THE EXCEPTIONAL SET IN VOJTA'S CONJECTURE FOR ALGEBRAIC POINTS OF BOUNDED DEGREE

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ABSTRACT. We study the dependence on various parameters of the exceptional set in Vojta's conjecture. In particular, by making use of certain elliptic surfaces, we answer in the negative the often-raised question of whether Vojta's conjecture holds when extended to all algebraic points (that is, if the conjecture holds without fixing a bound on the degree of the algebraic points).

1. INTRODUCTION

Inspired by results in Nevanlinna theory, Vojta [22, 24] made the following deep conjecture in Diophantine approximation for algebraic points of bounded degree in a nonsingular complete variety.

Conjecture 1.1 (Vojta's Conjecture). *Let X be a nonsingular complete variety with canonical divisor K . Let D be a normal crossings divisor on X , and let k be a number field over which X and D are defined. Let S be a finite set of places of k , A a big divisor on X , $\epsilon > 0$, and r a positive integer. Then there exists a proper Zariski-closed subset $Z = Z(r, \epsilon, X, D, A, k, S)$ of X such that*

$$(1.1) \quad m_{D,S}(P) + h_K(P) \leq d(P) + \epsilon h_A(P) + O(1)$$

for all points $P \in X(\bar{k}) \setminus Z$ with $[k(P) : k] \leq r$.

Vojta's Conjecture relates several fundamental arithmetic quantities associated to an algebraic point on a nonsingular complete variety. In the next section we will recall the definitions of several of the terms in the conjecture. Briefly, $m_{D,S}$ is a proximity function, measuring the part of the height relative to D coming from places lying above places in S ; h_K and h_A are absolute logarithmic height functions associated to the divisors K and A , respectively; and $d(P) = \frac{1}{[k(P):\mathbb{Q}]} \log |D_{k(P)/\mathbb{Q}}|$ is the absolute logarithmic discriminant. The set Z is called the *exceptional set*. It will be our primary object of study here.

Vojta's Conjecture has a wide range of important consequences (see [22]), of which we just mention the *abc* Conjecture and the Bombieri-Lang Conjecture on rational points on varieties of general type. Various strengthened versions of Vojta's Conjecture have frequently appeared in the literature, often asserting that the

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exceptional set Z in the conjecture can be chosen independently of some of the parameters appearing in the conjecture. It is this aspect of Vojta's Conjecture that we wish to study here, that is, the dependence of Z on the parameters r, ϵ, X, D, A, k , and S .

That the exceptional set Z must depend on X and D is trivial (indeed, $m_{D,S}(P)$ is not even well-defined for points in the support of D). The first nontrivial general observation regarding the dependence of Z on the parameters is due to Vojta himself, who showed in [23] that, in general, the set Z must be allowed to depend on the parameter ϵ . Vojta's example in [23] used quadratic points (i.e., $r = 2$). In [11], it was shown that Z must be allowed to depend on ϵ even when the conjecture is restricted to rational points ($r = 1$). Of course, in specific cases, it may happen that Z can be chosen independently of ϵ . For instance, in Schmidt's Subspace Theorem ($r = 1$, $X = \mathbb{P}^n$, D a sum of hyperplanes in general position), one of the few cases where Vojta's Conjecture is known, Vojta showed [23] that the exceptional set Z (which can be taken to consist only of hyperplanes) can in fact be taken to be independent of the parameter ϵ . We note also that the dependence of Z on ϵ has ramifications for the form of the error term in (1.1). In particular, contrary to a conjecture of Lang [8] for the error term in Roth's theorem ($X = \mathbb{P}^1$, $r = 1$), in higher dimensions one cannot replace the term $\epsilon h_A(P)$ in (1.1) by a term like $\log h_A(P)$. In fact, if one considers Vojta's Conjecture with "truncated counting functions", by a result of Stewart and Tijdeman [19] (see also [20]), this same phenomenon occurs even in the one-dimensional case, where the error term is essentially the same as the error term in the *abc* Conjecture (see [21]).

In Section 3, we show that Z must be allowed to depend on the big divisor A . In allowing arbitrary big divisors A , this is hardly surprising since the associated height h_A may not even be bounded from below for points in the base locus of the linear system $|A|$. However, we will show that even if one restricts to ample divisors A and rational points ($r = 1$), the set Z must be allowed to depend on A . Since for any two ample divisors A and A' on a variety X we have $h_A \gg h_{A'}$, this phenomenon is necessarily closely related to the dependence of Z on ϵ .

We have nothing new to offer regarding the dependence of Z on the number field k or the set of places S . If Z could not be taken independent of k and S , this would imply that the exceptional set Z is of a highly nongeometric nature, violating the spirit of Vojta's Conjecture. Consequently, Conjecture 1.1 is frequently stated with the additional hypothesis that Z can be chosen independently of k and S . Adopting this point of view, from now on we will notationally omit the (possible) dependence of Z on k and S .

Finally, we will give examples showing that the exceptional set Z in Vojta's Conjecture must be allowed to depend on the parameter r . Explicitly, we will show this for a large class of elliptic surfaces.

Theorem 1.2. *Let X be a semistable elliptic surface over $\mathbb{P}^1$ with a section over a number field k . Let D be a sum of fibers of X containing the singular fibers. If Vojta's Conjecture holds for X and D , then the exceptional set $Z(r, \epsilon, X, D, A)$ must depend on the parameter r .*

We can also produce similar examples (with a different, but related, class of surfaces X) where $D = 0$ (Theorem 4.7).

The rough idea of the construction behind Theorem 1.2 is as follows. We begin with a semistable elliptic surface X over $\mathbb{P}^1$ with zero section σ , everything defined

over some number field k . Let D be the sum of the singular fibers. We construct a set of integral points T on $\sigma \setminus D$ of some bounded degree (i.e., $[k(P) : k]$ is bounded for $P \in T$). For such points P in T we have $m_{D,S}(P) = h_D(P)$ and so the left-hand side of Vojta's Conjecture will be large. In fact, it will be large enough to force σ to be in the exceptional set (for an appropriate choice of r). Now we use the fiberwise multiplication-by- m map $[m]$ on the elliptic surface to pull back the points in T to obtain certain points on torsion multisections. Infinitely many of these points will again violate inequality (1.1) of Vojta's Conjecture. The basic intuitive reason behind this is that Vojta's Conjecture and the terms in Vojta's Conjecture are well-behaved with respect to unramified maps. We take advantage of a logarithmic version of this, using the facts that the points in T are integral with respect to D and $[m]$ is unramified on $X \setminus D$. We conclude that, for large enough r , $[m]^{-1}(\sigma \setminus D)$ must be contained in the exceptional set in Vojta's Conjecture. Now varying m gives the desired result.

As a consequence of Theorem 1.2, we obtain that Vojta's Conjecture does not hold uniformly for all algebraic points, answering a question that has frequently been raised (e.g., [22, p. 64] or [9, p. 223]). More precisely, in Section 4 we will show:

Theorem 1.3. *Vojta's Conjecture does not hold when extended to $r = \infty$ (i.e., to all algebraic points), even when allowing the $O(1)$ term to depend on $[k(P) : k]$. Specifically, there exists a nonsingular projective surface X , defined over a number field k , with canonical divisor K_X and big divisor A , such that for any $\delta > 0$, any constants $c_i \in \mathbb{R}$, $i \in \mathbb{N}$, and any sufficiently small $\epsilon > 0$, the inequality*

$$h_{K_X}(P) > \left(\frac{3}{2} - \delta\right) d(P) + \epsilon h_A(P) + c_{[k(P):k]}$$

holds for a Zariski-dense set of points P in $X(\bar{k})$.

This is somewhat surprising since the analogous function field version of Vojta's Conjecture has been proved for curves (independently by McQuillan [13] and Yamanoi [25]) with $r = \infty$. Moreover, previous results in the equidimensional Nevanlinna theory of coverings of $\mathbb{C}^n$ [3], [10, Ch. IV] also appeared to support the possibility that Vojta's Conjecture would hold uniformly for all algebraic points.

Despite Theorem 1.3, it is still quite plausible that Vojta's Conjecture may hold uniformly for all algebraic points after making some simple modifications to the terms in the inequality (1.1). One possibility is given by the original form of the conjecture proposed in [22]. In [22], Vojta included a $\dim X$ factor in the conjecture in front of the discriminant term $d(P)$, but this factor was later thought to be unnecessary (see, e.g., the comment after Conjecture 2.1 in [24]). None of the examples presented here prevent the possibility that Conjecture 1.1 could hold for all algebraic points with a $(\dim X)d(P)$ term in the inequality (1.1) (instead of $d(P)$). More precisely, our examples allow the possibility that Conjecture 1.1 could hold for all algebraic points when the $d(P)$ term is replaced by $\frac{3}{2}d(P)$ in the inequality (1.1). At the end of Section 4 we will also discuss some recent examples [1] of Autissier, Chambert-Loir, and Gasbarri concerning a geometric analogue of (1.1), which may provide further insight into the correct form of a uniform version of Vojta's Conjecture.

2. NOTATION AND DEFINITIONS

Throughout, k will denote a number field, $\mathcal{O}_k$ the ring of integers of k , and $\mathcal{O}_k^*$ the group of units of $\mathcal{O}_k$. Associated to k we have a canonical set M_k of places (or absolute values) of k consisting of one place for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, one place for each real embedding $\sigma : k \rightarrow \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \bar{\sigma} : k \rightarrow \mathbb{C}$. Let k_v denote the completion of k with respect to v . We normalize our absolute values so that for a prime $p \in \mathbb{Z}$, $|p|_v = p^{-[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$ if v corresponds to $\mathfrak{p}$ and $\mathfrak{p}|p$, and $|x|_v = |\sigma(x)|^{[k_v:\mathbb{R}]/[k:\mathbb{Q}]}$ if v corresponds to an embedding σ (in which case we say that v is Archimedean). If v is a place of k and w is a place of a field extension L of k , then we say that w lies above v , or $w|v$, if w and v define the same topology on k .

Let D be a divisor on a nonsingular complete variety X , with both X and D defined over a number field k . Let $v \in M_k$. Extend $|\cdot|_v$ to an absolute value on $\bar{k}_v$. Let $\text{Supp } D$ denote the support of D . We define a local Weil function (or local height function) for D relative to v to be a function $\lambda_{D,v} : X(\bar{k}_v) \setminus \text{Supp } D \rightarrow \mathbb{R}$ such that if D is represented locally by (f) on an open set U , then

$$\lambda_{D,v}(P) = -\log |f(P)|_v + \alpha_v(P),$$

where α_v is a continuous function on $U(\bar{k}_v)$ in the v -topology. Any two choices of $\lambda_{D,v}$ differ only by a bounded function on $X(\bar{k}_v) \setminus \text{Supp } D$. Viewing $\bar{k} \subset \bar{k}_v$, we may also view $\lambda_{D,v}$ as a function on $X(\bar{k}) \setminus \text{Supp } D$. The functions $\lambda_{D,v}$ give a measure of the v -adic distance of a point P from D (being large when P is v -adically close to D). If S is a finite set of places of k , we define the proximity function $m_{D,S}(P)$ (relative to D and S) for $P \in X(\bar{k}) \setminus \text{Supp } D$ by

$$m_{D,S}(P) = \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_{D,w}(P).$$

Associated to D we also have an absolute logarithmic height function on $X(\bar{k})$, denoted $h_D(P)$. In fact, up to a bounded function, h_D depends only on the linear equivalence class of D . For appropriately chosen local Weil functions $\lambda_{D,v}$, $v \in M_k$, we can decompose the height into a sum of local Weil functions:

$$h_D(P) = \sum_{v \in M_k} \lambda_{D,v}(P)$$

for P in $X(k) \setminus \text{Supp } D$. We refer the reader to [2], [5], and [22] for a more thorough discussion of the definitions and properties of local Weil functions and height functions.

For a point P in $X(\bar{k})$, we define the absolute logarithmic discriminant by

$$d(P) = \frac{1}{[k(P):\mathbb{Q}]} \log |D_{k(P)/\mathbb{Q}}|,$$

where $D_{k(P)/\mathbb{Q}}$ is the discriminant of $k(P)$ over $\mathbb{Q}$. As usual, we let $\mathcal{O}_X$ denote the structure sheaf of X and we let K_X denote a canonical divisor of X . We say that D is a big divisor on X if

$$\limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^{\dim X}} > 0.$$

In particular, an ample divisor is big. More generally, a well-known lemma of Kodaira [6, Addendum] gives that on a projective variety a divisor is big if and only if some positive multiple of the divisor can be written as the sum of an ample divisor and an effective divisor.

We say that an effective divisor D is a normal crossings divisor (or has normal crossings) if every point $P \in \text{Supp } D$ has an analytic open neighborhood in $X(\mathbb{C})$ with analytic local coordinates $z_1, \dots, z_{\dim X}$ such that D is locally defined by $z_1 \cdot z_2 \cdots z_i = 0$ for some i . In particular, a normal crossings divisor is a reduced effective divisor.

3. THE DEPENDENCE OF THE EXCEPTIONAL SET ON A

We now show that the exceptional set must be allowed to depend on the divisor A .

Theorem 3.1. *The exceptional set $Z = Z(r, \epsilon, X, D, A)$ cannot be chosen independently of A , even for $r = 1$, $D = 0$, and varying only over ample divisors A .*

As mentioned earlier, this theorem is very closely related to the fact that there are examples where the exceptional set $Z(r, \epsilon, X, D, A)$ must depend on ϵ .

Proof. Let X be a nonsingular projective surface over a number field k with infinitely many distinct curves $C \cong \mathbb{P}_k^1$ having intersection number $K_X.C > 0$. An example of such a surface was explicitly given in [11, Ex. 2.4]. Let $0 < \epsilon < \frac{1}{2}$. It suffices to show that for any such curve C , there exists an ample divisor A with $C \subset Z(r = 1, \epsilon, X, D = 0, A)$. Let C be such a curve. We first claim that there exists an ample divisor A' on X satisfying $\gcd(A'.C, C.C) \leq 2$. Note that since $C \cong \mathbb{P}^1$, by the adjunction formula, $K_X.C = -2 - C.C$ and $C.C < 0$. It follows that for any ample divisor A'' on X , either $A' = A''$ or $A' = nA'' + K_X$, for some sufficiently large integer n , will satisfy A' is ample and $\gcd(A'.C, C.C) \leq 2$. Since $A'.C > 0$ and $C.C < 0$, we may choose positive integers i and j such that $i(A'.C) + j(C.C) = \gcd(A'.C, C.C) \leq 2$. Let $A = iA' + jC$. We claim that A is ample. By construction, we have $A.C > 0$. For any curve $C' \neq C$, $C.C' \geq 0$ and $A'.C' > 0$, so $A.C' > 0$. Moreover, since $A'.A > 0$ and $C.A > 0$, we have $A^2 > 0$. It follows from the Nakai-Moishezon criterion [4, p. 365] that A is ample. Since $K_X.C - \epsilon A.C \geq 1 - 2\epsilon > 0$, infinitely many points P in $C(k)$ will satisfy $h_K(P) > \epsilon h_A(P)$. So we must have $C \subset Z(r = 1, \epsilon, X, D = 0, A)$, as was to be shown. $\square$

4. ELLIPTIC SURFACES

AND THE DEPENDENCE OF THE EXCEPTIONAL SET ON r

In this section we will study certain instances of Vojta's Conjecture on elliptic surfaces. We assume throughout that our (nonsingular, projective) elliptic surfaces are nonconstant and relatively minimal (i.e., no fiber contains an exceptional curve of the first kind). Recall that such an elliptic surface is called *semistable* if every singular fiber has Kodaira type I_n for some $n \geq 1$. In particular, on a semistable elliptic surface X , every fiber is a normal crossings divisor.

We will find it convenient to consider a modified version of Vojta's Conjecture where the coefficient of $d(P)$ is a function of the dimension of X . Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a fixed positive real-valued function.

Conjecture 4.1 (Modified Vojta Conjecture with function $f(n)$). *Let X be a nonsingular complete variety of dimension n with canonical divisor K . Let D be a normal crossings divisor on X , and let k be a number field over which X and D are defined. Let S be a finite set of places of k , A a big divisor on X , $\epsilon > 0$, and r a positive integer. Then there exists a proper Zariski closed subset $Z = Z(f, r, \epsilon, X, D, A)$ of X such that*

$$(4.1) \quad m_{D,S}(P) + h_K(P) \leq f(n)d(P) + \epsilon h_A(P) + O(1)$$

for all points $P \in X(\bar{k}) \setminus Z$ with $[k(P) : k] \leq r$.

We now prove the main result of this section.

Theorem 4.2. *Let X be a semistable elliptic surface over $\mathbb{P}^1$ with a section over a number field k . Let D be a sum of an even number N of fibers, containing the singular fibers. If Conjecture 4.1 holds for X , D , and a function f satisfying $f(2) < 1 + \frac{\chi(\mathcal{O}_X)}{N-2}$, then the exceptional set $Z(f, r, \epsilon, X, D, A)$ must depend on the parameter r .*

Proof. By enlarging k , we may assume that every fiber in D is defined over k and that the group of units $\mathcal{O}_k^*$ is infinite. Let $\pi : X \rightarrow \mathbb{P}^1$ be the fibration and let $\sigma : \mathbb{P}^1 \rightarrow X$ be a section over k (which we also identify with its image). We first claim that $\sigma \setminus D \cong \mathbb{P}^1 \setminus \{N \text{ points}\}$ contains an infinite set of integral points T with $[k(P) : k] \leq \frac{N}{2}$ for all $P \in T$. Let $R = \sigma \cap D = R_1 \cup R_2$, where R_1 and R_2 are disjoint sets each with $\frac{N}{2}$ elements of R . Let ψ be a rational function over k on $\mathbb{P}^1$ having zeros exactly at points in R_1 and poles exactly at points in R_2 . Then ψ defines a morphism $\psi : \sigma \setminus D \rightarrow \mathbb{G}_m$ of degree $\frac{N}{2}$. Since $\mathbb{G}_m$ contains an infinite set T' of integral points over k ($\mathcal{O}_k^*$ was assumed infinite), the set $T = \psi^{-1}(T')$ gives an infinite set of integral points on $\sigma \setminus D$ with $[k(P) : k] \leq \frac{N}{2}$ for all $P \in T$.

Taking σ to be the zero section on X , we have a rational map $[m] : X \rightarrow X$ induced by the fiberwise multiplication-by- m map on the nonsingular fibers of X . Let $C \neq \sigma$ be an irreducible component of the Zariski closure of $[m]^{-1}(\sigma \setminus D)$. Let $L \supset k$ be a number field that is a field of definition for C . Then $[m]$ induces a morphism (over L) $\phi : C \rightarrow \sigma$. Let $\tilde{T} = \phi^{-1}(T)$. Let $A = \sigma + F$, where F is some fiber. Then A is a big divisor on X . Indeed, from the intersection numbers $\sigma \cdot F = 1$ and $F^2 = F \cdot F = 0$, it is immediate that for some positive integer n we have $(\sigma + nF)^2 > 0$. Using Riemann-Roch (e.g., [4, pp. 364–365]), this implies that $\sigma + nF$ is big, which in turn implies that $n(\sigma + F) = (n-1)\sigma + (\sigma + nF)$ is big. Finally, since a positive multiple of $\sigma + F$ is big, it follows from the definition that $\sigma + F$ is itself big, as desired. Now let S be the set of Archimedean places of k . We claim that for any $\epsilon > 0$ and any constant c , all but finitely many points P in $\tilde{T}$ satisfy

$$(4.2) \quad m_{D,S}(P) + h_K(P) > \left(1 + \frac{\chi(\mathcal{O}_X) - 2\epsilon}{N-2}\right) d(P) + \epsilon h_A(P) + c.$$

Let us first show that this claim implies the theorem. Every point P in $\tilde{T}$ satisfies $[k(P) : k] \leq \frac{N}{2}[L : k] \deg \phi$. Since $f(2) < 1 + \frac{\chi(\mathcal{O}_X)}{N-2}$, it follows from (4.2) that for sufficiently small ϵ (depending only on $f(2)$), $C \subset Z(f, r, \epsilon, X, D, A)$, where $r = r(C) = \frac{N}{2}[L : k] \deg \phi$. As we vary the integer m , we obtain infinitely many distinct such curves C (with different values $r = r(C)$), obtaining the desired result.

We now show (4.2). First, since $\tilde{T} = \phi^{-1}(T)$ and T is a set of integral points on $\sigma \setminus D$, it follows that $\tilde{T}$ is a set of integral points on $C \setminus D$. Thus,

$$m_{D,S}(P) = h_D(P) + O(1), \quad \forall P \in \tilde{T}.$$

The canonical divisor is given by $K_X = (\chi(\mathcal{O}_X) - 2)F$ [7, Th. 12], where F is a fiber. We have $h_A = h_\sigma + h_F + O(1)$. It is known that $\sigma \cap C = \emptyset$ (see, e.g., [14, p. 72, Prop. (VII.3.2)]). It follows that h_σ is bounded on C . So

$$h_A(P) = h_F(P) + O(1), \quad \forall P \in C.$$

Since $D \sim NF$, it follows that for all P in $\tilde{T}$ we have

$$m_{D,S}(P) + h_K(P) - \epsilon h_A(P) = (N + \chi(\mathcal{O}_X) - 2 - \epsilon)h_F(P) + O(1).$$

Note that $F|_C$ is ample, and so for any infinite set of points of bounded degree on C we have $h_F(P) \rightarrow \infty$. Thus, for any constant c , all but finitely many points P in $\tilde{T}$ satisfy

$$m_{D,S}(P) + h_K(P) - \epsilon h_A(P) > (N + \chi(\mathcal{O}_X) - 2 - 2\epsilon)h_F(P) + c.$$

Comparing with (4.2), to finish the proof it suffices to show that

$$(4.3) \quad d(P) \leq (N - 2)h_F(P) + O(1), \quad \forall P \in \tilde{T}.$$

This will follow from the following two results.

Theorem 4.3 (Mahler [12]). *The inequality*

$$d(P) \leq (2r - 2)h(P) + O(1)$$

holds for all $P \in \mathbb{P}^1(\bar{k})$ with $[k(P) : k] \leq r$.

Theorem 4.4 (Chevalley-Weil). *Let $\phi : X \rightarrow Y$ be a finite morphism of nonsingular projective varieties, with ramification divisor R on X . Let T be a set of integral points of bounded degree on $X \setminus R$. Then*

$$d(P) \leq d(\phi(P)) + O(1)$$

for all P in T .

For a more general quantitative version of the Chevalley-Weil theorem, see [22, Th. 5.1.6].

Since $[k(P) : k] \leq \frac{N}{2}$ for points P in T , by Theorem 4.3 we have

$$(4.4) \quad d(P) \leq (N - 2)h(P) + O(1), \quad \forall P \in T.$$

The map $\phi : C \rightarrow \sigma$ is unramified outside of $D|_C$. So by Theorem 4.4

$$(4.5) \quad d(Q) \leq d(\phi(Q)) + O(1), \quad \forall Q \in \tilde{T}.$$

By functoriality, $h_F(P) = h(\pi(P)) + O(1)$ for all $P \in X(\bar{k})$. So

$$(4.6) \quad h_F(Q) = h_F(\phi(Q)) + O(1) = h(\phi(Q)) + O(1), \quad \forall Q \in C(\bar{k}).$$

Then combining (4.4), (4.5), and (4.6), we obtain (4.3) as desired. $\square$

To optimally apply Theorem 4.2, we want to minimize the number of singular fibers on a semistable elliptic surface relative to $\chi(\mathcal{O}_X)$. This problem was studied by Shioda in [17].

Theorem 4.5 (Shioda). *Let X be a semistable elliptic surface over $\mathbb{P}^1$ with a section. Let N be the number of singular fibers. Then $N \geq 2\chi(\mathcal{O}_X) + 2$. Furthermore, for any positive integer χ , there exists such an X with a section over a number field k , $\chi(\mathcal{O}_X) = \chi$, and exactly $N = 2\chi + 2$ singular fibers.*

The construction of semistable elliptic surfaces with a minimal number of singular fibers is related to so-called Davenport-Stothers triples and to polynomials which achieve equality in the polynomial abc theorem. As a consequence of Shioda's result, we obtain:

Corollary 4.6. *If Conjecture 4.1 holds for $r = \infty$, then we must have $f(2) \geq \frac{3}{2}$.*

A natural question to ask is whether Theorem 4.2 or Corollary 4.6 can be improved by looking at elliptic surfaces X (with a section) over a higher genus base curve B . In general, for such an X and a base curve B , we have Kodaira's formula [7, Th. 12]

$$K_X = \pi^*(K_B + E),$$

where E is some divisor of degree $\chi(\mathcal{O}_X)$ on B , and an inequality of Song and Tucker [18, Eq. (2.0.3)]

$$d(P) \leq (2r + \epsilon)h_{\text{pt}}(P) + h_{K_B}(P) + O(1)$$

for all P in $B(\bar{k})$ with $[k(P) : k] \leq r$, where pt is any point of $B(k)$. Mimicking the proof of Theorem 4.2, we find that the key quantity is the ratio

$$\frac{N + 2g - 2 + \chi(\mathcal{O}_X)}{N + 2g - 2} = 1 + \frac{\chi(\mathcal{O}_X)}{N + 2g - 2},$$

where N is the smallest even integer greater than or equal to the number of singular fibers of X . Unfortunately, we have the general inequality

$$N \geq 2\chi(\mathcal{O}_X) + 2 - 2g.$$

Thus, it does not appear that one can improve on the constant $\frac{3}{2}$ in Corollary 4.6 by using a base curve of higher genus. More generally, in the case X is not necessarily semistable, we have the inequality [16, Cor. 2.7]

$$\mu + 2\alpha \geq 2\chi(\mathcal{O}_X) + 2 - 2g,$$

where μ and α are the number of multiplicative and additive singular fibers, respectively. This gives some hope of using additive singular fibers to improve the number $\frac{3}{2}$ in Corollary 4.6. In this case, however, the singular fibers are no longer normal crossings divisors (as is required in Vojta's Conjecture), and this appears to prevent one, at least with our method, from improving on Corollary 4.6.

Using a trick of Vojta [22, Prop. 5.4.1], we can use our results on elliptic surfaces to construct varieties X such that $D = 0$ and the exceptional set $Z(r, \epsilon, X, D, A)$ must depend on the parameter r .

Theorem 4.7. *There exist a nonsingular projective surface V defined over a number field k ; curves $C_i \subset V$, $i \in \mathbb{N}$, with $\bigcup_i C_i$ Zariski-dense in V ; a big divisor A ; and a function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $i \in \mathbb{N}$, $\delta > 0$, sufficiently small $\epsilon > 0$, and constant c_i ,*

$$h_{K_V}(P) > \left(\frac{3}{2} - \delta\right)d(P) + \epsilon h_A(P) + c_i$$

for an infinite set of points $P \in C_i$ with $[k(P) : k] \leq g(i)$. In particular, for any set of constants c_i , $i \in \mathbb{N}$, $\delta > 0$, and any sufficiently small $\epsilon > 0$, the inequality

$$h_{K_V}(P) > \left(\frac{3}{2} - \delta\right) d(P) + \epsilon h_A(P) + c_{[k(P):k]}$$

holds for a Zariski-dense set of points $P \in V(\bar{k})$.

Proof. By Theorem 4.5, there exists a semistable elliptic surface X over $\mathbb{P}^1$ with a section over a number field k and $N = 2\chi(\mathcal{O}_X) + 2$ singular fibers. Let D be the sum of the singular fibers, which we may assume are all defined over k . By Theorem 4.2 and its proof, there exist infinitely many distinct curves $C'_i \subset X$, $i \in \mathbb{N}$, with an infinite set of D -integral points T_i of bounded degree, and an ample divisor A' , such that for any constant c_i and sufficiently small $\epsilon > 0$,

$$m_{D,S}(P) + h_{K_X}(P) > \left(\frac{3}{2} - \delta\right) d(P) + \epsilon h_{A'}(P) + c_i,$$

for all but finitely many points P in T_i .

We may assume one of the singular fibers lies over the point at infinity of $\mathbb{P}^1$. Let $t_1, \dots, t_{N-1} \in k$ be the projections of the other singular fibers. Let M be a large enough positive integer such that

$$m_{D,S}(P) < \frac{M\epsilon}{2} h_{A'}(P) + O(1)$$

for all P in $X(\bar{k}) \setminus D$. Let B be the nonsingular projective curve associated to $k(\sqrt[M]{t-t_1}, \dots, \sqrt[M]{t-t_{N-1}})$. We have a natural morphism $\phi : B \rightarrow \mathbb{P}^1$ of degree M^{N-1} , with ramification index M at each point above $t_1, \dots, t_{N-1}, \infty$ and unramified everywhere else. Let $V = X \times_{\mathbb{P}^1} B$ and let $\pi : V \rightarrow X$ be the projection map. From the above, it follows easily that the ramification divisor R of π is given by $R = \frac{M-1}{M} \pi^* D$. We also have the formula $K_V = \pi^* K_X + R$. Since the set of points T_i is D -integral, by Theorem 4.4 we have

$$d(Q) \leq d(\pi(Q)) + O(1)$$

for all points Q in $\pi^{-1}(T_i)$. Let $A = \pi^* A'$, a big divisor. Then for all $Q \in \pi^{-1}(T_i)$, setting $P = \pi(Q)$, we have (up to $O(1)$)

$$\begin{aligned} h_{K_V}(Q) &= h_{\pi^* K_X}(Q) + h_R(Q) = h_{K_X}(P) + \frac{M-1}{M} h_D(P) \\ &\geq h_{K_X}(P) + \frac{M-1}{M} m_{D,S}(P) \\ &\geq h_{K_X}(P) + m_{D,S}(P) - \frac{\epsilon}{2} h_{A'}(P) \\ &\geq \left(\frac{3}{2} - \delta\right) d(P) + \frac{\epsilon}{2} h_{A'}(P) \\ &\geq \left(\frac{3}{2} - \delta\right) d(Q) + \frac{\epsilon}{2} h_A(Q). \end{aligned}$$

This yields the theorem, taking C_i to be an appropriate curve in $\pi^{-1}(C'_i)$ for each $i \in \mathbb{N}$. $\square$

Finally, let us discuss some recent results in a related setting that are closely connected to the results of this section. In a geometric setting, Autissier, Chambert-Loir, and Gasbarri have recently shown [1]:

Theorem 4.8 (Autissier, Chambert-Loir, Gasbarri). *Let $n \geq 2$ be an integer and k an algebraically closed field of characteristic zero. There exists a nonsingular projective variety X over k of dimension n , with canonical divisor K , and projective curves $C_i \subset X$, $i \in \mathbb{N}$, such that the following are true:*

- (1) $\bigcup_{i \in \mathbb{N}} C_i$ is Zariski-dense in X .
- (2) Let A be an ample divisor on X and let $\delta > 0$. For all sufficiently small $\epsilon > 0$,

$$C_i.K > (n - \delta)\chi(C_i) + \epsilon C_i.A,$$

for all but finitely many $i \in \mathbb{N}$.

Here, for a projective curve C , we let $\chi(C) = 2g(C) - 2$ where $g(C)$ is the geometric genus of C . The terms $C.K$, $C.A$, and $\chi(C)$ are analogous to (the unnormalized versions of) $h_K(P)$, $h_A(P)$, and $d(P)$, respectively, in Conjecture 1.1. As noted in [1], the gonality of the curves C_i constructed in [1] in Theorem 4.8 go to infinity with i (the gonality is an analogue of the degree $[k(P) : k]$). Theorem 4.8 shows that in a geometric setting the analogue of inequality (1.1) in Vojta's Conjecture does not hold uniformly without at least a $\dim X$ factor in front of (the analogue of) the $d(P)$ term.

On the other hand, for nonsingular curves on minimal surfaces of general type, Miyaoka [15] proved:

Theorem 4.9 (Miyaoka). *Let X be a nonsingular minimal complex projective surface of general type with canonical divisor K . Let A be an ample divisor on X and let $\epsilon > 0$. Then*

$$C.K \leq \frac{3}{2}\chi(C) + \epsilon C.A + O(1),$$

for all nonsingular projective curves C in X .

It is unclear if the appearance of the coefficient $\frac{3}{2}$ in both Miyaoka's result and the results given here is coincidence or holds some deeper significance.

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