ASYMPTOTIC DISTRIBUTIONS OF THE ZEROS OF A FAMILY OF HYPERGEOMETRIC POLYNOMIALS

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ABSTRACT. The main object of this paper is to consider the asymptotic distribution of the zeros of certain classes of the Gauss hypergeometric polynomials. Some classical analytic methods and techniques are used here to analyze the behavior of the zeros of the Gauss hypergeometric polynomials,

$${}_2F_1(-n, a; -n + b; z),$$

where n is a nonnegative integer. Owing to the connection between the classical Jacobi polynomials and the Gauss hypergeometric polynomials, we prove a special case of a conjecture made by Martínez-Finkelshtein, Martínez-González and Orive. Numerical evidence and graphical illustrations of the clustering of the zeros on certain curves are generated by Mathematica (Version 4.0).

1. INTRODUCTION

The celebrated Gauss hypergeometric function is defined by (see, for details, [1] and [22])

$$(1.1) \quad {}_2F_1 \left[\begin{matrix} \alpha, \beta; \\ \delta; \end{matrix} \quad z \right] = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\delta)_k} \frac{z^k}{k!} \quad (|z| < 1),$$

provided that no zeros appear in the denominator of (1.1). Here, as usual, $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \quad (n \in \mathbb{N}; \mathbb{N} := \{1, 2, 3, \dots\}),$$

which is defined, in terms of the familiar Gamma function, by

$$(1.2) \quad (\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

with $\mathbb{C}$ being the set of complex numbers. It is understood (as usual) that $(0)_0 := 1$.

If one or both of the numerator parameters α and β is equal to a negative integer or zero, that is, if (for example)

$$\alpha = -n \quad (n \in \mathbb{N}_0 := \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}),$$

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then the series in (1.1) terminates and reduces to a polynomial of degree n in z . The natural question that arises in connection with any polynomials is the correlative properties of their zeros (see, for example, [5, 7, 8, 9, 10, 11, 21, 23, 24, 27]).

For the Gauss hypergeometric polynomials, the Euler integral representation, together with the saddle-point method, would generate the asymptotic zero distribution of some classes of ${}_2F_1$ polynomials [2, 3, 13, 26]. On the basis of the connection between the Gauss ${}_2F_1$ polynomial classes and the classical orthogonal Jacobi polynomials, a great deal of significant information about their zeros, including the location of zeros and the asymptotic zero distribution, has been obtained in [6]. Using different techniques which involve the direct investigation of the zero distribution of a ${}_3F_2$ polynomial, the asymptotic behavior of the zeros of the polynomial ${}_2F_1(-n, b; -2n; z)$ for $b > 0$ is obtained in [12].

In the present paper, we propose to derive the asymptotic results for the zeros of the following hypergeometric ${}_2F_1$ polynomials:

$${}_2F_1(-n, a; -n + b; z),$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; b \in \mathbb{C} \setminus \mathbb{N}_0),$$

where

$$\mathbb{Z}_0^- = \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\}.$$

By means of the hypergeometric identity (2.13) below, we also consider the asymptotic behavior of the zeros of the following Gauss hypergeometric polynomials:

$${}_2F_1(-n, d; f; z).$$

In addition, the classical Jacobi polynomials $P_n^{(-n+l, s)}(z)$ and $P_n^{(s, -n+l)}(z)$, which are connected by the well-known relationship (3.1) with the Gauss hypergeometric polynomials, are also investigated. Numerical evidence and graphical illustrations of the clustering of the zeros on certain curves are generated by Mathematica (Version 4.0).

2. A SET OF MAIN RESULTS

For notational simplicity, we write

$$(2.1) \quad {}_2F_1 \left[\begin{matrix} -n, a; \\ -n + b; \end{matrix} z \right] =: \sum_{k=0}^n a_{n,k} z^k \quad (a_{n,0} := 1).$$

Lemma 1. *Let*

$$a \in \mathbb{C} \setminus \mathbb{Z}_0^-; b \in \mathbb{C} \setminus \mathbb{N}_0.$$

Then each of the following inequalities holds true:

$$(2.2) \quad |a_{n,k}| \leq \mathfrak{M}_0 \cdot (k+1)^{|b|+|1-a|} \quad (0 \leq k \leq n; n, k \in \mathbb{N}_0),$$

where the constant $\mathfrak{M}_0 > 0$ depends on the parameters a and b , but not on n and k , and

$$(2.3) \quad \left| \frac{a_{n,n-k}}{a_{n,n}} \right| \leq \widetilde{\mathfrak{M}}_0 \cdot (k+1)^{|b|+|1-a|} \quad (0 \leq k \leq n; n, k \in \mathbb{N}_0),$$

where the constant $\widetilde{\mathfrak{M}}_0 > 0$ depends on the parameters a and b , but not on n and k .

Proof. We first consider the case when $0 < k < n$ ($n, k \in \mathbb{N}$). Then, according to the following identity for the Pochhammer symbol $(\lambda)_n$ defined by (1.2):

$$(2.4) \quad (-n)_j =: \begin{cases} \frac{(-1)^j n!}{(n-j)!} & (0 \leq j \leq n; n, j \in \mathbb{N}_0), \\ 0 & (j \geq n+1; n \in \mathbb{N}_0; j \in \mathbb{N}), \end{cases}$$

we have

$$(2.5) \quad \frac{a_{n,k+1}}{a_{n,k}} = \frac{(n-k)(a+k)}{(n-b-k)(1+k)}.$$

We also observe that

$$\frac{n-k}{n-b-k} = 1 + \frac{b}{n-b-k}$$

and

$$\frac{a+k}{1+k} = 1 + \frac{a-1}{1+k},$$

so that

$$\ln \left(\frac{n-k}{|n-b-k|} \right) \leq \frac{|b|}{|n-b-k|}$$

and

$$\ln \left(\frac{|a+k|}{1+k} \right) \leq \frac{|a-1|}{1+k}.$$

Consequently, we have

$$\begin{aligned} \ln |a_{n,k}| &= \ln \left(\prod_{l=0}^{k-1} \frac{|a_{n,l+1}|}{|a_{n,l}|} \right) = \sum_{l=0}^{k-1} \ln \left(\frac{|a_{n,l+1}|}{|a_{n,l}|} \right) \\ &\leq \sum_{l=0}^{k-1} \left(\frac{|b|}{|n-b-l|} + \frac{|a-1|}{1+l} \right) \\ &\leq \sum_{l=0}^{k-1} \left(\frac{|b|}{|n-b-k+1+l|} + \frac{|a-1|}{l+1} \right) \\ &\leq \sum_{l=0}^{\lfloor |b| \rfloor} \frac{|b|}{|n-b-k+1+l|} + \sum_{l=0}^{k-1} \frac{|b|+|a-1|}{l+1} \\ (2.6) \quad &\leq \mathfrak{M} + (|b|+|a-1|)[\ln(k+1) + \gamma], \end{aligned}$$

where $\mathfrak{M} > 0$ is a constant and γ is the Euler-Mascheroni constant given by

$$(2.7) \quad \gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right) \cong 0.57721\,56649\,01532\,8606\,06512\,09008\,24024\,31042 \dots$$

It follows from (2.6) that

$$|a_{n,k}| \leq \mathfrak{M}_0 \cdot (k+1)^{|b|+|a-1|} \quad (0 < k < n; n, k \in \mathbb{N}),$$

where the constant $\mathfrak{M}_0 > 0$ depends on the parameters a and b , but not on n and k . The cases of the assertion (2.2) of Lemma 1 when $k = 0$ and $k = n$ are trivial.

Next, by replacing k by $n-k$ in (2.5), we have

$$\frac{a_{n,n-k}}{a_{n,n-k+1}} = \frac{(k-b)(1+n-k)}{k(a+n-k)}.$$

Since

$$\frac{k-b}{k} = 1 - \frac{b}{k} \quad \text{and} \quad \frac{1+n-k}{a+n-k} = 1 + \frac{1-a}{a+n-k},$$

it follows that

$$\ln \left(\frac{|k-b|}{k} \right) \leq \frac{|b|}{k}$$

and

$$\ln \left(\frac{1+n-k}{|a+n-k|} \right) \leq \frac{|1-a|}{|a+n-k|},$$

respectively. Therefore, we obtain

$$\begin{aligned} \ln \left(\frac{|a_{n,n-k}|}{|a_{n,n}|} \right) &= \ln \left(\prod_{l=1}^k \frac{|a_{n,n-l}|}{|a_{n,n-l+1}|} \right) = \sum_{l=1}^k \ln \left(\frac{|a_{n,n-l}|}{|a_{n,n-l+1}|} \right) \\ &\leq \sum_{l=1}^k \left(\frac{|b|}{l} + \frac{|1-a|}{|a+n-l|} \right) \\ &\leq \sum_{l=1}^k \left(\frac{|b|}{l} + \frac{|1-a|}{|a+n-k-1+l|} \right) \\ &\leq \sum_{l=1}^{|a|+1} \frac{|1-a|}{|a+n-k-1+l|} + \sum_{l=1}^k \frac{|b| + |1-a|}{l} \\ (2.8) \quad &\leq \mathfrak{N} + (|b| + |1-a|)[\ln(k+1) + \gamma], \end{aligned}$$

where $\mathfrak{N} > 0$ is a constant and γ is the Euler-Mascheroni constant given by (2.7). It follows from (2.8) that

$$\frac{|a_{n,n-k}|}{|a_{n,n}|} \leq \widetilde{\mathfrak{M}}_0 \cdot (k+1)^{|b|+|1-a|},$$

where the constant $\widetilde{\mathfrak{M}}_0 > 0$ depends on the parameters a and b , but not on n and k .

Finally, since the cases of assertion (2.3) of Lemma 1 when $k = 0$ and $k = n$ are immediate, our proof of Lemma 1 is completed. $\square$

Theorem 1. For fixed parameters a and b constrained by

$$a \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{and} \quad b \in \mathbb{C} \setminus \mathbb{N}_0,$$

the zeros of the Gauss hypergeometric polynomials,

$${}_2F_1 \left[\begin{matrix} -n, a; \\ -n+b; \end{matrix} z \right],$$

approach the unit circle as $n \rightarrow \infty$.

Proof. From the assertion (2.2) of Lemma 1, it immediately follows that the sequence of polynomials

$$\left\{ \sum_{k=0}^n a_{n,k} z^k \right\}_{n \in \mathbb{N}_0}$$

is uniformly bounded on the disk Ω given by

$$\Omega = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < \rho \quad (0 < \rho < 1)\}.$$

Furthermore, since (for fixed k)

$$a_{n,k} = \frac{(-n)_k (a)_k}{k! (-n+b)_k} \rightarrow \frac{(a)_k}{k!} \quad (n \rightarrow \infty),$$

the sequence of functions

$$\left\{ \sum_{k=0}^n a_{n,k} z^k \right\}_{n \in \mathbb{N}_0}$$

converges pointwise for z ($|z| < 1$) and, therefore, also uniformly, by Vitali's theorem [15, p. 252], to the sum

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = {}_1F_0(a; \text{---}; z) = (1-z)^{-a} \quad (|z| < 1; a \in \mathbb{C}).$$

Since the function $(1-z)^{-a}$ does not have any zeros inside the unit disk $|z| = 1$, by Hurwitz's theorem [15, p. 205] there exists an index N_0 such that

$$\sum_{k=0}^n a_{n,k} z^k$$

does not have zeros on Ω for $n > N_0$. Hence there exist numbers ρ_n constrained by

$$0 < \rho_n < 1 \quad \text{so that} \quad \rho_n \rightarrow 1,$$

and we can ensure that

$$\rho_n \geq \rho \quad (n > N_0; 0 < \rho < 1).$$

Next, from assertion (2.3) of Lemma 1, it also follows that the sequence of polynomials

$$\frac{z^n}{a_{n,n}} \sum_{k=0}^n a_{n,k} \left(\frac{1}{z}\right)^k = \sum_{k=0}^n \frac{a_{n,n-k}}{a_{n,n}} z^k$$

is uniformly bounded on Ω . Moreover, since (for fixed k)

$$\frac{a_{n,n-k}}{a_{n,n}} = \frac{(-1)^k \Gamma(b) \Gamma(n+1) \Gamma(n+a-k)}{k! \Gamma(b-k) \Gamma(n+a) \Gamma(n-k+1)} \rightarrow \frac{(1-b)_k}{k!} \quad (n \rightarrow \infty),$$

the sequence of functions

$$\left\{ \sum_{k=0}^n \frac{a_{n,n-k}}{a_{n,n}} z^k \right\}_{n \in \mathbb{N}_0}$$

converges pointwise for z ($|z| < 1$) and therefore also uniformly, by Vitali's theorem [15, p. 252], to the sum

$$\sum_{k=0}^{\infty} \frac{(1-b)_k}{k!} z^k = {}_1F_0(1-b; \text{---}; z) = (1-z)^{b-1} \quad (|z| < 1; b \in \mathbb{C}).$$

Since the function $(1-z)^{b-1}$ does not have any zeros inside the unit disk $|z| = 1$, by Hurwitz's theorem [15, p. 205] there exists an index N_0 such that

$$\frac{z^n}{a_{n,n}} \sum_{k=0}^n a_{n,k} \left(\frac{1}{z}\right)^k$$

does not have zeros on Ω for $n > N_0$; that is,

$$\sum_{k=0}^n a_{n,k} z^k$$

does not have zeros $|z| > \frac{1}{\rho}$. Hence there exist numbers γ_n constrained by

$$\gamma_n > 1 \quad \text{so that} \quad \gamma_n \rightarrow 1,$$

and we can ensure that

$$\gamma_n \leq \frac{1}{\rho} \quad (n > N_0; 0 < \rho < 1).$$

Consequently, all zeros of the polynomial

$$\sum_{k=0}^n a_{n,k} z^k$$

lie in the annulus given by

$$\{z : z \in \mathbb{C} \quad \text{and} \quad \rho_n \leq |z| \leq \gamma_n\},$$

which completes the proof of Theorem 1. $\square$

Remark. In our proof of Theorem 1, we have used the fact that the sequences of functions

$$\left\{ \sum_{k=0}^n a_{n,k} z^k \right\}_{n \in \mathbb{N}_0} \quad \text{and} \quad \left\{ \sum_{k=0}^n \frac{a_{n,n-k}}{a_{n,n}} z^k \right\}_{n \in \mathbb{N}_0}$$

converge pointwise for z ($|z| < 1$) and therefore also uniformly, by Vitali's theorem [15, p. 252], to the sums

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = {}_1F_0(a; \text{---}; z) = (1-z)^{-a} \quad (|z| < 1; a \in \mathbb{C})$$

and

$$\sum_{k=0}^{\infty} \frac{(1-b)_k}{k!} z^k = {}_1F_0(1-b; \text{---}; z) = (1-z)^{b-1} \quad (|z| < 1; b \in \mathbb{C}),$$

respectively. Indeed, according to Lemma 1, we have

$$\left| \sum_{k=0}^n a_{n,k} z^k \right| \leq \left| \sum_{k=0}^n |a_{n,k}| \cdot |z|^k \right| \leq \mathfrak{M}_0 \cdot \sum_{k=0}^n (k+1)^{|b|+|1-a|} |z|^k.$$

Now, by applying *d'Alembert's Ratio Test*, we find that

$$\lim_{k \rightarrow \infty} \left\{ \frac{(k+2)^{|b|+|1-a|} |z|^{k+1}}{(k+1)^{|b|+|1-a|} |z|^k} \right\} = |z|,$$

so that, clearly, the radius of convergence of the *infinite* series

$$\sum_{k=0}^{\infty} (k+1)^{|b|+|1-a|} |z|^k$$

is 1 and the sequence of polynomials

$$\sum_{k=0}^n (k+1)^{|b|+|1-a|} |z|^k$$

is uniformly bounded on the set Ω given by

$$\Omega = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < \rho \quad (0 < \rho < 1)\}.$$

Consequently, the sequence of functions

$$\left\{ \sum_{k=0}^n a_{n,k} z^k \right\}_{n \in \mathbb{N}_0}$$

converges pointwise for arbitrary z_0 ($|z_0| < 1$). Similarly, the sequence of functions

$$\left\{ \sum_{k=0}^n \frac{a_{n,n-k}}{a_{n,n}} z^k \right\}_{n \in \mathbb{N}_0}$$

can be seen to be convergent pointwise for arbitrary z_0 ($|z_0| < 1$). Thus, just as we claimed in our proof of Theorem 1, the sequences of functions

$$\left\{ \sum_{k=0}^n a_{n,k} z^k \right\}_{n \in \mathbb{N}_0} \quad \text{and} \quad \left\{ \sum_{k=0}^n \frac{a_{n,n-k}}{a_{n,n}} z^k \right\}_{n \in \mathbb{N}_0}$$

are pointwise convergent for all z ($|z| < 1$).

Theorem 2. *If the fixed parameters a and b are constrained by*

$$0 < a \leq 1 \quad \text{and} \quad b < 0,$$

then the zeros of the Gauss hypergeometric polynomials

$${}_2F_1 \left[\begin{matrix} -n, a; \\ -n + b; \end{matrix} z \right]$$

lie outside the unit disk $|z| \leq 1$ and approach the unit circle $|z| = 1$ as $n \rightarrow \infty$.

The proof of Theorem 2 is based upon the following classical result [18, p. 136].

Lemma 2 (Eneström-Kakeya Theorem [18, p. 136]). *If*

$$0 < a_0 < a_1 < \cdots < a_n,$$

then all zeros of the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

lie in the unit disk $|z| \leq 1$.

Proof of Theorem 2. According to Theorem 1, we only need to prove that the zeros of the Gauss hypergeometric polynomials

$${}_2F_1(-n, a; -n + b; z)$$

lie outside the unit circle $|z| = 1$. We also find from (2.1) that

$$(2.9) \quad z^n {}_2F_1 \left[\begin{matrix} -n, a; \\ -n + b; \end{matrix} \frac{1}{z} \right] = \sum_{k=0}^n a_{n,k} z^{n-k} = \sum_{k=0}^n a_{n,n-k} z^k.$$

Thus, under the parametric constraints

$$0 < a \leq 1 \quad \text{and} \quad b < 0,$$

which are already mentioned in the hypothesis of Theorem 2, we find from (2.5) that

$$(2.10) \quad \frac{a_{n,n-(k+1)}}{a_{n,n-k}} = \frac{(k+1-b)(n-k)}{(k+1)(a+n-k-1)} > 1 \quad (k=0, 1, \dots, n-1),$$

which implies that the coefficients of the polynomial

$$(2.11) \quad \tilde{F}(z) := z^n {}_2F_1(-n, a; -n+b; z^{-1}) = \sum_{k=0}^n a_{n,n-k} z^k$$

are positive and increasing, that is,

$$0 < a_{n,n} < a_{n,n-1} < \dots < a_{n,0}.$$

It follows from Lemma 2 that the zeros of the polynomial $\tilde{F}(z)$ defined by (2.11) lie in unit disk $|z| \leq 1$. Hence the zeros of

$${}_2F_1(-n, a; -n+b; z)$$

lie outside the unit disk $|z| \leq 1$. This completes our proof of Theorem 2.

Theorem 3. *If the fixed parameters a and b are constrained by*

$$a \geq 1 \quad \text{and} \quad 0 < b < 1,$$

then the zeros of the Gauss hypergeometric polynomials

$${}_2F_1 \left[\begin{matrix} -n, a; \\ -n+b; \end{matrix} \middle| z \right]$$

lie inside the unit disk $|z| \leq 1$ and approach the unit circle $|z| = 1$ as $n \rightarrow \infty$.

Proof. By appealing appropriately to Theorem 1, we only need to prove that the zeros of

$${}_2F_1(-n, a; -n+b; z)$$

lie inside the unit disk $|z| \leq 1$. Thus, under the parametric constraints

$$a \geq 1 \quad \text{and} \quad 0 < b < 1,$$

which are already mentioned in the hypothesis of Theorem 3, we find from (2.5) that

$$(2.12) \quad \frac{a_{n,k+1}}{a_{n,k}} = \frac{(n-k)(a+k)}{(n-b-k)(1+k)} > 1 \quad (k=0, 1, \dots, n-1),$$

which implies that the coefficients of

$${}_2F_1(-n, a; -n+b; z)$$

are positive and increasing:

$$0 < a_{n,0} < a_{n,1} < \dots < a_{n,n}.$$

According to Lemma 2, we can see that the zeros of

$${}_2F_1(-n, a; -n+b; z)$$

lie inside the unit disk $|z| \leq 1$. This completes the proof of Theorem 3. $\square$

FIGURE 1. Zeros of ${}_2F_1(-n, a; -n+b; z)$ and the unit circle $|z| \leq 1$.

Figure 1 illustrates Theorems 1 to 3 with the help of the graphics in Mathematica (Version 4.0). It displays the zeros of the Gauss hypergeometric polynomials ${}_2F_1(-n, a; -n+b; z)$ and their asymptotic curve $|z| = 1$ for $n = 30$ for various values of a and b .

We shall make use of the following hypergeometric identity:

$$(2.13) \quad {}_2F_1 \left[\begin{matrix} -n, b; \\ c; \end{matrix} z \right] = \frac{(c-b)_n}{(c)_n} {}_2F_1 \left[\begin{matrix} -n, b; \\ b-c-n+1; \end{matrix} 1-z \right] \quad (c \notin \mathbb{Z}_0^-; c-b \notin \mathbb{Z}),$$

which is a rather immediate consequence of a familiar analytic continuation formula for the Gauss hypergeometric function in the form [14, p. 108, Equation 2.10(1)]

$$(2.14) \quad \begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z) \end{aligned}$$

$(|\arg(1-z)| < \pi; c \notin \mathbb{Z}_0^-; c-a-b \notin \mathbb{Z})$

when we set $a = -n$ ($n \in \mathbb{N}_0$).

In view of (2.13) and Theorems 1 to 3, we have the following corollary.

Corollary 1. *For fixed parameters b and c constrained by*

$$b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{and} \quad c-b \in \mathbb{C} \setminus \mathbb{Z},$$

all zeros of the hypergeometric polynomial ${}_2F_1(-n, b; c; z)$ approach the circle $|z-1| = 1$ as $n \rightarrow \infty$. Furthermore, if

$$0 < b \leq 1 \quad \text{and} \quad 1+b < c,$$

then these zeros lie outside the disk $|z-1| \leq 1$; and if

$$b \geq 1 \quad \text{and} \quad b < c < 1+b,$$

then these zeros lie inside the disk $|z-1| \leq 1$.

Figure 2 illustrates Corollary 1 by showing the zeros of the hypergeometric polynomial ${}_2F_1(-n, b; c; z)$ for $n = 40$ and various values of b and c with their asymptotic curve $|z-1| = 1$.

In the special case of Corollary 1 when $c = 2b$, Driver [6] obtained a more accurate result about the location of the zeros in Corollary 1 as follows.

FIGURE 2. Zeros of ${}_2F_1(-n, b; c; z)$ and the curve $|z - 1| = 1$.

Lemma 3 ([6, Theorem 1]). *For each $b > -\frac{1}{2}$, all zeros of the hypergeometric polynomial $F(-n, b; 2b; z)$ lie on the circle $|z - 1| = 1$.*

From the identity (2.13) and Lemma 3, we have the following corollary.

Corollary 2. *For fixed parameters a constrained by $a > -\frac{1}{2}$ and $a \neq 0$, the zeros of ${}_2F_1(-n, a, -n + 1 - a; z)$ lie on the unit circle $|z| = 1$.*

3. RELATIONSHIP WITH THE CLASSICAL JACOBI POLYNOMIALS

The classical Jacobi polynomials $P_n^{(\alpha, \beta)}(z)$ have the following hypergeometric representation (see, for details, [25, Chapter 4]):

$$(3.1) \quad (-1)^n P_n^{(\beta, \alpha)}(-z) = P_n^{(\alpha, \beta)}(z) = \frac{(1 + \alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha; \end{matrix} \frac{1 - z}{2} \right].$$

Our theorems can thus be applied to obtain asymptotic information about the zeros of the classical Jacobi polynomials (see also [4] for some recent results involving the Gauss hypergeometric polynomials and the classical Jacobi polynomials).

Theorem 4. (i) *For fixed parameters l and s constrained by*

$$l \in \mathbb{R} \setminus (\mathbb{N}_0 \cup \{-1\}) \quad \text{and} \quad 1 + l + s \in \mathbb{R} \setminus \mathbb{Z}_0^-,$$

all zeros of the Jacobi polynomials $P_n^{(-n+l, s)}(z)$ approach the circle $|z - 1| = 2$ as $n \rightarrow \infty$. Furthermore, if

$$l < -1 \quad \text{and} \quad -1 - l < s \leq -l,$$

then these zeros lie outside the disk $|z - 1| \leq 2$; and if

$$-1 < l < 0 \quad \text{and} \quad s \geq -l,$$

then these zeros lie inside the disk $|z - 1| \leq 2$.

(ii) *For fixed parameters l and s constrained by*

$$l \in \mathbb{R} \setminus (\mathbb{N}_0 \cup \{-1\}) \quad \text{and} \quad 1 + l + s \in \mathbb{R} \setminus \mathbb{Z}_0^-,$$

all zeros of the Jacobi polynomials $P_n^{(s, -n+l)}(z)$ approach the circle $|z + 1| = 2$ as $n \rightarrow \infty$. Furthermore, if

$$l < -1 \quad \text{and} \quad -1 - l < s \leq -l,$$

then these zeros lie outside the disk $|z + 1| \leq 2$; and if

$$-1 < l < 0 \quad \text{and} \quad s \geq -l,$$

then these zeros lie inside the disk $|z + 1| \leq 2$.

Proof. (i) In view of the last member in (3.1), we can write (see also [25, p. 62])

$$(3.2) \quad P_n^{(-n+l,s)}(z) = \frac{(1+l-n)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+l+s; \\ -n+1+l; \end{matrix} \quad \frac{1-z}{2} \right].$$

Also, upon setting

$$a = 1+l+s, \quad b = 1+l \quad \text{and} \quad u = \frac{1-z}{2}$$

in (3.2), we have

$$(3.3) \quad P_n^{(-n+l,s)}(z) = \frac{(1+l-n)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, a; \\ -n+b; \end{matrix} \quad u \right].$$

By applying Theorem 1, we can readily deduce that all zeros of the Jacobi polynomials $P_n^{(-n+l,s)}(z)$ approach the circle

$$\frac{|1-z|}{2} = |u| = 1 \quad \text{as} \quad n \rightarrow \infty,$$

provided that

$$1+l+s = a \notin \mathbb{Z}_0^- \quad \text{and} \quad 1+l = b \notin \mathbb{N}_0,$$

that is, for fixed

$$l \in \mathbb{R} \setminus (\mathbb{N}_0 \cup \{-1\}) \quad \text{and} \quad 1+l+s \in \mathbb{R} \setminus \mathbb{Z}_0^-.$$

Similarly, Theorems 2 and 3 can be applied to derive the remaining assertions of Theorem 4(i).

(ii) We now apply the symmetry relation in (3.1) in order to observe that

$$(3.4) \quad P_n^{(s,-n+l)}(z) = (-1)^n P_n^{(-n+l,s)}(-z).$$

From (3.4) and the assertions of Theorem 4(i), we can establish the following fact. For fixed

$$l \in \mathbb{R} \setminus (\mathbb{N}_0 \cup \{-1\}) \quad \text{and} \quad 1+l+s \in \mathbb{R} \setminus \mathbb{Z}_0^-,$$

all zeros of the Jacobi polynomials $P_n^{(s,-n+l)}(z)$ approach the circle $|z+1| = 2$ as $n \rightarrow \infty$. Furthermore, if

$$l < -1 \quad \text{and} \quad -1-l < s \leq -l,$$

then these zeros lie outside the disk $|z+1| \leq 2$; and if

$$-1 < l < 0 \quad \text{and} \quad s \geq -l,$$

then these zeros lie inside the disk $|z+1| \leq 2$. This proves the assertions of Theorem 4(ii), thereby completing the proof of Theorem 4. $\square$

Figure 3 and Figure 4 illustrate Theorem 4(i) and Theorem 4(ii), respectively. The zeros of the Jacobi polynomials $P_n^{(-n+l,s)}(z)$ and their asymptotic curve $|z-1| = 2$ are shown in Figure 3. Figure 4 shows the zeros of the Jacobi polynomials $P_n^{(s,-n+l)}(z)$ for $n = 50$ and various values of s and l , together with their asymptotic curve $|z+1| = 2$.

FIGURE 3. Zeros of $P_n^{(-n+l, s)}(z)$ and the curve $|z-1|=2$. FIGURE 4. Zeros of $P_n^{(s, -n+l)}(z)$ and the curve $|z+1|=2$.

4. CONCLUDING REMARKS AND OBSERVATIONS

It is interesting to note in conclusion that one of our results (Theorem 4 above) proves a special case of the conjecture made by Martínez-Finkelshtein *et al.* [19]. Moreover, in [16, 17, 19, 20], the asymptotic distribution of the zeros of the Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}(z)$ was investigated when

$$\frac{\alpha_n}{n} \rightarrow A \quad \text{and} \quad \frac{\beta_n}{n} \rightarrow B \quad (n \rightarrow \infty),$$

and the results presented here deal with the following two particular transitional cases:

$$A+1=B=0 \quad \text{and} \quad A=B+1=0.$$

Finally, by applying (3.1) and Corollary 2, we can deduce Corollary 3.

Corollary 3. *For fixed parameters l constrained by $l < \frac{1}{2}$ ($l \neq 0$), all zeros of the Jacobi polynomials $P_n^{(-n+l, -1-2l)}(z)$ lie on the circle $|z-1|=2$ and all zeros of the Jacobi polynomials $P_n^{(-1-2l, -n+l)}(z)$ lie on the circle $|z+1|=2$.*

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