

be an equivariant (or Tucker) labelling, i.e. $L(T(v)) = -L(v)$. Then there exists a complementary edge in Λ such that its two vertices are labelled by opposite numbers (see [9, Theorem 2.3.1]).

Let M be as in Theorem 2. Using the same arguments as in [9, Theorem 2.3.2], it can be proved that the following statement is equivalent to (a):

(d) For any equivariant labelling of an equivariant triangulation of M there is a complementary edge.

In fact, any equivariant labelling L defines a simplicial map $f_L : M \rightarrow \mathbb{R}^n$. Indeed, let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. We define $f_L : \Lambda \rightarrow \mathbb{R}^n$ for $v \in V(\Lambda)$ by $f_L(v) = e_i$ if $L(v) = i$ and $f_L(v) = -e_i$ if $L(v) = -i$. In the paper [11], it is shown that M is a BUT manifold if and only if

(e) There exist an equivariant triangulation Λ of M and an equivariant labelling of $V(\Lambda)$ such that $f_L : \Lambda \rightarrow \mathbb{R}^n$ is transversal to zeros and the number of complementary edges is $4k + 2$, where $k = 0, 1, 2, \dots$.

Lyusternik and Shnirel'man proved in 1930 that for any cover $F_1, \dots, F_{n+1}$ of the sphere $\mathbb{S}^n$ by $n + 1$ closed sets, there is at least one set containing a pair of antipodal points (that is, $F_i \cap (-F_i) \neq \emptyset$). Equivalently, for any cover $U_1, \dots, U_{n+1}$ of $\mathbb{S}^n$ by $n + 1$ open sets, there is at least one set containing a pair of antipodal points [9, Theorem 2.1.1]. By the same arguments it can be shown that M is a BUT manifold if and only if

(f) M is a Lyusternik-Shnirel'man type manifold; i.e. for any cover $F_1, \dots, F_{n+1}$ of M^n by $n + 1$ closed (respectively, by $n + 1$ open) sets, there is at least one set containing a pair $(x, T(x))$.

Denote by $\text{cat}(X)$ the Lyusternik-Shnirel'man category of a space X , i.e. the smallest m such that there exists an open covering $U_1, \dots, U_{m+1}$ of X with each U_i contractible to a point in X . It is not hard to prove the last statement in this remark:

(g) M is a BUT manifold if and only if $\text{cat}(M/T) = \text{cat}(\mathbb{R}\mathbb{P}^n) = n$.

Usually, it is not easy to find $\text{cat}(X)$. Here, Theorems 2–4 yield interesting possibilities to find lower bounds for $\text{cat}(M/G)$.

²I would like to thank Alexey Volovikov, who noted this theorem. He as well as Pavle Blagojević and Roman Karasev also sent me another proof of [12, Theorem 3].

Corollary 1. *Let M be any closed manifold. Then for the connected sum $M \# M$ there exists a “centrally symmetric” free involution such that $M \# M$ is a BUT manifold.*

Proof. The Whitney embedding theorem states that any smooth or simplicial n -dimensional manifold can be embedded in Euclidean $2n$ -space. Consider an embedding of M in $\mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_{2n})$. Let M_- (respectively, M_+) denote the set of points in $M \subset \mathbb{R}^{2n}$ with $x_1 < 0$ (respectively, $x_1 > 0$). Let M_0 be the set of points in M with $x_1 = 0$. Without loss of generality we can assume that M is embedded in $\mathbb{R}^{2n}$ in such a way that M_- is homeomorphic to an open n -ball and M_0 is a sphere $x_1^2 + \dots + x_{n+1}^2 = 1$ with $x_{n+2} = \dots = x_{2n} = 0$. Then the central symmetry $s(s(x) = -x)$ is well defined on $X := s(M_-) \cup M_0 \cup M_+$ as a free involution, and X is homeomorphic to $M \# M$. Consider the projection h of X onto the n -plane $x_{n+1} = \dots = x_{2n} = 0$, i.e. $h(x_1, \dots, x_{2n}) = (x_1, \dots, x_n)$. Since $Z_h \subset M_0$, we have $Z_h = Z_t$, where $t := h|_{M_0}$. On the other hand, $t : M_0 \rightarrow \mathbb{R}^n$ is an orthogonal projection of $\mathbb{S}^n$. Thus, $|Z_h| = 2$ and Theorem 2(b) implies that $M \# M$ is a BUT manifold. $\square$

5. OBSTRUCTIONS FOR G -MAPS IN COBORDISMS

In the previous sections, we considered equivariant maps $f : M^m \rightarrow \mathbb{R}^n$ with $m = n$. Now we extend this approach to the case $m \geq n$.

Consider closed PL manifolds with an H -structure (such as an orientation). One can define a “cobordism with H -structure”, but there are various technicalities. In each particular case, cobordism is an equivalence relation on manifolds. A basic question is to determine the equivalence classes for this relationship, called the cobordism classes of manifolds. These form a graded ring called the cobordism ring Ω_*^H , with grading by dimension, addition by disjoint union, and multiplication by Cartesian product.

Let $\Omega_*^H(G)$ denote the PL cobordism group with H -structure of free simplicial actions of a finite group G . Let $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{R})$ be a representation of a group G on $\mathbb{R}^n$ which also has H -structure. Lemma 4 shows that for a generic simplicial equivariant map $f : M^m \rightarrow \mathbb{R}^n$ the cobordism class of the manifold Z_f is uniquely defined up to cobordisms and so well defines a homomorphism

$$\mu_\rho^G : \Omega_m^H(G) \rightarrow \Omega_{m-n}^H(G).$$

Remark. Note that the homomorphism $\mu_\rho^G : \Omega_m^H(G) \rightarrow \Omega_{m-n}^H(G)$ depends only on a representation of a group G on $\mathbb{R}^n$. For some groups, this homomorphism is known in algebraic topology. For instance, if $G = \mathbb{Z}_2$ and $H = O$ (unoriented cobordisms, i.e. $\Omega_*^O(\mathbb{Z}_2) = \mathfrak{N}_*(\mathbb{Z}_2)$), then $\mu = \Delta^k$, where Δ is called the *Smith homomorphism*. Conner and Floyd [5, Theorem 26.1] give the following definition of Δ : Let T be a free involution on a closed manifold M . For $n \geq m$ there exists an antipodal generic map $f : M^m \rightarrow \mathbb{S}^n$. Let $X^{m-1} = f^{-1}(\mathbb{S}^{n-1})$. Then the map $\Delta : \Omega_m^O(\mathbb{Z}_2) \rightarrow \Omega_{m-1}^O(\mathbb{Z}_2)$ which is defined by $[M^m, T] \rightarrow [X^{m-1}, T|_X]$ is a homomorphism and Δ does not depend on n and f .

The invariant μ_ρ^G is an obstruction for the existence of equivariant maps $f : M \rightarrow \mathbb{R}^n \setminus \{0\}$. Namely, we have the following theorem.

Theorem 3. *Let M^m be a closed PL G -manifold with a free action τ . Let ρ be a linear action of G on $\mathbb{R}^n$. Let us assume that actions, manifolds, and maps are with*

H-structure. Suppose that $\mu_\rho^G([M, \tau]) \neq 0$ in $\Omega_{m-n}^H(G)$. Then for any continuous equivariant map $f : M^m \rightarrow \mathbb{R}^n$ the set of zeros Z_f is not empty.

Proof. Actually the proof of the theorem and of the corollary are almost word for word the same as the proof of Theorem 1. Let us suppose that $f^{-1}(0) = \emptyset$. Since M is compact, there is an $\varepsilon > 0$ such that $\|f(x)\| \geq \varepsilon$ for all $x \in M$. From Lemma 1 and Lemma 3, it follows that there exists a generic h such that $\|f(x) - h(x)\| \leq \varepsilon/2$ for all $x \in M$. By Lemma 4 we have $\mu_\rho^G([M, \tau]) = [Z_h]_G \neq 0$ in $\Omega_{m-n}^H(G)$. Then Z_h is a submanifold of M of dimension $m - n$.

Since $Z_h \neq \emptyset$, there is an $x \in M$ such that $h(x) = 0$. Thus, $\varepsilon/2 \geq \|f(x) - h(x)\| = \|f(x)\| \geq \varepsilon > 0$, a contradiction. $\square$

Corollary 2. Let manifolds, actions, and maps be as above. Suppose that there is a continuous equivariant transversal (in zeros) map $h : M^m \rightarrow \mathbb{R}^n$ such that the set Z_h is a closed submanifold in M of codimension n and $[Z_h]_G \neq 0$ in $\Omega_{m-n}^H(G)$. Then for any continuous equivariant map $f : M \rightarrow \mathbb{R}^n$ the set of zeros Z_f is not empty.

Proof. By Lemma 3, for any $\varepsilon > 0$ there is a generic map a_ε such that $\|a_\varepsilon(x) - h(x)\| \leq \varepsilon$ for all $x \in M$. If $\varepsilon \rightarrow 0$, then $Z_\varepsilon := a_\varepsilon^{-1}(0) \rightarrow Z$. This implies that for a sufficiently small ε , there is a homeomorphism between Z_ε and Z_h . Therefore, $\mu_\rho^G([M, \tau]) = [Z_h]_G \neq 0$ in $\Omega_{m-n}^H(G)$. $\square$

Let $G = \mathbb{Z}_2$ and $\rho(x) = -x$. Denote by $\mu_n := \mu_\rho^{\mathbb{Z}_2}$. Then we have

$$\mu_n : \mathfrak{N}_m(\mathbb{Z}_2) \rightarrow \mathfrak{N}_{m-n}(\mathbb{Z}_2).$$

If $u \in \mathfrak{N}_m(\mathbb{Z}_2)$ is given explicitly, then we can easily find $\mu_n(u)$.

Lemma 6.

$$\mu_n([M^m, T]) = \mu_n \left(\sum_{k=0}^m [V^k] [\mathbb{S}^{m-k}, A] \right) = \sum_{k=0}^{m-n} [V^k] [\mathbb{S}^{m-n-k}, A].$$

Proof. Since $Z_{f_i} = F^{-1}(0) \cap M_i$ in Lemma 4 are cobordant to each other, we can consider the standard projection $Pr : \mathbb{S}^d \rightarrow \mathbb{R}^n$. Then $Pr^{-1}(0) = \mathbb{S}^{d-n}$. This yields $\mu_n([\mathbb{S}^d, A]) = [\mathbb{S}^{d-n}, A]$. It is clear that the lemma follows from this equality. $\square$

Lemma 6 and Theorem 3 imply the following corollary.

Corollary 3. Let $[M^m, T] = [V^{m-n}][\mathbb{S}^n, A] + [V^{m-n+1}][\mathbb{S}^{n-1}, A] + \dots + [V^m][\mathbb{S}^0, A]$ in $\mathfrak{N}_m(\mathbb{Z}_2)$ with $[V^{m-n}] \neq 0$ in $\mathfrak{N}_{m-n}$. Then for any antipodal continuous map $f : M^m \rightarrow \mathbb{R}^n$ the set Z_f is not empty.

This corollary can be extended for the case $G = (\mathbb{Z}_2)^k = \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$. Note that $\mathfrak{N}_*((\mathbb{Z}_2)^k) = \mathfrak{N}_*(\mathbb{Z}_2) \otimes \dots \otimes \mathfrak{N}_*(\mathbb{Z}_2)$ [5, Sec. 29]. In other words, $\mathfrak{N}_*((\mathbb{Z}_2)^k)$ is a free $\mathfrak{N}_*$ -module with generators $\{\gamma_{i_1} \otimes \dots \otimes \gamma_{i_k}\}$, $i_1, \dots, i_k = 0, 1, \dots$, where $\gamma_i := [\mathbb{S}^i, A] \in \mathfrak{N}_i(\mathbb{Z}_2)$.

Corollary 4. Let M be a closed PL-manifold with a free action Ψ of $(\mathbb{Z}_2)^k$. Let $[M^m, \Psi] = \sum [V^{i_1, \dots, i_k}] \gamma_{i_1} \otimes \dots \otimes \gamma_{i_k}$ in $\mathfrak{N}_m((\mathbb{Z}_2)^k)$ with $[V^{i_1, \dots, i_k}] \neq 0$ in $\mathfrak{N}_*$. Consider $(\mathbb{Z}_2)^k$ with generators $T_1, \dots, T_k$ acting on $\mathbb{R}^{i_1} \oplus \dots \oplus \mathbb{R}^{i_k}$ by $T_\ell(x) = -x$ for $x \in \mathbb{R}^{i_\ell}$. Then for any continuous equivariant map $f : M^m \rightarrow \mathbb{R}^{i_1} \oplus \dots \oplus \mathbb{R}^{i_k}$ the set Z_f is not empty.

Let $\tilde{G}(4, 2)$ be the oriented Grassmann manifold, that is, the space of all oriented 2-dimensional subspaces of $\mathbb{R}^4$. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\tilde{G}(4, 2)$, where for $x \in \tilde{G}(4, 2)$, $T_1(x)$ changes the orientation of x and $T_2(x)$ is the oriented orthogonal complement of an oriented 2-plane x in $\mathbb{R}^4$.

It is well known that $\tilde{G}(4, 2)$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ [7, 3.2.3]. For $(u, v) \in \mathbb{S}^2 \times \mathbb{S}^2$, we have $T_1(u, v) = (-u, -v)$ and $T_2(u, v) = (u, -v)$. Then for $\Psi = (T_1, T_2)$ we have $[\tilde{G}(4, 2), \Psi] = \gamma_2 \otimes \gamma_2$ in $\mathfrak{N}_4(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then Corollary 4 (or Theorem 1) yields:

Corollary 5. *Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ act on $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ by $T_1(u, v) = (-u, -v)$ and $T_2(u, v) = (u, -v)$. Then for any equivariant continuous $f : \tilde{G}(4, 2) \rightarrow \mathbb{R}^4$ the set Z_f is not empty.*

ACKNOWLEDGMENTS

I wish to thank Arseniy Akopyan, Imre Bárány, Pavle Blagojević, Mike Field, Jiří Matoušek, Alexey Volovikov, and Günter Ziegler for helpful discussions and comments. I am most grateful to Roman Karasev and to the anonymous referee for several critical comments and corrections.

REFERENCES

- [1] I. Bárány, Borsuk's theorem through complementary pivoting, *Math. Programming*, **18** (1980), 84-88. MR557116 (81d:90099)
- [2] E. Bierstone, General position of equivariant maps, *Trans. Amer. Math. Soc.*, **234** (1977), 447-466. MR0464287 (57:4221)
- [3] G. E. Bredon, Introduction to compact transformation groups, Academic Press, 1972. MR0413144 (54:1265)
- [4] P. E. Conner and E. E. Floyd, Fixed point free involutions and equivariant maps, *Bull. Amer. Math. Soc.*, **66** (1960), 416-441. MR0163310 (29:613)
- [5] P. E. Conner and E. E. Floyd, Differentiable periodic maps, Springer-Verlag, 1964. MR0176478 (31:750)
- [6] M. J. Field, Transversality in G -manifolds, *Trans. Amer. Math. Soc.*, **231** (1977), 429-450. MR0451276 (56:9563)
- [7] D. B. Fuks and V. A. Rokhlin, Beginner's course in topology. Geometric chapters, Springer-Verlag, Berlin-New York, 1984. MR759162 (86a:57001)
- [8] S. Illman, Smooth equivariant triangulations of G -manifolds for G a finite group, *Math. Ann.*, **233** (1978), 199-220. MR0500993 (58:18474)
- [9] J. Matoušek, Using the Borsuk-Ulam theorem, Springer-Verlag, Berlin, 2003. MR1988723 (2004i:55001)
- [10] M. D. Meyerson and A. H. Wright, A new and constructive proof of the Borsuk-Ulam theorem, *Proc. Amer. Math. Soc.*, **73** (1979), 134-136. MR512075 (80e:55006)
- [11] O. R. Musin, Sperner and Tucker's lemmas and its generalizations, in preparation.
- [12] D. Repovš and A. Skopenkov, On projected embeddings and desuspending the α -invariant, *Topology and its Applications*, **124** (2002) 69-75. MR1926135 (2003g:57046)
- [13] H. Steinlein, Borsuk's antipodal theorem and its generalizations and applications: A survey, *Méth. Topol. en Anal. Non Linéaire, Sémin. Math. Sup.*, **95** (1985), Presses Univ. Montréal, Montréal, Quebec, 166-235. MR801938 (86k:55002)
- [14] A. S. Svarc, The genus of a fibered space, *Trudy Moskov. Mat. Obsc.*, **10** (1961), 217-272; and **11** (1962), 99-126 (Russian). English translation in Amer. Math. Soc. Translat., II. Ser., **55** (1966), 49-140. MR0154284 (27:4233)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT BROWNSVILLE, 80 FORT BROWN, BROWNSVILLE, TEXAS 77820

E-mail address: oleg.musin@utb.edu