UNIFORM EQUICONTINUITY OF SEQUENCES OF MEASURABLE OPERATORS AND NON-COMMUTATIVE ERGODIC THEOREMS

SEMYON LITVINOV

(Communicated by Marius Junge)

ABSTRACT. The notion of uniform equicontinuity in measure at zero for sequences of additive maps from a normed space into the space of measurable operators associated with a semifinite von Neumann algebra is discussed. It is shown that uniform equicontinuity in measure at zero on a dense subset implies the uniform equicontinuity in measure at zero on the entire space, which is then applied to derive some non-commutative ergodic theorems.

1. INTRODUCTION AND PRELIMINARIES

Let (Ω, μ) be a measure space. Denote by $\mathcal{L}_0(\mathcal{L})$ the set of equivalence classes of all (respectively, almost everywhere finite) real or complex valued measurable functions on Ω . If X is a given set and $a_n : X \rightarrow \mathcal{L}$, $n = 1, 2, \dots$, let

$$a^*(x) = \sup_n |a_n(x)|, \quad x \in X.$$

The function $a^* : X \rightarrow \mathcal{L}_0$ is called a *maximal operator* associated with the family $\{a_n\}$.

Now let $(X, \|\cdot\|)$ be a normed space, $a_n : X \rightarrow \mathcal{L}$ a sequence of additive maps, and τ_μ the measure topology in $\mathcal{L}$. The τ_μ -continuity of a^* at zero means that, for any given $\epsilon > 0$, $\delta > 0$, it is possible to find such $\gamma > 0$ that $x \in X$ and $\|x\| < \gamma$ would imply that there exists a measurable set $E \subset \Omega$ for which the conditions

$$\mu(\Omega \setminus E) \leq \epsilon \quad \text{and} \quad \|a^*(x)\chi_E\|_\infty \leq \delta$$

hold. (Here $\|\cdot\|_\infty$ is the uniform norm in $\mathcal{L}^\infty = \mathcal{L}^\infty(\Omega)$ and χ_E is the characteristic function of the set E .)

Proposition 1.1. *If $\{a_n(x)\chi_E\}_1^\infty \subset \mathcal{L}^\infty$, then*

$$\|a^*(x)\chi_E\|_\infty = \sup_n \|a_n(x)\chi_E\|_\infty.$$

Proof. It is enough to show and easy to see that, given $\lambda > 0$, $\sup_n \|a_n(x)\chi_E\|_\infty > \lambda$ if and only if $\|a^*(x)\chi_E\|_\infty > \lambda$. □

Therefore, τ_μ -continuity at zero of the maximal operator associated with a sequence of additive maps $a_n : X \rightarrow \mathcal{L}$ is, in fact, the *uniform equicontinuity*

Received by the editors February 20, 2011.

2010 *Mathematics Subject Classification.* Primary 46L51; Secondary 47A35.

Key words and phrases. Semifinite von Neumann algebra, uniform equicontinuity, non-commutative ergodic theorem.

in measure of the family $\{a_n\}$ at zero. This will allow us to extend the notion of τ_μ -continuity of the maximal operator at zero to the non-commutative setting (see the definition below). The importance of this notion can be illustrated, for example, by the fact that τ_μ -continuity at zero of the maximal operator associated with a sequence $a_n : X \rightarrow \mathcal{L}$ of additive maps implies that the set $\{x \in X : \{a_n(x)\} \text{ converges almost everywhere}\}$ is closed in X (see, for instance, [4], p. 3; also [6], Theorem 2). Thus, almost everywhere convergence of the sequence $\{a_n(x)\}$ for every x in a dense subset of X entails its convergence for all $x \in X$.

Let $(H, (\cdot, \cdot))$ be a Hilbert space, $\|\cdot\|_0 = (\cdot, \cdot)^{1/2}$, $B(H)$ the algebra of all bounded linear operators in H , and $\mathbb{I}$ the unit of $B(H)$. Assume that $M \subset B(H)$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ . We denote by $P(M)$ the complete lattice of all projections in M and set $e^\perp = \mathbb{I} - e$ whenever $e \in P(M)$. If $\{e_i\}_{i \in I} \subset P(M)$, then $\bigwedge_{i \in I} e_i$ is the projection on the subspace $\bigcap_{i \in I} e_i H$.

A densely-defined closed operator x in H is said to be *affiliated* with the algebra M if $x'x \subset xx'$ for every $x' \in B(H)$ such that $x'x = xx'$ for all $x \in M$. An operator x affiliated with M is called τ -*measurable* if for each $\epsilon > 0$ there exists such an $e \in P(M)$ with $\tau(e^\perp) \leq \epsilon$ that the subspace eH belongs to the domain of x . (In this case of course $xe \in M$.) Let $L = L(M, \tau)$ be the set of all measurable operators affiliated with the algebra M . Denote $\|\cdot\|_\infty$ as the uniform norm in $B(H)$. The topology t_τ defined in L by the family

$$\{V(\epsilon, \delta) = \{x \in L : \|xe\|_\infty \leq \delta \text{ for some } e \in P(M) \text{ with } \tau(e^\perp) \leq \epsilon\} : \epsilon > 0, \delta > 0\}$$

of (closed) neighborhoods of zero is called a *measure topology*. Note that if one replaces the condition $\|xe\|_\infty \leq \delta$ above by the generally weaker condition $\|exe\|_\infty \leq \delta$, then the corresponding family of neighborhoods of zero generates the same topology t_τ ([3], Theorem 2.2).

We say that a sequence $\{x_n\} \subset L$ converges to $x \in L$ *almost uniformly (a.u.)* (*bilaterally almost uniformly (b.a.u.)*) if for every $\epsilon > 0$ there exists such an $e \in P(M)$ that $\tau(e^\perp) \leq \epsilon$ and $\|(x - x_n)e\|_\infty \rightarrow 0$ (respectively, $\|e(x - x_n)e\|_\infty \rightarrow 0$). Note that the algebra L is complete with respect to both a.u. and b.a.u. convergences ([3], Theorem 2.3 and Remark 2.4).

For a positive self-adjoint operator $x = \int_0^\infty \lambda de_\lambda$ affiliated with M one can define

$$\tau(x) = \sup_n \tau \left(\int_0^n \lambda de_\lambda \right) = \int_0^\infty \lambda d\tau(e_\lambda).$$

If $1 \leq p < \infty$, then the non-commutative L^p -space associated with (M, τ) is defined as

$$L^p = L^p(M, \tau) = \{x \in L : \|x\|_p = (\tau(|x|^p))^{1/p} < \infty\},$$

where $|x| = (x^*x)^{1/2}$, the absolute value of x . Naturally, $L^\infty = M$.

In what follows, we shall use the following well-known properties of non-commutative L^p -spaces without reference:

- (i) If $1 \leq p \leq \infty$, $x \in L^\infty$ and $y \in L^p$, then $\|xy\|_p \leq \|x\|_\infty \|y\|_p$.
- (ii) Given $1 \leq p < \infty$, both $L^p \cap M$ and $L^1 \cap M$ are dense in L^p .
- (iii) If $1 \leq p \leq \infty$, $x, y \in L^p$ and $0 \leq x \leq y$, then $\|x\|_p \leq \|y\|_p$.

If $X \subset L$, we denote

$$X_h = \{x \in X : x^* = x\} \text{ and } X_+ = \{x \in X : x \geq 0\}.$$

The following fundamental result of Kadison [9] will be used extensively in the sequel.

Theorem 1.1 (Kadison's inequality). *Let $a : M \rightarrow M$ be a positive linear map such that $a(\mathbb{I}) \leq \mathbb{I}$. Then $a(x)^2 \leq a(x^2)$ for every $x \in M_h$.*

2. UNIFORM EQUICONTINUITY IN NON-COMMUTATIVE SETTING

Let X be any set. If $a_i : X \rightarrow L$, $i \in I$, and $e \in P(M)$ are such that $\{a_i(x)e\}_{i \in I} \subset M$ ($\{ea_i(x)e\}_{i \in I} \subset M$), we denote

$$a^*(x, e) = \sup_{i \in I} \|a_i(x)e\|_\infty \quad \left(\text{respectively, } a_b^*(x, e) = \sup_{i \in I} \|ea_i(x)e\|_\infty \right).$$

Definition 2.1. Let $(X, \|\cdot\|)$ be a normed space and X_0 a subset of X such that $0 \in X_0$, where 0 is the neutral element of X . A family $a_i : X \rightarrow L$, $i \in I$, of additive maps is called *uniformly equicontinuous in measure (u.e.m.)* (bilaterally uniformly equicontinuous in measure (b.u.e.m.)) at 0 on X_0 if, given $\epsilon > 0$, $\delta > 0$, there is such a $\gamma > 0$ that for every $x \in X_0$ with $\|x\| < \gamma$ there exists $e \in P(M)$ for which

$$\tau(e^\perp) \leq \epsilon \quad \text{and} \quad a^*(x, e) \leq \delta \quad (\text{respectively, } a_b^*(x, e) \leq \delta).$$

Remark 2.1. As we explained in the previous section, in the commutative case with a countable I , the notion of uniform equicontinuity (bilateral uniform equicontinuity) in measure at 0 of a family $\{a_i\}_{i \in I}$ reduces to τ_μ -continuity at zero of the maximal operator associated with this family.

Lemma 2.1 ([2, Lemma 1.6]). *Let $(X, +)$ be a semigroup, and let $a_n : X \rightarrow L$ be a sequence of additive maps. Assume that $x \in X$ is such that for every $\epsilon > 0$ there exist a sequence $\{x_k\} \subset X$ and a projection $e \in P(M)$ such that the following conditions hold:*

- (i) $\{a_n(x + x_k)\}$ converges a.u. (b.a.u.) as $n \rightarrow \infty$ for each k ;
- (ii) $\tau(e^\perp) \leq \epsilon$;
- (iii) $a^*(x_k, e) \rightarrow 0$ (respectively, $a_b^*(x_k, e) \rightarrow 0$).

Then the sequence $\{a_n(x)\}$ also converges a.u. (b.a.u., respectively).

Theorem 2.1 (see [2], Theorem 2.3). *Assume that $(X, \|\cdot\|)$ is a normed space and $a_n : X \rightarrow L$ is a sequence of additive maps. If the family $\{a_n\}$ is u.e.m. (b.u.e.m.) at 0 on X , then the set $C = \{x \in X : \{a_n(x)\} \text{ converges a.u. (respectively, b.a.u.)}\}$ is closed in X .*

Proof. We shall consider the case of b.a.u. convergence; proof for the a.u. convergence is similar.

Fix $\epsilon > 0$. Since the family $\{a_n\}$ is b.u.e. at zero on X , for every given $k \in \mathbb{N}$, there is such a $\gamma_k > 0$ that $x \in X$ and $\|x\| < \gamma_k$ imply that it is possible to find $e_{k,x} \in P(M)$ for which

$$\tau(e_{k,x}^\perp) \leq \frac{\epsilon}{2^k} \quad \text{and} \quad a_b^*(x, e_{k,x}) \leq \frac{1}{k}.$$

Pick $x \in \overline{C}$. Given $k \in \mathbb{N}$, let $y_k \in C$ be such that $\|y_k - x\| < \gamma_k$. Then, denoting $y_k - x = x_k$, let a sequence $\{e_k\} \subset P(M)$ be such that

$$\tau(e_k^\perp) \leq \frac{\epsilon}{2^k} \quad \text{and} \quad a_b^*(x_k, e_k) \leq \frac{1}{k}, \quad k = 1, 2, \dots$$

We have $x + x_k = y_k \in C$ for every k . Also, if $e = \bigwedge e_k$, then

$$\tau(e^\perp) \leq \epsilon \quad \text{and} \quad a_b^*(x_k, e) \leq \frac{1}{k} \rightarrow 0.$$

Consequently, Lemma 2.1 yields $x \in C$. $\square$

Let a positive linear map $\alpha : L^1 \rightarrow L^1$ be such that $\alpha(x) \leq \mathbb{I}$ and $\tau(\alpha(x)) \leq \tau(x)$ for every $x \in L^1 \cap M$ with $0 \leq x \leq \mathbb{I}$. Note that, as it is shown in [12], $\|\alpha(x)\|_p \leq \|x\|_p$ for each $x \in L_h^1 \cap M$ and all $1 \leq p \leq \infty$. Besides, there exist unique continuous extensions $\alpha : L^p \rightarrow L^p$ for every $1 \leq p < \infty$ and a unique ultra-weakly continuous extension $\alpha : M \rightarrow M$.

If $x \in L^1$, denote

$$(1) \quad a_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(x), \quad n = 1, 2, \dots$$

The following theorem due to Yeadon [12] plays an important role in this article.

Theorem 2.2. *If the maps $a_n : L^1 \rightarrow L^1$ are given by (1), then, for every $x \in L^1$ and $\nu > 0$, there is such an $e \in P(M)$ that*

$$\tau(e^\perp) \leq \frac{4\|x\|_1}{\nu} \quad \text{and} \quad a_b^*(x, e) \leq 4\nu.$$

Remark 2.2. If $x \in L_+^1$, then $e \in P(M)$ in Theorem 2.2 can be chosen to satisfy

$$\tau(e^\perp) \leq \frac{\|x\|_1}{\nu} \quad \text{and} \quad a_b^*(x, e) \leq \nu.$$

Corollary 2.1. *The family $\{a_n\}$ given by formula (1) is b.u.e.m. at 0 on $(L^1, \|\cdot\|_1)$.*

Corollary 2.2. *If $X_0 = L^1 \cap \{x \in L^\infty : \|x\|_\infty \leq 1\}$, then the family $\{a_n\}$ given by (1) is u.e.m. at 0 on $(X_0, \|\cdot\|_1)$.*

Proof. Fix $\epsilon > 0$ and $\delta > 0$, and let $\nu > 0$ and $\gamma > 0$ be such that $2\nu^{1/2} \leq \delta$ and $\frac{4\gamma}{\nu} \leq \epsilon$. Take $x \in X_0$ with $\|x\|_1 < \gamma$. Since $x^2 \in L^1$, Theorem 2.2 implies that there exists such an $e \in P(M)$ that $\tau(e^\perp) \leq \frac{4\|x^2\|_1}{\nu}$ and $a_b^*(x^2, e) \leq 4\nu$. Then we have $\tau(e^\perp) < \frac{4\gamma}{\nu} \leq \epsilon$ and, by virtue of Kadison's inequality,

$$\|a_n(x)e\|_\infty^2 = \|ea_n(x)^2e\|_\infty \leq \|ea_n(x^2)e\|_\infty \leq 4\nu \quad \text{for every } n,$$

implying that $a^*(x, e) \leq \delta$. $\square$

3. REDUCTION TO A DENSE SUBSET

In this section we will show that the problem of determining whether a sequence of t_τ -continuous additive maps is u.e.m. (b.u.e.m.) at 0 on a given set can be reduced to considering a dense subset of this set.

Lemma 3.1 ([5, Lemma 3]). *Let $b \in M$, $0 \leq b \leq \mathbb{I}$, and let e be the spectral projection of b corresponding to the interval $[1/2, 1]$. Then:*

- (i) $\tau(e^\perp) \leq 2\tau(\mathbb{I} - b)$;
- (ii) *there exists $b^- \in M$ such that $\|b^-\|_\infty \leq 2$ and $e = bb^-$.*

Proposition 3.1 ([10, see also [5, Proposition 1]). *If $\{y_n\} \subset L$ and $y_n \rightarrow 0$ in t_τ , then $y_{n_k} \rightarrow 0$ a.u. for some subsequence $\{y_{n_k}\} \subset \{y_n\}$.*

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space, and let $a_n : X \rightarrow (L, t_\tau)$ be a sequence of continuous additive maps. If the family $\{a_n\}$ is b.u.e.m. at 0 on a set $X_0 \subset X$ such that $\{a_n(x)\} \subset L_+$ for all $x \in \overline{X_0}$, where $\overline{X_0}$ is the closure of X_0 in X , then it is also b.u.e.m. at 0 on $\overline{X_0}$.*

Proof. We employ the scheme of the proof of closedness of the sets $X_{L,k}$ in [3], Theorem 2.7. Fix $\epsilon > 0$ and $\delta > 0$. Since $\{a_n\}$ is b.u.e.m. at 0 on X_0 , there exists a $\gamma > 0$ such that for every $x \in X_0$ with $\|x\| < \gamma$ it is possible to find a projection $e_x \in P(M)$ satisfying

$$\tau(e_x^\perp) \leq \frac{\epsilon}{2} \quad \text{and} \quad a_b^*(x, e_x) = \sup_n \|e_x a_n(x) e_x\|_\infty \leq \frac{\delta}{4}.$$

Take any $x \in X_0$, $\|x\| < \gamma$. As $|a_n(x)^{1/2} e_x|^2 = e_x a_n(x) e_x \in M$, we have $|a_n(x)^{1/2} e_x| \in M$, hence $a_n(x)^{1/2} e_x \in M$, for every n , so we can write

$$\sup_n \|a_n(x)^{1/2} e_x\|_\infty = [\sup_n \|e_x a_n(x) e_x\|_\infty]^{1/2} \leq \frac{\sqrt{\delta}}{2}.$$

Pick $x \in \overline{X_0}$ such that $\|x\| < \gamma$. Then there is a sequence $\{y_m\} \subset X_0$ for which we have $\|x - y_m\| \rightarrow 0$ and $\|y_m\| < \gamma$, $m = 1, 2, \dots$. It follows that $a_1(y_m) \rightarrow a_1(x)$ in t_τ implying $a_1(y_m)^{1/2} \rightarrow a_1(x)^{1/2}$ in t_τ [11]. Therefore, due to Proposition 3.1, $a_1(y_m^{(1)})^{1/2} \rightarrow a_1(x)^{1/2}$ a.u. for some subsequence $\{y_m^{(1)}\} \subset \{y_m\}$. By the same argument, one can find a subsequence $\{y_m^{(2)}\} \subset \{y_m^{(1)}\}$ such that $a_2(y_m^{(2)})^{1/2} \rightarrow a_2(x)^{1/2}$ a.u. Repeating this procedure, given $n \geq 3$, we choose a subsequence $\{y_m^{(n)}\} \subset \{y_m^{(n-1)}\}$ for which the convergence $a_n(y_m^{(n)})^{1/2} \rightarrow a_n(x)^{1/2}$ a.u., $m \rightarrow \infty$, takes place. If we define $x_m = y_m^{(n)}$, $m = 1, 2, \dots$, then, taking into account that, for every n , $\{x_m\}_{m \geq n}$ is a subsequence of $\{y_m^{(n)}\}$, we conclude that

$$a_n(x_m)^{1/2} \rightarrow a_n(x)^{1/2} \quad \text{a.u. as } m \rightarrow \infty, \quad n = 1, 2, \dots$$

Now, because $\{x_m\} \subset X_0$ and $\|x_m\| < \gamma$ for all m , one can present a sequence $\{e_m\} \subset P(M)$ such that

$$(2) \quad \sup_n \|a_n(x_m)^{1/2} e_m\|_\infty \leq \frac{\sqrt{\delta}}{2} \quad \text{and} \quad \tau(e_m^\perp) \leq \frac{\epsilon}{2}, \quad m = 1, 2, \dots$$

Since the unit ball of M is compact in the weak operator topology, there are a subnet $\{e_\alpha\} \subset \{e_m\}$ and $b \in M$ such that $e_\alpha \rightarrow b$ weakly. Then we clearly have $0 \leq b \leq \mathbb{I}$. Moreover, by the well-known inequality (see [1], for example),

$$\tau(\mathbb{I} - b) \leq \liminf_\alpha \tau(e_\alpha^\perp) \leq \frac{\epsilon}{2}.$$

Show that $\sup_n \|a_n(x)^{1/2} b\|_\infty \leq \sqrt{\delta}/2$. For that, fix n and denote $a_n(x)^{1/2} = z$ and $a_n(x_m)^{1/2} = z_m$, $m = 1, 2, \dots$. Since $z_m \rightarrow z$ a.u., given $k \in \mathbb{N}$, there exists $p_k \in P(M)$ such that $\tau(p_k^\perp) \leq 1/k$ and

$$\|p_k(z - z_m)\|_\infty = \|(z - z_m)p_k\|_\infty \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Let us show first that $\|p_k z b\|_\infty \leq \sqrt{\delta}/2$ for any fixed k . Pick $\nu > 0$ and find m_0 such that

$$(3) \quad \|p_k(z - z_m)\|_\infty < \nu \quad \text{whenever } m \geq m_0.$$

Next, fix $\xi, \eta \in H$ such that $\|\xi\|_0 = \|\eta\|_0 = 1$. Since $e_\alpha \rightarrow b$ weakly, we can find such an index $\alpha(\nu)$ that

$$(4) \quad |((b - e_\alpha)\xi, zp_k\eta)| < \nu \quad \text{for all } \alpha > \alpha(\nu).$$

Furthermore, because $\{e_\alpha\}$ is a subnet of $\{e_m\}$, it is possible to choose an index $\alpha(m_0)$ for which $\{e_\alpha\}_{\alpha > \alpha(m_0)} \subset \{e_m\}_{m \geq m_0}$. In particular, if $\alpha_0 \geq \max\{\alpha(\nu), \alpha(m_0)\}$, then $e_{\alpha_0} = e_{m_1}$ for some $m_1 \geq m_0$. It follows now from (2)-(4) that

$$\begin{aligned} |(p_k zb\xi, \eta)| &\leq |(p_k z_{m_1} e_{m_1} \xi, \eta)| + |(p_k (z - z_{m_1}) e_{m_1} \xi, \eta)| \\ &\quad + |((b - e_{m_1})\xi, zp_k\eta)| \leq \frac{\sqrt{\delta}}{2} + 2\nu. \end{aligned}$$

Remembering that $\nu > 0$ was taken arbitrarily, we arrive at

$$\|p_k zb\|_\infty = \sup_{\|\xi\|_0 = \|\eta\|_0 = 1} |(p_k zb\xi, \eta)| \leq \frac{\sqrt{\delta}}{2}, \quad k = 1, 2, \dots$$

As $p_k \rightarrow \mathbb{I}$ weakly, $p_k zb \rightarrow zb$ weakly, and we obtain

$$\|a_n(x)^{1/2}b\|_\infty = \|zb\|_\infty \leq \liminf_k \|p_k zb\|_\infty \leq \frac{\sqrt{\delta}}{2}, \quad n = 1, 2, \dots$$

Now let e be the spectral projection of b corresponding to the interval $[1/2, 1]$. Then, by Lemma 3.1, we have $\tau(e^\perp) \leq \epsilon$ and

$$\sup_n \|a_n(x)^{1/2}e\|_\infty \leq 2 \sup_n \|a_n(x)^{1/2}b\|_\infty \leq \sqrt{\delta}.$$

Finally, we can write

$$a_b^*(x, e) = \sup_n \|ea_n(x)e\| = \sup_n \|a_n(x)^{1/2}e\|_\infty^2 \leq \delta;$$

thus, the family $\{a_n\}$ is b.u.e.m. at 0 on $\overline{X_0}$. □

Repeating the procedure in the proof of Theorem 3.1 with obvious simplifications, we obtain the following.

Theorem 3.2 (cf. [7], Theorem 3.1.1). *Let X be a Banach space, and let $a_n : X \rightarrow (L, t_\tau)$ be a sequence of continuous additive maps. If the family $\{a_n\}$ is u.e.m. at 0 on a set $X_0 \subset X$, then it is also u.e.m. at 0 on the closure of X_0 in X .*

4. INDIVIDUAL ERGODIC THEOREMS

Individual ergodic theorems in L^p -spaces associated with a semifinite von Neumann algebra for $1 < p < \infty$ were established in [8] as an application of their maximal ergodic inequalities. Note that the settings of [8] seem to be different from that of [12]: a map α in [8] is initially defined on M , and it is assumed that $\|\alpha(x)\|_\infty \leq \|x\|_\infty$ for each $x \in M$; whereas in [12] α is initially defined on L^1 , and it is shown that it can be uniquely extended to an ultra-weakly continuous map on M . In what follows we show how such ergodic theorems, in the set-up of [12], can be obtained directly from Theorem 2.2 with the help of Kadison's inequality and Theorems 3.1 and 3.2. Everywhere in this section, unless specified differently, $1 \leq p < \infty$.

We start with the following.

Lemma 4.1. *Let $a_n : L^p \rightarrow L$, $n = 1, 2, \dots$. Then the family $\{a_n\}$ is u.e.m. (b.u.e.m.) at 0 on $(L^p, \|\cdot\|_p)$ if and only if it is u.e.m. (respectively, b.u.e.m.) at 0 on $(L^p_+, \|\cdot\|_p)$.*

Proof. Let us consider the bilateral case. The “only if” part is obvious. To prove the “if” part, fix $\epsilon > 0$ and $\delta > 0$. Then there is such a $\gamma > 0$ that $x \in L^p_+$ and $\|x\|_p < \gamma$ imply that $a_b^*(x, e) \leq \frac{\delta}{4}$ for some $e \in P(M)$ with $\tau(e^\perp) \leq \frac{\epsilon}{4}$.

Take $x \in L^p$ such that $\|x\|_p < \gamma$. Then $x = x_1 - x_2 + i(x_3 - x_4)$, where $x_i \in L^p_+$ and $\|x_i\|_p \leq \|x\|_p < \gamma$, $i = 1, 2, 3, 4$. Therefore, there exists $e_i \in P(M)$ such that $\tau(e_i^\perp) \leq \frac{\epsilon}{4}$ and $a_b^*(x_i, e_i) \leq \frac{\delta}{4}$, $i = 1, 2, 3, 4$. Now, if $e = \bigwedge_{i=1}^4 e_i$, then $\tau(e^\perp) \leq \epsilon$ and

$$a_b^*(x, e) \leq \sum_{i=1}^4 a_b^*(x_i, e_i) \leq \delta.$$

Thus, the assertion holds. $\square$

Assume now that the family $\{a_n\}$ is always given by (1).

Proposition 4.1. *The family $\{a_n\}$ is u.e.m. at 0 on $(L^2, \|\cdot\|_2)$.*

Proof. Since $\overline{L^1_+ \cap M}^{\|\cdot\|_2} = L^2_+$, due to Theorem 3.2 (with $X = (L^2, \|\cdot\|_2)$ and $X_0 = L^1_+ \cap M$) and Lemma 4.1, it is enough to show that $\{a_n\}$ is u.e.m. at 0 on $(L^1_+ \cap M, \|\cdot\|_2)$.

Fix $\epsilon > 0$ and $\delta > 0$, and let $\nu > 0$ and $\gamma > 0$ be such that $\nu \leq \delta^2$ and $\frac{\gamma}{\nu} \leq \epsilon$. Take $x \in L^1_+ \cap M$ with $\|x\|_2 < \gamma$. Since $x^2 \in L^1_+$, Theorem 2.2 (Remark 2.2) implies that $a_b^*(x^2, e) \leq \nu$ for some $e \in P(M)$ with $\tau(e^\perp) \leq \frac{\|x^2\|_1}{\nu}$. Then we have $\tau(e^\perp) \leq \frac{\|x\|_2}{\nu} < \frac{\gamma}{\nu} \leq \epsilon$ and, by Kadison’s inequality,

$$a^*(x, e)^2 = \sup_n \|a_n(x)e\|_\infty^2 = \sup_n \|ea_n(x)^2e\|_\infty \leq \sup_n \|ea_n(x^2)e\|_\infty \leq \nu.$$

Hence $a^*(x, e) \leq \delta$, and the proof is complete. $\square$

Theorem 4.1. *For each $x \in L^2$, the averages (1) converge a.u. to some $\hat{x} \in L^2$.*

Proof. Since the map α generates a contraction in the real Hilbert space $L^2_h = \{x \in L^2 : x^* = x\}$, where $(x, y)_\tau = \tau(xy)$, $x, y \in L^2_h$, it is easy to verify that the set

$$H_0 = \{x \in L^2_h : \alpha(x) = x\} + \{x - \alpha(x) : x \in L^2_h \cap M\}$$

is dense in $(L^2_h, \|\cdot\|_2)$. Besides, the sequence $\{a_n(x)\}$ converges uniformly, hence a.u., for every $x \in H_0$.

Next, by Proposition 4.1, $\{a_n\}$ is u.e.m. at 0 on $(L^2_h, \|\cdot\|_2)$. Therefore, Theorem 2.1 implies that the averages a_n converge a.u. for all $x \in L^2_h$, hence for all $x \in L^2$. Thus, it only remains to notice that its limit $\hat{x} \in L^2$ (see, for example, [3], Remark 2.4 and Theorem 1.2). $\square$

Corollary 4.1. *The averages (1) converge a.u. for all $x \in L^1 \cap M$.*

Now we can present an alternative ending of the proof of the main result of [12].

Theorem 4.2. [12] *For each $x \in L^1$, the averages (1) converge b.a.u. to some $\hat{x} \in L^1$.*

Proof. Since the set $L^1 \cap M$ is dense in L^1 and, by Corollary 4.1, the averages a_n converge a.u., hence b.a.u., on this set, Corollary 2.1 and Theorem 2.1 entail that the sequence $\{a_n(x)\}$ converges b.a.u. for all $x \in L^1$. Besides, by Theorems 2.3 and 1.2 in [3], its limit $\hat{x} \in L^1$. $\square$

Proposition 4.2. *The family $\{a_n\}$ is b.u.e.m. at 0 on $(L^p, \|\cdot\|_p)$.*

Proof. Due to Lemma 4.1, it is sufficient to show that $\{a_n\}$ is b.u.e.m. at 0 on $(L^p_+, \|\cdot\|_p)$, which in turn, by Theorem 3.1, would follow from that of $(L^1_+ \cap M, \|\cdot\|_p)$.

So, let us fix $\epsilon > 0$ and $\delta > 0$. There exists such a $t = t(p, \delta)$ that $t \cdot \lambda^p \geq \lambda$ whenever $\lambda \geq \frac{\delta}{2t}$. Let $\nu > 0$ and $\gamma > 0$ be such that $\nu \leq \frac{\delta}{2t}$ and $\frac{\gamma}{\nu} \leq \epsilon$.

Take $x \in L^1_+ \cap M$ such that $\|x\|_p < \gamma$. We have

$$x = \int_0^{\|x\|_\infty} \lambda de_\lambda = \int_0^{\delta/2} \lambda de_\lambda + \int_{\delta/2}^{\|x\|_\infty} \lambda de_\lambda \leq x_\delta + t \int_{\delta/2}^{\|x\|_\infty} \lambda^p de_\lambda \leq x_\delta + t \cdot x^p,$$

where $x_\delta = \int_0^{\delta/2} \lambda de_\lambda$, so that $\|x_\delta\|_\infty \leq \frac{\delta}{2}$ and $0 \leq x_\delta \leq x$, implying that $x_\delta \in L^1_+ \cap M$, which in turn entails that $\|a_n(x_\delta)\|_\infty \leq \|x_\delta\|_\infty \leq \frac{\delta}{2}$ for all n .

Next, since $x \in L^p_+$, it follows that $x^p \in L^1_+$. Therefore, by Theorem 2.2 (Remark 2.2), one can find such an $e \in P(M)$ that $\tau(e^\perp) \leq \frac{\|x^p\|_1}{\nu} = \frac{\|x\|_p}{\nu} < \frac{\gamma}{\nu} \leq \epsilon$ and $a_b^*(x^p, e) \leq \nu \leq \frac{\delta}{2t}$. Then we have

$$a_b^*(x, e) \leq a_b^*(x_\delta, e) + t \cdot a_b^*(x^p, e) \leq \sup_n \|a_n(x_\delta)\|_\infty + \frac{\delta}{2} \leq \delta,$$

which completes the proof. $\square$

Employing Proposition 4.2 and repeating the argument of Theorem 4.2, we derive the following.

Theorem 4.3. *For each $x \in L^p$, the averages (1) converge b.a.u. to some $\hat{x} \in L^p$.*

Proposition 4.3. *Let $2 < p < \infty$. Then the family $\{a_n\}$ is u.e.m. at 0 on $(L^p, \|\cdot\|_p)$.*

Proof. By Lemma 4.1 and Theorem 3.2, it is enough to show that $\{a_n\}$ is u.e.m. at 0 on $(L^p_+ \cap M, \|\cdot\|_p)$.

Fix $\epsilon > 0$ and $\delta > 0$. Since, due to Proposition 4.2, $\{a_n\}$ is b.u.e.m. at 0 on $(L^{\frac{p}{2}}, \|\cdot\|_{\frac{p}{2}})$, there exists such a $\gamma > 0$ that $y \in L^{\frac{p}{2}}$ and $\|y\|_{\frac{p}{2}} < \gamma$ imply that $a_b^*(y, e) \leq \delta^2$ for some $e \in P(M)$ with $\tau(e^\perp) \leq \epsilon$.

If one takes $x \in L^p_+ \cap M$ with $\|x\|_p < \gamma^{1/2}$, then $x^2 \in L^{\frac{p}{2}}$ and $\|x^2\|_{\frac{p}{2}} = \|x\|_p^2 < \gamma$. Therefore, there is $e \in P(M)$ such that $\tau(e^\perp) \leq \epsilon$ and $a_b^*(x^2, e) \leq \delta^2$, and, applying Kadison's inequality again, we arrive at $a^*(x, e) \leq \delta$. $\square$

Theorem 4.4. *If $2 \leq p < \infty$, then, for every $x \in L^p$, the averages (1) converge a.u. to some $\hat{x} \in L^p$.*

Proof. Since the set $L^1 \cap M$ is dense in L^p and, by Corollary 4.1, the sequence $\{a_n(x)\}$ converges a.u. for every $x \in L^1 \cap M$, applying Proposition 4.3 and then Theorem 2.1, we infer that the averages a_n converge a.u. for all $x \in L^p$. The fact that the corresponding limits $\hat{x}$ belong to L^p follows as in Theorem 4.2. $\square$

REFERENCES

- [1] O. Bratelli, D.N. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer, Berlin, 1979.
- [2] V. Chilin, S. Litvinov, Uniform equicontinuity for sequences of homomorphisms into the ring of measurable operators, *Methods of Funct. Anal. Top.*, 12 (2)(2006), 124-130. MR2238034 (2007b:46112)
- [3] V. Chilin, S. Litvinov, A. Skalski, A few remarks in non-commutative ergodic theory, *J. Operator Theory*, 53 (2)(2005), 331-350. MR2153152 (2006c:46053)
- [4] A. Garsia, *Topics in Almost Everywhere Convergence*, Markham: Lectures in Advanced Mathematics, 4, 1970. MR0261253 (41:5869)
- [5] M. Goldstein, S. Litvinov, Banach principle in the space of τ -measurable operators, *Studia Math.*, 143(2000), 33-41. MR1814479 (2002b:46107)
- [6] S. Litvinov, The Banach principle for topological groups, *Atti Sem. Mat. Fis. Univ. Modena e Reggio Emilia*, 53 (2005), 323-330. MR2289403 (2007j:22004)
- [7] M. de Guzman, *Real Variable Methods in Fourier Analysis*, North-Holland: Math. Studies, 46, 1981. MR596037 (83j:42019)
- [8] M. Junge, Q. Xu, Noncommutative maximal ergodic theorems, *J. Amer. Math. Soc.*, 20 (2)(2007), 385-439. MR2276775 (2007k:46109)
- [9] R. V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. of Math.* (2), 56(1952), 494-503. MR0051442 (14:481c)
- [10] I. Segal, A non-commutative extension of abstract integration, *Ann. of Math.* (2), 57(1953), 401-457. MR0054864 (14:991f)
- [11] O.E. Tichonov, Continuity of operator functions in topologies connected with a trace on a von Neumann algebra [Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1(1987), 77-79. MR892008 (88h:46120)
- [12] F. J. Yeadon, Ergodic theorems for semifinite von Neumann algebras. I, *J. London Math. Soc.*, 16 (2)(1977), 326-332. MR0487482 (58:7111)

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, 76 UNIVERSITY DRIVE,
HAZLETON, PENNSYLVANIA 18202

E-mail address: `sn12@psu.edu`