ASYMPTOTIC PERIODICITY OF GRADE ASSOCIATED TO MULTIGRADED MODULES

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ABSTRACT. Let R be a Noetherian $\mathbb{N}^r$ -graded ring generated in degrees $\mathbf{d}_1, \dots, \mathbf{d}_r$ which are linearly independent vectors over $\mathbb{R}$, and let $\mathfrak{a}$ be an ideal in R_0 . In this paper, we investigate the asymptotic behavior of the grade of the ideal $\mathfrak{a}$ on the homogeneous components $M_{\mathbf{n}}$ of a finitely generated $\mathbb{Z}^r$ -graded R -module M and show that the periodicity occurs in a cone.

1. INTRODUCTION

Let A be a commutative Noetherian ring and $\mathfrak{a}$ an ideal in A . Let R be a Noetherian $\mathbb{N}^r$ -graded ring with $R_0 = A$, and let M be a finitely generated $\mathbb{Z}^r$ -graded R -module. In this paper, we study the asymptotic behavior of the numerical function $\text{grade}(\mathfrak{a}, M_{\mathbf{n}})$ associated to M .

The first result in this setting is due to McAdam and Eakin [7]. They proved that if R is a standard $\mathbb{N}$ -graded ring, then the set of primes $\text{Ass}_A M_n$ is stable for all large n , and hence, by using the technique due to Brodmann [3], we have that $\text{grade}(\mathfrak{a}, M_n)$ is constant for all large n . They also showed that Brodmann's result [2] about the asymptotic prime divisors of an ideal followed from their results as a direct consequence. Afterwards, a number of authors have extended these results to multigraded cases, especially in connection with the study of the asymptotic prime divisors and the analytic spread of ideals. In particular, West [8] extended McAdam-Eakin's results to multigraded cases. He proved that if R is a standard $\mathbb{N}^r$ -graded ring, then the set of primes $\text{Ass}_A M_{\mathbf{n}}$ is eventually stable, and hence $\text{grade}(\mathfrak{a}, M_{\mathbf{n}})$ is constant for all large $\mathbf{n}$. He also considered several interesting nonstandard multigraded cases. For more general results in the standard graded cases, see [1, 5].

On the other hand, when R is a standard $\mathbb{N}$ -graded ring with a local ring A , Herzog and Hibi [6] gave a direct proof of the stability of $\text{depth}_A M_n$ by using the Hilbert polynomial of Koszul homology modules of M with respect to the maximal ideal of A instead of the asymptotic stability of $\text{Ass}_A M_n$. More recently, Colomn and Elias [4] investigated in a similar way the asymptotic behavior of the depth associated to graded modules over certain nonstandard multigraded rings.

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The purpose of this paper is to give a common generalization to all of the results concerning the asymptotic grade stated above with a more direct approach. The result is the following:

Theorem 1.1. *Let R be a Noetherian $\mathbb{N}^r$ -graded ring generated in degrees $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$, where $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r \in \mathbb{N}^r$ are linearly independent vectors over $\mathbb{R}$, with $R_0 = A$. Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$ -graded R -module. Then, for any ideal $\mathfrak{a}$ in A , there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that, in the cone $C_{\mathbf{k}}$ with vertex $\mathbf{k}$ generated by $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$, $\text{grade}(\mathfrak{a}, M_{\mathbf{n}})$ is periodic with respect to $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$. Namely, the equality*

$$\text{grade}(\mathfrak{a}, M_{\mathbf{n}}) = \text{grade}(\mathfrak{a}, M_{\mathbf{n}+\mathbf{m}})$$

holds true for all $\mathbf{n} \in C_{\mathbf{k}}$ and all $\mathbf{m} \in \Gamma$, where Γ is the semigroup generated by $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$.

In the next section, we fix our notation and give some facts about cones and graded modules. In section 3, we will give a proof of Theorem 1.1.

Throughout this paper, A is a commutative Noetherian ring with identity. $\mathbb{N}$ (resp. $\mathbb{R}$) denotes the set of nonnegative integers (resp. real numbers), and r is any fixed positive integer. Vectors will always be written in bold-faced letters, e.g., $\mathbf{a}$, and they will be represented by row vectors, e.g., $\mathbf{a} = (a_1, a_2, \dots, a_r)$.

2. PRELIMINARIES

Let $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r \in \mathbb{N}^r$ be any fixed linearly independent vectors over $\mathbb{R}$. We denote by $\Gamma \subseteq \mathbb{N}^r$ the semigroup generated by $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$, i.e.,

$$\Gamma = \left\{ \sum_{i=1}^r c_i \mathbf{d}_i \mid c_i \in \mathbb{N} \right\}.$$

For any vector $\mathbf{k} \in \mathbb{N}^r$, let

$$C_{\mathbf{k}} := \left\{ \mathbf{k} + \sum_{i=1}^r c_i \mathbf{d}_i \mid c_i \in \mathbb{R}_{\geq 0} \right\} \cap \mathbb{N}^r$$

be the cone with vertex $\mathbf{k}$ generated by $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$ and

$$\Delta_{\mathbf{k}} := \left\{ \mathbf{k} + \sum_{i=1}^r c_i \mathbf{d}_i \mid 0 \leq c_i < 1, c_i \in \mathbb{R} \right\} \cap \mathbb{N}^r$$

be the basic cell of the cone $C_{\mathbf{k}}$. Then it is easy to see that

- (i) $\Delta_{\mathbf{k}}$ is a finite subset of $C_{\mathbf{k}}$,
- (ii) for any $\mathbf{n} \in C_{\mathbf{k}}$, there is a unique expression $\mathbf{n} = \delta + \mathbf{m}$ with $\delta \in \Delta_{\mathbf{k}}$ and $\mathbf{m} \in \Gamma$, and
- (iii) $C_{\mathbf{k}} = \bigcup_{\delta \in \Delta_{\mathbf{k}}} (\delta + \Gamma)$.

Moreover, we have the following:

Lemma 2.1. *For any vectors $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that*

$$\mathbf{k}' + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \dots + \mathbf{d}_r) \in C_{\mathbf{k}} \text{ for all } \ell \geq \ell_0.$$

In particular, $C_{\mathbf{k}} \cap C_{\mathbf{k}'} \neq \emptyset$, and hence there exists a cone $C_{\mathbf{k}''}$ such that $C_{\mathbf{k}''} \subseteq C_{\mathbf{k}} \cap C_{\mathbf{k}'}$.

Proof. Consider the system

$$(1) \quad \mathbf{x}D = \mathbf{k} - \mathbf{k}',$$

where D is a square matrix of size r whose i -th row is $\mathbf{d}_i$. Since $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$ are linearly independent, there is a unique solution $\mathbf{a} = (a_1, a_2, \dots, a_r) \in \mathbb{R}^r$ of the system (1). Let

$$\ell_0 := \max \{ \lceil |a_i| \rceil \mid i = 1, 2, \dots, r \},$$

where $\lceil * \rceil$ denotes the least integer $\geq *$. Take $\ell \geq \ell_0$ and put $c_i := \ell - a_i \in \mathbb{R}_{\geq 0}$. Then $\mathbf{a} = (\ell - c_1, \ell - c_2, \dots, \ell - c_r)$ and we have

$$\mathbf{k}' + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \dots + \mathbf{d}_r) = \mathbf{k} + c_1\mathbf{d}_1 + c_2\mathbf{d}_2 + \dots + c_r\mathbf{d}_r \in C_{\mathbf{k}}.$$

The last assertions follow from the fact that the above vector is in $C_{\mathbf{k}'}$. $\square$

Let $R = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} R_{\mathbf{n}}$ be a Noetherian $\mathbb{N}^r$ -graded ring with $R_0 = A$. Assume that R is generated in degrees $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$, i.e., $R = A[R_{\mathbf{d}_1}, R_{\mathbf{d}_2}, \dots, R_{\mathbf{d}_r}]$. Let $R_{++} = (R_{\mathbf{d}_1}R_{\mathbf{d}_2} \cdots R_{\mathbf{d}_r})R$ be the irrelevant ideal of R , which is the ideal generated by homogeneous components of degree $\mathbf{d}_1 + \mathbf{d}_2 + \dots + \mathbf{d}_r$. For a finitely generated $\mathbb{Z}^r$ -graded R -module $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$, we define the *homogeneous support* of M as

$$\text{Supp}_{++} M := \{P \in \text{Spec } R \mid P \text{ is a graded ideal, } M_P \neq (0), \text{ and } R_{++} \not\subseteq P\}.$$

For any vector $\delta \in \mathbb{N}^r$, we set $M^{(\delta+\Gamma)} = \bigoplus_{\mathbf{m} \in \Gamma} M_{\delta+\mathbf{m}}$, which is a graded submodule of M .

Lemma 2.2. *Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$ -graded R -module. Then the following statements are equivalent:*

- (1) $\text{Supp}_{++} M = \emptyset$;
- (2) *there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$.*

Proof. Suppose $\text{Supp}_{++} M = \emptyset$. Let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. Then we claim the following:

Claim. For any $\delta \in \Delta_{\mathbf{k}_0}$, there exists $\mathbf{k} = \mathbf{k}(\delta) \in \mathbb{N}^r$ such that $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$.

Let $\delta \in \Delta_{\mathbf{k}_0}$. The assertion is clear if $M^{(\delta+\Gamma)} = (0)$. Assume $M^{(\delta+\Gamma)} \neq (0)$ and write $M^{(\delta+\Gamma)} = Rm_1 + Rm_2 + \dots + Rm_t$, where $m_i \in M_{\mathbf{k}_i}$ and $\mathbf{k}_i \in \delta + \Gamma$. Since $\text{Supp}_{++} M = \emptyset$, $\text{Supp}_{++} M^{(\delta+\Gamma)} = \emptyset$ so that $R_{++} \subseteq \sqrt{\text{Ann}_R(M^{(\delta+\Gamma)})}$. Therefore there exists an integer $\ell \geq 0$ such that $R_{++}^\ell \cdot m_i = (0)$ for all $i = 1, 2, \dots, t$. This implies that

$$[Rm_i]_{\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \dots + \mathbf{d}_r)} = (0)$$

for all $i = 1, 2, \dots, t$. Let $\mathbf{l}_i := \mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \dots + \mathbf{d}_r) \in \delta + \Gamma$. Then $[Rm_i]_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{l}_i} \cap (\delta + \Gamma)$. Hence, by taking a vector $\mathbf{k} = \mathbf{k}(\delta) \in \mathbb{N}^r$ such that

$$C_{\mathbf{k}} \subseteq C_{\mathbf{l}_1} \cap C_{\mathbf{l}_2} \cap \dots \cap C_{\mathbf{l}_t},$$

we have that $M_{\mathbf{n}} = [M^{(\delta+\Gamma)}]_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$. This completes the proof of the claim.

Now, let $\mathbf{k} \in \mathbb{N}^r$ be a vector such that

$$C_{\mathbf{k}} \subseteq C_{\mathbf{k}_0} \cap \left[\bigcap_{\delta \in \Delta_{\mathbf{k}_0}} C_{\mathbf{k}(\delta)} \right],$$

where $\mathbf{k}(\delta)$ is the vector in the claim. Then $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$. Indeed, since $\mathbf{n} \in C_{\mathbf{k}_0}$, there is a unique expression $\mathbf{n} = \delta + \mathbf{m}$ for some $\delta \in \Delta_{\mathbf{k}_0}$ and $\mathbf{m} \in \Gamma$. Hence $\mathbf{n} \in C_{\mathbf{k}(\delta)} \cap (\delta + \Gamma)$ so that $M_{\mathbf{n}} = (0)$ by the claim.

We prove the other implication. Suppose that there exists $\mathbf{k} \in \mathbb{N}^r$ such that $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$. We write $M = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M_{\mathbf{k}_i}$. Then it is enough to show that for any $i = 1, 2, \dots, t$, there exists an integer $\ell \geq 0$ such that $R_{++}^{\ell} \cdot m_i = (0)$. For the vectors $\mathbf{k}, \mathbf{k}_i \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that

$$\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r) \in C_{\mathbf{k}} \text{ for all } \ell \geq \ell_0$$

by Lemma 2.1. Thus

$$[Rm_i]_{\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r)} \subseteq M_{\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r)} = (0)$$

and hence $R_{++}^{\ell} \cdot m_i = (0)$. $\square$

Lemma 2.3. *Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$ -graded R -module and let $\mathbf{k}_0 \in \mathbb{N}^r$ and $\delta \in \Delta_{\mathbf{k}_0}$. Then there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that*

$$M_{\mathbf{n}} \neq (0) \text{ for all } \mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$$

if $\text{Supp}_{++} M^{(\delta + \Gamma)} \neq \emptyset$.

Proof. Suppose $\text{Supp}_{++} M^{(\delta + \Gamma)} \neq \emptyset$. Assume the contrary, so that for any vector $\mathbf{k} \in \mathbb{N}^r$, there exists $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$ such that $M_{\mathbf{n}} = (0)$. Write $M^{(\delta + \Gamma)} = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M_{\mathbf{k}_i}$ and $\mathbf{k}_i \in \delta + \Gamma$. Then, for each $\mathbf{k}_i$, there exists $\mathbf{n}_i \in C_{\mathbf{k}_i} \cap (\delta + \Gamma)$ such that $M_{\mathbf{n}_i} = (0)$ by the assumption. Therefore $[Rm_i]_{\mathbf{n}_i} = (0)$ and hence $[Rm_i]_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{n}_i}$. By taking a cone $C_{\mathbf{k}}$ such that

$$C_{\mathbf{k}} \subseteq C_{\mathbf{n}_1} \cap C_{\mathbf{n}_2} \cap \cdots \cap C_{\mathbf{n}_t},$$

we have $[M^{(\delta + \Gamma)}]_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$, which implies that $\text{Supp}_{++} M^{(\delta + \Gamma)} = \emptyset$ by Lemma 2.2. This is a contradiction. $\square$

3. PROOF OF THEOREM 1.1

We are now ready to prove Theorem 1.1. Recall that R is a Noetherian $\mathbb{N}^r$ -graded ring generated in degrees $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r$, where $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r \in \mathbb{N}^r$ are linearly independent vectors over $\mathbb{R}$, with $R_0 = A$. Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$ -graded R -module and let $\mathfrak{a}$ be an ideal in A .

Proof of Theorem 1.1. Let $\mathfrak{a} = (a_1, a_2, \dots, a_p)A$ and let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. Then we claim the following:

Claim. For any $\delta \in \Delta_{\mathbf{k}_0}$, there exist a vector $\mathbf{k} = \mathbf{k}(\delta) \in \mathbb{N}^r$ and a constant $c = c(\delta) \in \mathbb{N} \cup \{\infty\}$ such that $\text{grade}(\mathfrak{a}, M_{\mathbf{n}}) = c$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$.

Let $\delta \in \Delta_{\mathbf{k}_0}$ and $L := M/\mathfrak{a}M$. If $\text{Supp}_{++} L^{(\delta + \Gamma)} = \emptyset$, then there exists $\mathbf{k} \in \mathbb{N}^r$ such that $L_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$ by Lemma 2.2. Therefore $M_{\mathbf{n}} = \mathfrak{a}M_{\mathbf{n}}$ so that $\text{grade}(\mathfrak{a}, M_{\mathbf{n}}) = \infty$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$. Suppose $\text{Supp}_{++} L^{(\delta + \Gamma)} \neq \emptyset$. By Lemma 2.3, there exists $\mathbf{k}_1 \in \mathbb{N}^r$ such that for any $\mathbf{n} \in C_{\mathbf{k}_1} \cap (\delta + \Gamma)$, $L_{\mathbf{n}} \neq (0)$ so that $M_{\mathbf{n}} \neq \mathfrak{a}M_{\mathbf{n}}$. Thus, by the grade sensitivity of the Koszul complex, we have that for any $\mathbf{n} \in C_{\mathbf{k}_1} \cap (\delta + \Gamma)$,

$$\text{grade}(\mathfrak{a}, M_{\mathbf{n}}) = p - \max\{i \mid H_i(\underline{a}; M_{\mathbf{n}}) \neq (0)\},$$

where $H_i(\underline{a}; *)$ denotes the i -th Koszul homology module of $*$ with respect to the sequence $\underline{a} = a_1, a_2, \dots, a_p$. Let

$$q := \max \left\{ i \mid \text{Supp}_{++} \left(H_i \left(\underline{a}; M^{(\delta+\Gamma)} \right) \right) \neq \emptyset \right\}.$$

For any $i > q$, since $\text{Supp}_{++} \left(H_i \left(\underline{a}; M^{(\delta+\Gamma)} \right) \right) = \emptyset$, there exists $\mathbf{l}_i \in \mathbb{N}^r$ such that

$$H_i(\underline{a}; M_{\mathbf{n}}) = (0) \text{ for all } \mathbf{n} \in C_{\mathbf{l}_i} \cap (\delta + \Gamma)$$

by Lemma 2.2. On the other hand, since $\text{Supp}_{++} \left(H_q(\underline{a}; M^{(\delta+\Gamma)}) \right) \neq \emptyset$, there exists $\mathbf{l}_q \in \mathbb{N}^r$ such that

$$H_q(\underline{a}; M_{\mathbf{n}}) \neq (0) \text{ for all } \mathbf{n} \in C_{\mathbf{l}_q} \cap (\delta + \Gamma)$$

by Lemma 2.3. Thus, by taking a cone $C_{\mathbf{k}_2}$ such that

$$C_{\mathbf{k}_2} \subseteq C_{\mathbf{l}_q} \cap C_{\mathbf{l}_{q+1}} \cap \dots \cap C_{\mathbf{l}_p},$$

we have that for all $\mathbf{n} \in C_{\mathbf{k}_2} \cap (\delta + \Gamma)$,

$$H_q(\underline{a}; M_{\mathbf{n}}) \neq (0), \text{ and } H_i(\underline{a}; M_{\mathbf{n}}) = (0) \text{ if } i > q.$$

By taking a cone $C_{\mathbf{k}}$ such that $C_{\mathbf{k}} \subseteq C_{\mathbf{k}_1} \cap C_{\mathbf{k}_2}$, we get that

$$\text{grade}(\mathbf{a}, M_{\mathbf{n}}) = p - q \text{ for all } \mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma).$$

This completes the proof of the claim.

Now let us take a cone $C_{\mathbf{k}}$ such that

$$C_{\mathbf{k}} \subseteq C_{\mathbf{k}_0} \cap \left[\bigcap_{\delta \in \Delta_{\mathbf{k}_0}} C_{\mathbf{k}(\delta)} \right],$$

where $\mathbf{k}(\delta)$ is the vector in the claim. Then we have the equality

$$\text{grade}(\mathbf{a}, M_{\mathbf{n}}) = \text{grade}(\mathbf{a}, M_{\mathbf{n}+\mathbf{m}})$$

for all $\mathbf{n} \in C_{\mathbf{k}}$ and all $\mathbf{m} \in \Gamma$. Indeed, since $\mathbf{n} \in C_{\mathbf{k}_0}$, there is a unique $\delta \in \Delta_{\mathbf{k}_0}$ such that $\mathbf{n} \in \delta + \Gamma$. Thus $\mathbf{n}, \mathbf{n} + \mathbf{m} \in C_{\mathbf{k}(\delta)} \cap (\delta + \Gamma)$ and hence $\text{grade}(\mathbf{a}, M_{\mathbf{n}}) = \text{grade}(\mathbf{a}, M_{\mathbf{n}+\mathbf{m}})$ by the claim. $\square$

If we take each $\mathbf{d}_i = (0, \dots, 0, 1, 0, \dots, 0)$ to be the i -th standard basis element of $\mathbb{N}^r$, then we can readily get the known results in the standard graded cases [6, 7, 8]. Moreover, as a direct consequence, we have the stability of the grade in the special nonstandard graded cases considered in [8].

Corollary 3.1. *Let $R, M, \mathbf{a}$ be the same as in Theorem 1.1. Assume that each vector $\mathbf{d}_i$ has the form $\mathbf{d}_i = (*, \dots, *, 1, 0, \dots, 0)$, where 1 is in the i -th column. Then there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that $\text{grade}(\mathbf{a}, M_{\mathbf{n}})$ is constant in the cone $C_{\mathbf{k}}$. Namely, the equality*

$$\text{grade}(\mathbf{a}, M_{\mathbf{n}}) = \text{grade}(\mathbf{a}, M_{\mathbf{k}})$$

holds true for all $\mathbf{n} \in C_{\mathbf{k}}$.

Proof. Let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. Then $\Delta_{\mathbf{k}_0} = \{\mathbf{k}_0\}$ because of the form of the vectors $\mathbf{d}_i$. Thus we have the assertion as a direct consequence of Theorem 1.1. $\square$

Colomé-Nin and Elias studied in [4] the asymptotic behavior of $\text{depth}_A M_{\mathbf{n}}$ in the case where each $\mathbf{d}_i$ has the form $(*, \dots, *, \lambda_i, 0, \dots, 0)$ with $\lambda_i \neq 0$. This case is also a special case of Theorem 1.1.

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