NONLINEAR PERTURBATIONS OF LINEAR ELLIPTIC SYSTEMS AT RESONANCE

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ABSTRACT. We consider a semilinear system

$$\begin{aligned}\Delta u + \lambda v + b_1(v) &= f(x), \quad x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega, \\ \Delta v + \frac{\lambda_1^2}{\lambda} u + b_2(u) &= g(x), \quad x \in \Omega, \quad v = 0 \quad \text{for } x \in \partial\Omega,\end{aligned}$$

whose linear part is at resonance. Here $\lambda > 0$ and the functions $b_1(t)$ and $b_2(t)$ are bounded and continuous. Assuming that $tb_i(t) > 0$ for all $t \in R$, $i = 1, 2$, and that the first harmonics of $f(x)$ and $g(x)$ lie on a certain straight line, we prove the existence of solutions. This extends a similar result for one equation, due to D.G. de Figueiredo and W.-M. Ni.

1. INTRODUCTION

Following the publication of the classical paper of E.M. Landesman and A.C. Lazer [7], there has been enormous interest in nonlinear perturbations of linear equations at resonance of the type

$$(1.1) \quad \Delta u + \lambda_1 u + b(u) = f(x), \quad x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega,$$

where Ω is a bounded smooth domain in R^n and λ_1 is the principal eigenvalue of the Laplacian $-\Delta$ on Ω , with zero at the boundary condition (we shall denote by $\phi_1(x)$ the corresponding eigenfunction). Early contributions included the other classics, such as A. Ambrosetti and G. Prodi [1] and M.S. Berger and E. Podolak [3]; see a nice presentation in the book of A. Ambrosetti and G. Prodi [2]. Recently, the present author [6] has suggested a unified approach to these results. The function $b(u)$ is usually assumed to be bounded and continuous, and the famous E.M. Landesman and A.C. Lazer [7] conditions required that it had limits at $\pm\infty$. In an elegant paper, D.G. de Figueiredo and W.-M. Ni [5] proved the existence of solutions assuming that $ub(u) > 0$ for all $u \in R$ and the forcing term $f(x)$ has zero first harmonic; i.e., $\int_{\Omega} f(x)\phi_1(x) dx = 0$. Their proof involved the establishment of an a priori estimate, which was remarkable because such estimates usually require some conditions on $b(u)$ at infinity.

In this paper we extend the result of D.G. de Figueiredo and W.-M. Ni [5] to a system of two equations. The system

$$(1.2) \quad \begin{aligned}\Delta u + \lambda v + b_1(v) &= f(x), \quad x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega, \\ \Delta v + \frac{\lambda_1^2}{\lambda} u + b_2(u) &= g(x), \quad x \in \Omega, \quad v = 0 \quad \text{for } x \in \partial\Omega,\end{aligned}$$

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with any $\lambda > 0$ can be seen as the case of resonance at the principal eigenvalue, similar to (1.1). Similarly to [5], we assume that $tb_i(t) > 0$ for all $t \in R$, $i = 1, 2$. We prove the existence of solutions, provided that the first harmonics of $f(x)$ and $g(x)$ lie on a certain straight line. There is considerable interest in systems of this type; see, e.g., the recent surveys of D.G. de Figueiredo [4] and B. Ruf [9].

2. EXISTENCE OF SOLUTIONS

On a smooth domain $\Omega \subset R^n$, we consider a weakly coupled linear system

$$(2.1) \quad \begin{aligned} \Delta u + \lambda v &= f(x), \quad x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega, \\ \Delta v + \bar{\lambda} u &= g(x), \quad x \in \Omega, \quad v = 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

with given functions $f(x)$ and $g(x)$, and parameters λ and $\bar{\lambda}$. The following proposition identifies the set of nonresonant parameters λ and $\bar{\lambda}$. We denote by λ_n the eigenvalues of $-\Delta$ on Ω , with zero boundary condition, and by $\phi_n(x)$ the corresponding eigenfunctions.

Proposition 1. *Assume that $\lambda\bar{\lambda} \neq \lambda_n^2$ for all $n \geq 1$. Then for any pair $(f(x), g(x)) \in L^2(\Omega) \times L^2(\Omega)$ there exists a unique solution $(u(x), v(x)) \in \left(W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\right)^2$.*

Proof. Existence of a solution in $L^2(\Omega) \times L^2(\Omega)$ follows by using the Fourier series in $\phi_n(x)$, written for u, v, f , and g , and then the standard elliptic estimates provide the extra regularity of a solution. $\square$

The *resonance* case is when $\lambda\bar{\lambda} = \lambda_n^2$. We shall consider the *principal* resonance case $\lambda\bar{\lambda} = \lambda_1^2$, i.e., $\bar{\lambda} = \frac{\lambda_1^2}{\lambda}$. We shall prove solvability for the system

$$(2.2) \quad \begin{aligned} \Delta u + \lambda v + b_1(v) &= f(x), \quad x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega, \\ \Delta v + \frac{\lambda_1^2}{\lambda} u + b_2(u) &= g(x), \quad x \in \Omega, \quad v = 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

with given functions $f(x), g(x) \in L^2(\Omega)$, and a constant $\lambda > 0$. The following is a system analog of the result of D.G. de Figueiredo and W.-M. Ni [5]. We denote $\phi_1^\perp = \{f \in L^2(\Omega) : \int_\Omega f \phi_1 dx = 0\}$.

Theorem 2.1. *Assume that $b_1(t)$ and $b_2(t)$ are bounded and continuous functions such that*

$$(2.3) \quad tb_i(t) > 0 \quad \text{for all } t \in R, \quad i = 1, 2.$$

Decompose $f(x) = \mu_1 \phi_1(x) + e_1(x)$, $g(x) = \nu_1 \phi_1(x) + e_2(x)$, with $e_1(x), e_2(x) \in \phi_1^\perp$. Then the system (2.2) is solvable for any (μ_1, ν_1) satisfying

$$(2.4) \quad \lambda_1 \mu_1 + \lambda \nu_1 = 0$$

with $u, v \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, for all $p > 2$.

The proof will be based on the following lemmas. The first one follows immediately by considering the Fourier series in $\phi_n(x)$.

Lemma 2.1. *The solution set of the linear system*

$$(2.5) \quad \begin{aligned} \Delta u + \lambda v &= 0, \quad x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega, \\ \Delta v + \frac{\lambda_1^2}{\lambda} u &= 0, \quad x \in \Omega, \quad v = 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

is $(u, v) = c(\phi_1, \frac{\lambda_1}{\lambda} \phi_1)$, where c is an arbitrary constant. In particular, the only solution of (2.5) in $\phi_1^\perp \times \phi_1^\perp$ is $(0, 0)$.

Lemma 2.2. *Let $U, V \in \phi_1^\perp$ be solutions of*

$$(2.6) \quad \begin{aligned} \Delta U + \lambda V &= f(x), \quad x \in \Omega, \quad u = 0 \text{ for } x \in \partial\Omega, \\ \Delta V + \frac{\lambda_1^2}{\lambda} U &= g(x), \quad x \in \Omega, \quad v = 0 \text{ for } x \in \partial\Omega, \end{aligned}$$

with $f(x), g(x) \in L^\infty(\Omega)$. Then for any $p > 1$ one can find a constant $c > 0$ such that

$$(2.7) \quad \|U\|_{W^{2,p}(\Omega)} + \|V\|_{W^{2,p}(\Omega)} \leq c(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}).$$

By the Sobolev imbedding this implies that $\|U\|_{L^\infty(\Omega)} + \|V\|_{L^\infty(\Omega)} \leq c_1$ for some constant $c_1 > 0$.

Proof. Standard elliptic estimates imply that

$$\|U\|_{W^{2,p}} + \|V\|_{W^{2,p}} \leq c(\|f\|_{L^p} + \|g\|_{L^p} + \|U\|_{L^p} + \|V\|_{L^p}).$$

The estimate (2.7) will follow, once we prove that

$$(2.8) \quad \|U\|_{L^p} + \|V\|_{L^p} \leq c(\|f\|_{L^p} + \|g\|_{L^p}).$$

Assume for definiteness that $\|f\|_{L^p} \geq \|g\|_{L^p}$. Dividing both equations in (2.7) by the same constant $\|f\|_{L^p}$ and redefining U and V , we may assume that $\|f\|_{L^p} = 1$ and $\|g\|_{L^p} \leq 1$. Assuming that the estimate (2.8) is not possible with any constant c , we could find a sequence $\{f_n, g_n\}$, with $\|f_n\|_{L^p} = 1$ and $\|g_n\|_{L^p} \leq 1$, and the corresponding solutions of (2.6) $\{U_n, V_n\} \in \phi_1^\perp \times \phi_1^\perp$ so that

$$\|U_n\|_{L^p} + \|V_n\|_{L^p} \geq n(1 + \|g_n\|_{L^p}).$$

In particular, $\|U_n\|_{L^p} + \|V_n\|_{L^p} \rightarrow \infty$, as $n \rightarrow \infty$. Define $u_n = \frac{U_n}{\|U_n\|_{L^p} + \|V_n\|_{L^p}}$ and $v_n = \frac{V_n}{\|U_n\|_{L^p} + \|V_n\|_{L^p}}$. They satisfy

$$(2.9) \quad \begin{aligned} \Delta u_n + \lambda v_n &= \frac{f_n(x)}{\|U_n\|_{L^p} + \|V_n\|_{L^p}}, \\ \Delta v_n + \frac{\lambda_1^2}{\lambda} u_n &= \frac{g_n(x)}{\|U_n\|_{L^p} + \|V_n\|_{L^p}}. \end{aligned}$$

Since $\|u_n\|_{L^p} < 1$, $\|v_n\|_{L^p} < 1$, we get uniform in n bounds for $\|u_n\|_{W^{2,p}}$ and $\|v_n\|_{W^{2,p}}$. In a standard way, along a subsequence $\{u_n, v_n\} \rightarrow (u, v) \in \phi_1^\perp \times \phi_1^\perp$, with (u, v) solving (2.5). Hence $u = v = 0$ by Lemma 2.2, but $\|u + v\|_{L^p} = 1$, a contradiction. $\square$

The following lemma provides the crucial a priori estimate. As mentioned by D.G. de Figueiredo and W.-M. Ni, it is remarkable that this estimate does not require any conditions on $b_i(t)$ at infinity (which are usually needed to get a priori estimates).

Lemma 2.3. *In the conditions of the Theorem 2.1, there is a constant $c > 0$ so that any solution of (2.2) satisfies*

$$\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq c.$$

Proof. Decompose $u(x) = \xi_1 \phi_1(x) + U(x)$, $v(x) = \eta_1 \phi_1(x) + V(x)$, with $U(x), V(x) \in \phi_1^\perp$. Then system (2.2) becomes

$$(2.10) \quad \begin{aligned} \Delta U + \lambda V + (-\lambda_1 \xi_1 + \lambda \eta_1) \phi_1 + b_1(\eta_1 \phi_1(x) + V(x)) &= \mu_1 \phi_1(x) + e_1(x), \\ \Delta U + \frac{\lambda_1^2}{\lambda} U + \frac{\lambda_1}{\lambda} (\lambda_1 \xi_1 - \lambda \eta_1) \phi_1 + b_2(\xi_1 \phi_1(x) + U(x)) &= \nu_1 \phi_1(x) + e_2(x). \end{aligned}$$

We claim that

$$(2.11) \quad |-\lambda_1 \xi_1 + \lambda \eta_1| \leq c,$$

for some constant $c > 0$. Indeed, multiply the first equation in (2.10) by ϕ_1 and integrate over Ω . Since $\int_{\Omega} \Delta U \phi_1 dx = \int_{\Omega} V \phi_1 dx = 0$, while b_1 is a bounded function, the claim follows. By Lemma 2.2, it follows that

$$(2.12) \quad \|U\|_{C^1(\Omega)} + \|V\|_{C^1(\Omega)} \leq c_1,$$

for some constant $c_1 > 0$.

To complete the proof, we need an a priori estimate of the first harmonics ξ_1 and η_1 . By (2.11), if either one of ξ_1 and η_1 is large and positive (negative), so is the other one. Assume for definiteness that ξ_1 and η_1 are both negative and large in absolute value. Multiply the first equation in (2.10) by $\lambda_1 \phi_1$, the second one by $\lambda \phi_1$, integrate over Ω , and add the results. We may assume that $\int_{\Omega} \phi_1^2 dx = 1$. By our condition (2.4)

$$0 = \lambda_1 \mu_1 + \lambda \nu_1 = \int_{\Omega} [\lambda_1 b_1(\eta_1 \phi_1(x) + V(x)) + \lambda b_2(\xi_1 \phi_1(x) + U(x))] \phi_1(x) dx.$$

We claim that the integral on the right is negative, which gives us a contradiction. Indeed, by (2.12), $\eta_1 \phi_1(x) + V(x) < 0$ and $\xi_1 \phi_1(x) + U(x) < 0$ over Ω , and then, by condition (2.3), the functions b_1 and b_2 are negative. $\square$

Proof of the Theorem 2.1. Letting $w = (u, v)$, we rewrite the system (2.2) in the operator form

$$w = T(w),$$

where $T(w) = (\Delta^{-1}(-\lambda v - b_1(v) + \mu_1 \phi_1 + e_1), \Delta^{-1}(-\frac{\lambda_1^2}{\lambda} u - b_2(u) + \nu_1 \phi_1 + e_2))$. T is a compact map $L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega) \times L^2(\Omega)$. We define $\mathbf{L}^2 = L^2(\Omega) \times L^2(\Omega)$, with the norm $\|w\|_{\mathbf{L}^2}^2 = \|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2$. Following D.G. de Figueiredo and W.-M. Ni [5], we consider the operator

$$T_k(w) = \frac{1}{k+1} T(w) - \frac{k}{k+1} T(-w), \quad 0 \leq k \leq 1,$$

which is compact for all k , $T_0 = T$, and T_1 is an odd operator. It is known (see, e.g., L. Nirenberg [8]) that the Leray-Schauder degree

$$\deg(I - T_1, B_R, 0) \neq 0$$

for any ball $B_R = \{w \in \mathbf{L}^2 : \|w\|_{\mathbf{L}^2} \leq R\}$. We claim that there is an R such that

$$w - T_k(w) \neq 0, \quad \text{for } \|w\|_{\mathbf{L}^2} = R, \quad 0 \leq k \leq 1.$$

Then by the homotopy invariance of the degree, $\deg(I - T, B_R, 0) \neq 0$, which implies that system (2.2) has a solution. To prove the claim, we need a uniform in k a priori bound for

$$w - T_k(w) = 0,$$

which is equivalent to

$$(2.13) \quad \begin{aligned} \Delta u + \lambda v + \frac{1}{k+1} b_1(v) - \frac{k}{k+1} b_1(-v) &= \frac{1-k}{1+k} (\mu_1 \phi_1 + e_1), \\ \Delta v + \frac{\lambda_1^2}{\lambda} u + \frac{1}{k+1} b_2(u) - \frac{k}{k+1} b_2(-u) &= \frac{1-k}{1+k} (\nu_1 \phi_1 + e_2). \end{aligned}$$

Clearly, the condition (2.4) on the first harmonics is satisfied for all k . Letting $b_i^k(t) = \frac{1}{k+1} b_i(t) - \frac{k}{k+1} b_i(-t)$, $i = 1, 2$, we see that these functions are uniformly

bounded in k and satisfy the condition (2.3). By Lemma 2.3, we conclude a uniform in k a priori bound for solutions of (2.13), completing the proof. $\square$

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