WEIGHTED L^p BOUNDEDNESS OF CARLESON TYPE MAXIMAL OPERATORS

YONG DING AND HONGHAI LIU

(Communicated by Richard Rochberg)

ABSTRACT. In 2001, E. M. Stein and S. Wainger gave the L^p boundedness of the Carleson type maximal operator $\mathcal{T}^*$, which is defined by

$$\mathcal{T}^*f(x) = \sup_{\lambda} \left| \int_{\mathbb{R}^n} e^{iP_{\lambda}(y)} K(y) f(x-y) dy \right|.$$

In this paper, the authors show that if K is a homogeneous kernel, i.e. $K(y) = \Omega(y')|y|^{-n}$, then Stein-Wainger's result still holds on the weighted L^p spaces when Ω satisfies only an L^q -Dini condition for some $1 < q \leq \infty$.

1. INTRODUCTION

For $f \in L^2([-\pi, \pi])$ and $x \in [-\pi, \pi]$, the Carleson operator $\mathcal{C}^*$ is defined by

$$(1.1) \quad \mathcal{C}^*f(x) = \sup_{\lambda \in \mathbb{R}} \left| \int_{-\pi}^{\pi} \frac{e^{-i\lambda t} f(t)}{x-t} dt \right|.$$

In 1966, using the weak type $(2, 2)$ of $\mathcal{C}^*$, Carleson [2] proved his celebrated theorem on almost everywhere convergence of the Fourier series of the functions in $L^2([-\pi, \pi])$. Following that, Hunt [7] extended Carleson's theorem to $L^p([-\pi, \pi])$ for $1 < p < \infty$.

In 1970, Sjölin [11] studied an analogue of the Carleson operator $\mathcal{C}^*$ on $\mathbb{R}^n$. Suppose that K is an appropriate Calderón-Zygmund kernel. Then the Carleson type maximal operator $\mathcal{S}^*$ on $\mathbb{R}^n$ is defined by

$$(1.2) \quad \mathcal{S}^*(f)(x) = \sup_{\lambda \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\lambda \cdot y} K(x-y) f(y) dy \right|,$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

Theorem A (Sjölin, [11]). *Let K satisfy the following conditions:*

- (1) $K(tx) = t^{-n}K(x)$, for $t > 0$;
- (2) $\int_{S^{n-1}} K(x') d\sigma(x') = 0$;
- (3) $K \in C^{n+1}(\mathbb{R}^n \setminus \{0\})$.

Then $\|\mathcal{S}^(f)\|_{L^p} \leq C_p \|f\|_{L^p}$ for $1 < p < \infty$.*

Received by the editors May 29, 2010 and, in revised form, March 6, 2011.

2010 *Mathematics Subject Classification*. Primary 42B20, 42B25; Secondary 42B99.

Key words and phrases. Carleson operator, homogeneous kernel, L^q -Dini condition, A_p weight.

The first author was supported by the NSF of China (Grant 10931001), SRFDP of China (Grant 20090003110018) and Program for Changjiang Scholars and Innovative Research Team in University.

In 2001, Stein and Wainger [13] considered a more general case. Denote by $P_\lambda(x) = \sum_{1 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$ the polynomial in $\mathbb{R}^n$ with real coefficients $\lambda := (\lambda_\alpha)_{1 \leq |\alpha| \leq d}$. Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^n} e^{iP_\lambda(y)} K(y) f(x-y) dy.$$

Then the Carleson type maximal operator $\mathcal{T}^*$ is defined by

$$(1.3) \quad \mathcal{T}^* f(x) = \sup_{\lambda} |T_\lambda(f)(x)|,$$

where the supremum is taken over all the real coefficients λ of P_λ . Stein and Wainger got the following result:

Theorem B (Stein and Wainger, [13]). *Suppose that $P_\lambda(x) = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$ and K satisfies the following conditions:*

- (1) K is a tempered distribution and agrees with a C^1 function $K(x)$ for $x \neq 0$;
- (2) $\widehat{K} \in L^\infty$;
- (3) $|\partial_x^\gamma K(x)| \leq A|x|^{-n-|\gamma|}$ for $0 \leq |\gamma| \leq 1$.

Then $\|\mathcal{T}^*(f)\|_{L^p} \leq C_p \|f\|_{L^p}$ for $1 < p < \infty$.

Remark 1.1. We should point out that Stein and Wainger's result is a meaningful extension of Theorem A. In fact, on the one hand, they replaced the linear phase in the definition of $\mathcal{S}^*$ by the more general polynomial phase with fixed degree. On the other hand, the homogeneous kernel of Theorem A is also replaced by the more general distribution kernel in Theorem B. Therefore, the conditions of the kernel K in Theorem B are more general for the applicability of Theorem B.

In 1974, Hunt and Yong [8] established a weighted norm inequality for the Carleson operator $\mathcal{C}^*$. In 2000, Prestini and Sjölin [10] gave the weighted analogue of Theorem A.

Motivated by the above results, in this note we give the weighted L^p boundedness for the Carleson type maximal operator $\mathcal{T}^*$ defined in (1.3). Before stating our result, let us first give some definitions.

Suppose that Ω is a measurable function on $\mathbb{R}^n \setminus \{0\}$ and satisfies the following conditions:

$$(1.4) \quad \Omega(tx) = \Omega(x) \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\} \text{ and } t > 0;$$

$$(1.5) \quad \Omega \in L^1(S^{n-1}),$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) with area measure $d\sigma$;

$$(1.6) \quad \int_{S^{n-1}} \Omega(x') d\sigma(x') = 0.$$

Definition 1.2. Suppose that Ω satisfies (1.4) and $\Omega \in L^q(S^{n-1})$ for $1 \leq q \leq \infty$. Then Ω is said to satisfy an L^q -Dini condition if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty,$$

where $\omega_q(\delta)$ ($0 < \delta \leq 1$) is called the integral continuous modulus of Ω of degree q , which is defined by

$$\omega_q(\delta) = \sup_{\|\rho\| < \delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{1/q} \quad \text{for } 1 \leq q < \infty$$

and

$$\omega_\infty(\delta) = \sup_{\|\rho\| < \delta} |\Omega(\rho x') - \Omega(x')|,$$

where ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup \{|\rho x' - x'| : x' \in S^{n-1}\}$.

A function Ω is said to be a Lipschitz function on S^{n-1} , which is denoted by $\Omega \in \text{Lip}(S^{n-1})$, if there exists a constant $L > 0$ such that for any $x', y' \in S^{n-1}$,

$$|\Omega(x') - \Omega(y')| \leq L|x' - y'|.$$

Obviously, if $\Omega \in \text{Lip}(S^{n-1})$, then Ω also satisfies the L^∞ -Dini condition.

A locally integrable nonnegative function ω on $\mathbb{R}^n$ is said to belong to A_p ($1 < p < \infty$) if there is a constant $C > 0$ such that

$$\sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C,$$

where Q denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes (see [6] for the properties of A_p).

Our main result in this paper is as follows:

Theorem 1.3. *Suppose that $P_\lambda(x) = \sum_{2^{\leq |\alpha| \leq d}} \lambda_\alpha x^\alpha$ and $K(x) = \Omega(x)|x|^{-n}$, where Ω satisfies (1.4)~(1.6). If Ω satisfies the L^q -Dini condition for some $1 < q \leq \infty$, then for $1 \leq q' < p < \infty$ and $w \in A_{p/q'}$, the Carleson type maximal operator $\mathcal{T}^*$ is a bounded operator on the weighted space $L^p(w)$. That is, there exists a constant $C > 0$ such that for all $f \in L^p(w)$*

$$(1.7) \quad \|\mathcal{T}^* f\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

Remark 1.4. Note that

$$C^1(S^{n-1}) \subset \text{Lip}(S^{n-1}) \subset L^q(S^{n-1}) \ (1 < q \leq \infty) \subset H^1(S^{n-1}) \subset L^1(S^{n-1}),$$

where $H^1(S^{n-1})$ denotes the Hardy space on S^{n-1} (see [3] for the definition of $H^1(S^{n-1})$). Therefore, Theorem 1.3 is an improvement only when Theorem B is applied to the homogeneous kernel. Recently, we [4] considered the L^p boundedness of the Carleson type maximal operator $\mathcal{T}^*$ with $\Omega \in H^1(S^{n-1})$.

The proof of Theorem 1.3 is based on an idea of linearized maximal operators and Stein and Wainger's TT^* method in [13]. However, because the kernel of our objective operator lacks smoothness on the unit sphere, we need to develop some new techniques for dealing with the roughness of the kernel.

2. THE PROOF OF THEOREM 1.3

It is obvious that

$$(2.1) \quad \mathcal{T}^*(f)(x) = \sup_\lambda |T_\lambda(f)(x)| \leq \sup_{\lambda \neq 0} |T_\lambda(f)(x)| + |T_\Omega(f)(x)|,$$

where T_Ω is given by

$$T_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y) dy.$$

By the weighted L^p boundedness of T_Ω (see [9]), we may assume that the supremum is taken over all vectors $\lambda = (\lambda_\alpha)_{2 \leq |\alpha| \leq d}$ with $|\lambda| > 0$, where

$$|\lambda| = \sum_{2 \leq |\alpha| \leq d} |\lambda_\alpha|.$$

Let $P_\lambda(x) = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$ be the real polynomial of degree d with coefficients $\lambda = (\lambda_\alpha)$. For $f \in L^p$ and $x \in \mathbb{R}^n$, denote by $\lambda(x)$ the nonzero vector $(\lambda_\alpha(x))_{2 \leq |\alpha| \leq d}$ such that

$$(2.2) \quad |T_{\lambda(x)}(f)(x)| \geq \frac{1}{2} \sup_{\lambda} |T_\lambda(f)(x)|.$$

Thus by (2.2), to prove (1.7), it is sufficient to prove that, under the conditions of Theorem 1.3, there exists a constant C , independent of the choice of $\lambda(\cdot)$, such that

$$(2.3) \quad \|T_{\lambda(\cdot)}(f)\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

Take a nonnegative function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}(\psi) \subseteq \{\frac{1}{4} < |y| \leq 1\}$ and ψ satisfies

$$\sum_{j=-\infty}^{\infty} \psi_j(y) = 1 \quad \text{for } y \neq 0,$$

where $\psi_j(y) = \psi(2^{-j}y)$. Denote $N(\lambda) = \sum_{2 \leq |\alpha| \leq d} |\lambda_\alpha|^{\frac{1}{|\alpha|}}$. Then $N(\lambda(x)) > 0$ for $x \in \mathbb{R}^n$. It is trivial that

$$\sum_{j=-\infty}^{\infty} \psi_j(N(\lambda(x))y) = 1 \quad \text{for } y \neq 0.$$

For $x \in \mathbb{R}^n$, let $\psi_{j,\lambda}(y) = \psi_j(N(\lambda(x))y)$ for short. Write K as

$$K(y) = \sum_{j=0}^{\infty} K_j(y),$$

where $K_0(y) = \sum_{j=-\infty}^0 \psi_{j,\lambda}(y)K(y)$ and $K_j(y) = \psi_{j,\lambda}(y)K(y)$ for $j \geq 1$. Hence we have

$$(2.4) \quad \begin{aligned} |T_{\lambda(x)}(f)(x)| &\leq \left| \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} K_0(y) f(x-y) dy \right| \\ &\quad + \sum_{j=1}^{\infty} \left| \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} K_j(y) f(x-y) dy \right| \\ &:= T_{\lambda(x)}^0(f)(x) + \sum_{j=1}^{\infty} T_{\lambda(x)}^j(f)(x). \end{aligned}$$

2.1. The weighted estimate of $T_{\lambda(\cdot)}^0$. Note that $\text{supp}(K_0) \subseteq \{|y| \leq \frac{1}{N(\lambda(x))}\}$ and $K_0(y) = K(y)$ for $|y| \leq \frac{1}{2N(\lambda(x))}$. We get

$$(2.5) \quad \begin{aligned} |T_{\lambda(x)}^0(f)(x)| &\leq \left| \int_{|y| \leq \frac{1}{2N(\lambda(x))}} e^{iP_{\lambda(x)}(y)} K(y) f(x-y) dy \right| \\ &\quad + C \int_{\frac{1}{2N(\lambda(x))} \leq |y| \leq \frac{1}{N(\lambda(x))}} |K(y)| |f(x-y)| dy. \end{aligned}$$

It is easy to see that the second term on the right side of (2.5) is controlled pointwise by a function $M_\Omega(f)$, where M_Ω denotes the maximal operator with homogeneous kernel defined by

$$M_\Omega(f)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\leq r} |\Omega(y)| |f(x-y)| dy.$$

Since $N(\lambda(x))|y| \leq 1$ and $|\lambda_\alpha(x)| \leq N(\lambda(x))^{|\alpha|}$, we get

$$|e^{iP_{\lambda(x)}(y)} - 1| \leq C \sum_{2 \leq |\alpha| \leq d} |\lambda_\alpha(x)| |y|^\alpha \leq C \sum_{2 \leq |\alpha| \leq d} N(\lambda(x))^{|\alpha|} |y|^{|\alpha|} \leq CN(\lambda(x))|y|.$$

Thus, the first term on the right side of (2.5) can be dominated by

$$\begin{aligned} & \left| \int_{|y| \leq \frac{1}{2N(\lambda(x))}} K(y)f(x-y)dy \right| + \left| \int_{|y| \leq \frac{1}{2N(\lambda(x))}} (e^{iP_{\lambda(x)}(y)} - 1)K(y)f(x-y)dy \right| \\ & \leq \left| \text{p.v.} \int_{\mathbb{R}^n} K(y)f(x-y)dy \right| + \sup_{\varepsilon>0} \left| \int_{|y| \geq \varepsilon} K(y)f(x-y)dy \right| \\ & \quad + CN(\lambda(x)) \int_{|y| \leq \frac{1}{2N(\lambda(x))}} \frac{|\Omega(y)|}{|y|^{n-1}} |f(x-y)| dy \\ & \leq |T_\Omega(f)(x)| + T_\Omega^*(f)(x) + CM_\Omega(f)(x), \end{aligned}$$

where T_Ω^* denotes the maximal operator of truncated singular integrals with homogeneous kernel, which is defined by

$$T_\Omega^*(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy \right|.$$

Hence,

$$(2.6) \quad |T_{\lambda(x)}^0(f)(x)| \leq |T_\Omega(f)(x)| + T_\Omega^*(f)(x) + CM_\Omega(f)(x),$$

where C is independent of the choice of $\lambda(\cdot)$. Using the weighted L^p boundedness of T_Ω , T_Ω^* and M_Ω (see [9] and [5]), we know that, under the conditions of Theorem 1.3,

$$(2.7) \quad \|T_{\lambda(\cdot)}^0(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)},$$

where the constant C is independent of $\lambda(\cdot)$.

2.2. The weighted estimates of $T_{\lambda(\cdot)}^j$. Below we estimate $T_{\lambda(\cdot)}^j$ for $j = 1, 2, \dots$.

Take a nonnegative $C_c^\infty(\mathbb{R}^n)$ function ϕ such that $\text{supp}(\phi) \subseteq \{|y| \leq 2^{-5}\}$ and $\|\phi\|_{L^1} = 1$. For $a > 0$, denote $\phi_a(x) = a^{-n}\phi(x/a)$. For each $j \in \mathbb{N}$ and some $\sigma > 0$ small enough, which will be chosen later, let

$$L_{j,\lambda(x)}(y) = K_j * \phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(y) \quad \text{and} \quad R_{j,\lambda(x)}(y) = K_j(y) - L_{j,\lambda(x)}(y).$$

Define operators $\mathcal{L}_{\lambda(\cdot)}^j$ and $\mathcal{R}_{\lambda(\cdot)}^j$ by

$$\mathcal{L}_{\lambda(x)}^j(f)(x) = \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} L_{j,\lambda(x)}(y) f(x-y) dy$$

and

$$\mathcal{R}_{\lambda(x)}^j(f)(x) = \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} R_{j,\lambda(x)}(y) f(x-y) dy,$$

respectively. It is clear that

$$|T_{\lambda(x)}^j(f)(x)| \leq |\mathcal{L}_{\lambda(x)}^j(f)(x)| + |\mathcal{R}_{\lambda(x)}^j(f)(x)|.$$

Hence, if we can show that there exist $C, \theta > 0$, independent of the choice of $\lambda(\cdot)$, such that for $j = 1, 2, \dots$,

$$(2.8) \quad \|\mathcal{L}_{\lambda(\cdot)}^j(f)\|_{L^p(w)} \leq C2^{-j\theta}\|f\|_{L^p(w)}$$

and

$$(2.9) \quad \|\mathcal{R}_{\lambda(\cdot)}^j(f)\|_{L^p(w)} \leq C(2^{-j\sigma} + \omega_q(2^{-j\sigma-2}))\|f\|_{L^p(w)},$$

then we have

$$(2.10) \quad \left\| \sum_{j=1}^{\infty} T_{\lambda(\cdot)}^j(f) \right\|_{L^p(w)} \leq \sum_{j=1}^{\infty} \|T_{\lambda(\cdot)}^j(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.$$

In the above estimate, we use the fact that Ω satisfies the L^q -Dini condition and

$$\begin{aligned} \sum_{j \geq 1} \omega_q(2^{-j\sigma-2}) &= (\log 2)^{-1} \sum_{j \geq 1} \omega_q(2^{-j\sigma-2}) \int_{2^{-j\sigma-2}}^{2^{-j\sigma-1}} \frac{d\delta}{\delta} \\ &\leq (\log 2)^{-1} \sum_{j \geq 1} \int_{2^{-j\sigma-2}}^{2^{-j\sigma-1}} \frac{\omega_q(\delta)}{\delta} d\delta < \infty. \end{aligned}$$

Thus, Theorem 1.3 follows from (2.3), (2.4), (2.7) and (2.10). Hence, to complete the proof of Theorem 1.3, it remains to verify (2.8) and (2.9).

2.2.1. *Proof of (2.8). The weighted estimates of $\mathcal{L}_{\lambda(\cdot)}^j$.* Obviously, $L_{j,\lambda}$ is smooth by its definition and

$$(2.11) \quad \text{supp } (L_{j,\lambda}) \subseteq \left\{ \frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))} \right\}.$$

Let

$$L_j(y) = \int_{\mathbb{R}^n} \psi(y-z)K(y-z)2^{jn\sigma}\phi(2^{j\sigma}z)dz.$$

Then by a simple calculation, we have

$$(2.12) \quad L_{j,\lambda}\left(\frac{2^j y}{N(\lambda(x))}\right) = (2^{-j}N(\lambda(x)))^n L_j(y).$$

It is trivial that

$$(2.13) \quad |L_{j,\lambda}(y)| \leq C(2^{-j}N(\lambda(x)))^n 2^{jn\sigma},$$

where C is independent of j and $\lambda(\cdot)$. By (2.11) and (2.13), we have

$$(2.14) \quad \mathcal{L}_{\lambda(x)}^j(f)(x) \leq C2^{jn\sigma}M(f)(x),$$

where, and in the sequel, M denotes the usual Hardy-Littlewood maximal operator. Recall $\lambda(x) = (\lambda_\alpha(x))_{2 \leq |\alpha| \leq d}$. For $j \in \mathbb{N}$, let

$$A_{j,\lambda} \circ \lambda = \left(\left(\frac{2^j}{N(\lambda(x))} \right)^{|\alpha|} \lambda_\alpha(x) \right)_{2 \leq |\alpha| \leq d}$$

for convenience. Note that

$$\begin{aligned} (2.15) \quad P_{\lambda(x)}(y) &= \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha(x)y^\alpha = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha(x) \left(\frac{2^j}{N(\lambda(x))} \right)^{|\alpha|} (2^{-j}N(\lambda(x))y)^\alpha \\ &= P_{A_{j,\lambda} \circ \lambda}(2^{-j}N(\lambda(x))y). \end{aligned}$$

By (2.12), we have

$$\mathcal{L}_{\lambda(x)}^j(f)(x) = \int_{\mathbb{R}^n} e^{iP_{A_{j,\lambda} \circ \lambda}(2^{-j}N(\lambda(x))y)} (2^{-j}N(\lambda(x)))^n L_j(2^{-j}N(\lambda(x))y) f(x-y) dy.$$

From now on, we denote $\mathcal{L}_{\lambda(x)}^j(f)(x)$ by $\mathcal{L}_j(f)(x)$ for simplicity. Note that for a nonzero vector $(\nu_\alpha)_{2 \leq |\alpha| \leq d}$,

$$N(A_{j,\nu} \circ \nu) = \sum_{2 \leq |\alpha| \leq d} \left(\left(\frac{2^j}{N(\nu)} \right)^{|\alpha|} |\nu_\alpha| \right)^{\frac{1}{|\alpha|}} = \frac{2^j}{N(\nu)} \sum_{2 \leq |\alpha| \leq d} |\nu_\alpha|^{\frac{1}{|\alpha|}} = 2^j.$$

Since there exists a constant $c_0 > 0$ such that $N(\nu) \leq c_0 |\nu|$ for any vector ν satisfying $N(\nu) \geq 1$, we get

$$(2.16) \quad |A_{j,\lambda} \circ \lambda| \geq 2^j/c_0 \quad \text{for all } \lambda(x), x \in \mathbb{R}^n.$$

For $r \geq 2^j/c_0$, let

$$U_{j,r} = \{x : r \leq |A_{j,\lambda} \circ \lambda| < 2r\}$$

and

$$\mathcal{L}_{j,r}(f)(x) = \mathcal{L}_j(f)(x) \chi_{U_{j,r}}(x).$$

$\mathcal{L}_{j,r}^*$ denotes the adjoint operator of $\mathcal{L}_{j,r}$. It is easy to check that

$$\mathcal{L}_{j,r}^*(g)(y) = \int_{\mathbb{R}^n} e^{-iP_{\lambda(x)}(x-y)} L_{j,\lambda(x)}(x-y) g(x) \chi_{U_{j,r}}(x) dx$$

and

$$(\mathcal{L}_{j,r} \mathcal{L}_{j,r}^*)(f)(x) = \int_{\mathbb{R}^n} \mathcal{K}(x, z) f(z) dz,$$

where

$$\begin{aligned} \mathcal{K}(x, z) &= \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} e^{-iP_{\lambda(z)}(z-x+y)} L_{j,\lambda(x)}(y) L_{j,\lambda(z)}(z-x+y) dy \chi_{U_{j,r}}(x) \chi_{U_{j,r}}(z) \\ &= (e^{iP_{\lambda(x)}(\cdot)} L_{j,\lambda(x)}(\cdot)) * (e^{-iP_{\lambda(z)}(\cdot)} L_{j,\lambda(z)}(\cdot))(x-z) \chi_{U_{j,r}}(x) \chi_{U_{j,r}}(z). \end{aligned}$$

We claim that for $r \geq 2^j/c_0$ and fixed $x, z \in U_{j,r}$,

$$\begin{aligned} |\mathcal{K}(x, z)| &\leq C(2^{-j}N(\lambda(z)))^n 2^{2jn\sigma} [r^{-2\delta} \chi_{B_3}(2^{-j}N(\lambda(z))(x-z)) \\ &\quad + \chi_{E_{\lambda(z)}^j}(2^{-j}N(\lambda(z))(x-z))] \\ (2.17) \quad &+ C(2^{-j}N(\lambda(x)))^n 2^{2jn\sigma} [r^{-2\delta} \chi_{B_3}(2^{-j}N(\lambda(x))(x-z)) \\ &\quad + \chi_{E_{\lambda(x)}^j}(2^{-j}N(\lambda(x))(x-z))], \end{aligned}$$

where the sets $E_{\lambda(x)}^j, E_{\lambda(z)}^j \subset B_3 = \{|y| \leq 3\}$ satisfy $|E_{\lambda(x)}^j|, |E_{\lambda(z)}^j| \leq r^{-4\delta}$ with $\delta = (6d)^{-1}$.

Accepting (2.17), we first show how to deduce (2.8) from (2.17). To do this, we need a variant of the Hardy-Littlewood maximal operator M and its L^p boundedness. Let $B_3 = \{x \in \mathbb{R}^n : |x| \leq 3\}$. For a measurable set $E \subset B_3$, denote by χ_E the characteristic function of E . For $\varepsilon > 0$, the maximal operator $\mathcal{M}_\varepsilon$ is defined by

$$\mathcal{M}_\varepsilon(f)(x) = \sup_{\substack{E \subset B_3, |E| \leq \varepsilon \\ a > 0}} |f| * (\chi_E)_a(x),$$

where $(\chi_E)_a(x) = a^{-n} \chi_E(x/a)$.

Lemma 2.1 (see [13]). *For $1 < p < \infty$, there exists a constant $C > 0$, independent of ε , such that*

$$\|\mathcal{M}_\varepsilon(f)\|_{L^p} \leq C\varepsilon^{1-1/p}\|f\|_{L^p}.$$

Let us return to the proof of (2.8). Denote $\varepsilon = r^{-4\delta}$ and by (2.17), we have

$$\begin{aligned} & |(\mathcal{L}_{j,r}\mathcal{L}_{j,r}^*f, g)| \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{K}(x, z)| |f(z)| |g(x)| dx dz \\ & \leq Cr^{-2\delta} 2^{2jn\sigma} \int_{\mathbb{R}^n} |f(z)| (2^{-j}N(\lambda(z)))^n \int_{|x-z| \leq \frac{3 \cdot 2^j}{N(\lambda(z))}} |g(x)| dx dz \\ & \quad + C2^{2jn\sigma} \int_{\mathbb{R}^n} |f(z)| (2^{-j}N(\lambda(z)))^n \int_{\mathbb{R}^n} \chi_{E_{\lambda(z)}^j} (2^{-j}N(\lambda(z))(x-z)) |g(x)| dx dz \\ & \quad + Cr^{-2\delta} 2^{2jn\sigma} \int_{\mathbb{R}^n} |g(x)| (2^{-j}N(\lambda(x)))^n \int_{|x-z| \leq \frac{3 \cdot 2^j}{N(\lambda(x))}} |f(z)| dz dx \\ & \quad + C2^{2jn\sigma} \int_{\mathbb{R}^n} |g(x)| (2^{-j}N(\lambda(x)))^n \int_{\mathbb{R}^n} \chi_{E_{\lambda(x)}^j} (2^{-j}N(\lambda(x))(x-z)) |f(z)| dz dx \\ & \leq Cr^{-2\delta} 2^{2jn\sigma} \int_{\mathbb{R}^n} |f(z)| M(g)(z) dz + C2^{2jn\sigma} \int_{\mathbb{R}^n} |f(z)| \mathcal{M}_\varepsilon(g)(z) dz \\ & \quad + Cr^{-2\delta} 2^{2jn\sigma} \int_{\mathbb{R}^n} |g(x)| M(f)(x) dx + C2^{2jn\sigma} \int_{\mathbb{R}^n} |g(x)| \mathcal{M}_\varepsilon(f)(x) dx. \end{aligned}$$

Using Hölder's inequality, the L^2 boundedness of M (see [12]) and Lemma 2.1, we get

$$(2.18) \quad |(\mathcal{L}_{j,r}\mathcal{L}_{j,r}^*f, g)| \leq Cr^{-2\delta} 2^{2jn\sigma} \|f\|_{L^2} \|g\|_{L^2}.$$

Note that

$$\left\{x \in \mathbb{R}^n : |A_{j,\lambda} \circ \lambda| \geq \frac{2^j}{c_0}\right\} = \bigcup_{k=0}^{\infty} \left\{x : \frac{2^{j+k}}{c_0} \leq |A_{j,\lambda} \circ \lambda| < \frac{2^{j+k+1}}{c_0}\right\}.$$

Now we take $r = 2^{j+k}/c_0$ successively for $k = 0, 1, \dots$, and denote $\mathcal{L}_j^{(k)} := \mathcal{L}_{j,r}$. Then $\mathcal{L}_j(f)(x) = \sum_{k=0}^{\infty} \mathcal{L}_j^{(k)}(f)(x)$ and by (2.18) we have

$$(2.19) \quad \|\mathcal{L}_j\|_{L^2 \rightarrow L^2} \leq \sum_{k=0}^{\infty} \|\mathcal{L}_j^{(k)}\|_{L^2 \rightarrow L^2} \leq C2^{jn\sigma} \sum_{k=0}^{\infty} (2^{j+k}/c_0)^{-\delta} \leq C2^{jn\sigma} 2^{-j\delta}.$$

By (2.14), we also have

$$(2.20) \quad \|\mathcal{L}_j(f)\|_{L^s} \leq C2^{jn\sigma} \|f\|_{L^s} \quad \text{for } 1 < s < \infty.$$

Using the Riesz-Thörin interpolation theorem between (2.19) and (2.20), we get

$$(2.21) \quad \|\mathcal{L}_j(f)\|_{L^p} \leq C2^{jn\sigma} 2^{-j\gamma(p)\delta} \|f\|_{L^p}, \quad q' < p < \infty,$$

where $\gamma(p) = \min\{2/p, 2/p'\}$.

On the other hand, using (2.14) again and the weighted L^p boundedness of M (see [6]), for $1 < p < \infty$ and $w \in A_p$, we have

$$(2.22) \quad \|\mathcal{L}_j(f)\|_{L^p(w)} \leq C2^{jn\sigma} \|f\|_{L^p(w)}.$$

Since $w \in A_{p/q'}$, there is an $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_{p/q'} \subset A_p$ (see [6]). Thus by (2.22), we get

$$(2.23) \quad \|\mathcal{L}_j(f)\|_{L^p(w^{1+\varepsilon})} \leq C2^{jn\sigma}\|f\|_{L^p(w^{1+\varepsilon})}.$$

Applying the Stein-Weiss interpolation theorem with change of measure (see [14]) between (2.21) and (2.23), we get for $q' < p < \infty$ and $w \in A_{p/q'}$,

$$(2.24) \quad \|\mathcal{L}_j(f)\|_{L^p(w)} \leq C2^{jn\sigma}2^{-j\gamma(p)\delta\frac{\varepsilon}{1+\varepsilon}}\|f\|_{L^p(w)}.$$

Hence, taking $0 < \sigma < \gamma(p)\delta\varepsilon/n(1+\varepsilon)$ and letting $\theta = \gamma(p)\delta\frac{\varepsilon}{1+\varepsilon} - n\sigma$, we obtain (2.8).

Finally, we give the proof of (2.17). Define $\mathcal{F}_j^{\mu,\nu}$ by

$$\mathcal{F}_j^{\mu,\nu}(u) = (e^{iP_\nu(\cdot)}L_{j,\nu}(\cdot)) * (e^{-iP_\mu(\cdot)}L_{j,\mu}(\cdot))(u),$$

where $\nu = (\nu_\alpha)_{2 \leq |\alpha| \leq d}$ and $\mu = (\mu_\alpha)_{2 \leq |\alpha| \leq d}$, to satisfy

$$r \leq |A_{j,\nu} \circ \nu|, \quad |A_{j,\mu} \circ \mu| < 2r.$$

Let $h = \frac{N(\mu)}{N(\nu)}$. We may assume that $h = \frac{N(\mu)}{N(\nu)} \leq 1$. Hence, by (2.12) and (2.15) we have

$$(2.25) \quad \mathcal{F}_j^{\mu,\nu}\left(\frac{2^j u}{N(\mu)}\right) = \int_{\mathbb{R}^n} e^{i[P_{A_{j,\nu} \circ \nu}(y) - P_{A_{j,\mu} \circ \mu}(-u+hy)]} L_j(y) \left(\frac{2^j}{N(\mu)}\right)^{-n} L_j(hy - u) dy.$$

Below we give the estimate of $\mathcal{F}_j^{\mu,\nu}\left(\frac{2^j u}{N(\mu)}\right)$ by dividing into two cases: h is near the origin and h is away from the origin.

Case 1. $0 < h \leq \eta \ll 1$, where η will be chosen later. Notice that

$$\text{supp}(L_j) \subseteq \{1/8 < |y| \leq 3/2\} \quad \text{and} \quad |u| \leq |hy - u| + h|y| \leq 3.$$

By a trivial calculation, we have

$$\begin{aligned} P_{A_{j,\nu} \circ \nu}(y) - P_{A_{j,\mu} \circ \mu}(-u + hy) &= \sum_{2 \leq |\alpha| \leq d} ((A_{j,\nu} \circ \nu)_\alpha + O(h|A_{j,\mu} \circ \mu|)) y^\alpha \\ &\quad - P_{A_{j,\mu} \circ \mu}(-u). \end{aligned}$$

Note that

$$r \leq |A_{j,\nu} \circ \nu| = \sum_{2 \leq |\alpha| \leq d} |(A_{j,\nu} \circ \nu)_\alpha| < 2r$$

and $|A_{j,\mu} \circ \mu| < 2r$. Hence if η is chosen small enough, then

$$\begin{aligned} \sum_{2 \leq |\alpha| \leq d} |((A_{j,\nu} \circ \nu)_\alpha + O(h|A_{j,\mu} \circ \mu|))| &\geq \sum_{2 \leq |\alpha| \leq d} |(A_{j,\nu} \circ \nu)_\alpha| - C\eta|A_{j,\mu} \circ \mu| \\ &\geq C \sum_{2 \leq |\alpha| \leq d} |(A_{j,\nu} \circ \nu)_\alpha| \geq Cr. \end{aligned}$$

Thus, by the van der Corput lemma in n dimensions (see Proposition 2.1 in [13, p. 791]) and (2.25), we have

$$(2.26) \quad \left| \mathcal{F}_j^{\mu,\nu}\left(\frac{2^j u}{N(\mu)}\right) \right| \leq C(2^{-j}N(\mu))^n 2^{2jn\sigma} r^{-1/d} \chi_{B_3}(u).$$

Case 2. $\eta < h \leq 1$. By the assumption on polynomials in Theorem 1.3, there was no first-order term in y in $P_{A_{j,\nu} \circ \nu}(y)$. Thus, the first-order term in y in $P_{A_{j,\nu} \circ \nu}(y) - P_{A_{j,\mu} \circ \mu}(-u + hy)$ is

$$-h \sum_{k=1}^n P_{A_{j,\mu} \circ \mu}^{(k)}(u) y_k,$$

where

$$P_{A_{j,\mu} \circ \mu}^{(k)}(u) = \sum_{2 \leq |\alpha| \leq d} \alpha_k \left(\frac{2^j}{N(\mu)} \right)^{|\alpha|} \mu_\alpha u^{\alpha - e_k}$$

and $e_k = (0, \dots, 1, 0, \dots)$ with 1 in the k^{th} component. Applying Proposition 2.1 in [13] and (2.25) again, we get

$$\left| \mathcal{F}_j^{\mu, \nu} \left(\frac{2^j u}{N(\mu)} \right) \right| \leq C(2^{-j} N(\mu))^n 2^{2jn\sigma} \left(\sum_{k=1}^n |P_{A_{j,\mu} \circ \mu}^{(k)}(u)| \right)^{-1/d} \chi_{B_3}(u).$$

For $\rho > 0$, denote

$$E_\mu^j = \left\{ u \in B_3 : \sum_{k=1}^n |P_{A_{j,\mu} \circ \mu}^{(k)}(u)| \leq \rho \right\}.$$

Thus, for $u \in (E_\mu^j)^c$, it is obvious that

$$(2.27) \quad \left| \mathcal{F}_j^{\mu, \nu} \left(\frac{2^j u}{N(\mu)} \right) \right| \leq C(2^{-j} N(\mu))^n 2^{2jn\sigma} \rho^{-1/d} \chi_{B_3}(u).$$

By Proposition 2.2 in [13, p. 791], we obtain

$$|E_\mu^j| \leq C_{n,d} \left(\sum_{k=1}^n \sum_{2 \leq |\alpha| \leq d} \alpha_k \left(\frac{2^j}{N(\mu)} \right)^{|\alpha|} |\mu_\alpha| \right)^{-1/d} \rho^{1/d}.$$

Note that

$$\sum_{k=1}^n \sum_{2 \leq |\alpha| \leq d} \alpha_k \left(\frac{2^j}{N(\mu)} \right)^{|\alpha|} |\mu_\alpha| \geq \sum_{2 \leq |\alpha| \leq d} \left(\frac{2^j}{N(\mu)} \right)^{|\alpha|} |\mu_\alpha| = |A_{j,\mu} \circ \mu| \geq r.$$

Thus for $u \in E_\mu^j$, we have

$$\left| \mathcal{F}_j^{\mu, \nu} \left(\frac{2^j u}{N(\mu)} \right) \right| \leq C(2^{-j} N(\mu))^n 2^{2jn\sigma} \chi_{E_\mu^j}(u),$$

with $|E_\mu^j| \leq C_{n,d}(\rho/r)^{1/d}$. We now take $\rho = (C_{n,d})^{-d} r^{1/3}$ and denote $\delta = \frac{1}{6d}$. Then, for $u \in E_\mu^j$, we have

$$(2.28) \quad \left| \mathcal{F}_j^{\mu, \nu} \left(\frac{2^j u}{N(\mu)} \right) \right| \leq C(2^{-j} N(\mu))^n 2^{2jn\sigma} \chi_{E_\mu^j}(u)$$

and $|E_\mu^j| \leq r^{-4\delta}$. Note that $r \geq 2^j/c_0$. It follows from (2.26), (2.27) and (2.28) that

$$\left| \mathcal{F}_j^{\mu, \nu} \left(\frac{2^j u}{N(\mu)} \right) \right| \leq C(2^{-j} N(\mu))^n 2^{2jn\sigma} [r^{-2\delta} \chi_{B_3}(u) + \chi_{E_\mu^j}(u)].$$

Then, for μ and ν with

$$r \leq |A_{j,\nu} \circ \nu| \leq 2r, \quad r \leq |A_{j,\mu} \circ \mu| \leq 2r \quad \text{and} \quad h \leq 1,$$

we have

$$(2.29) \quad |\mathcal{F}_j^{\mu,\nu}(u)| \leq C(2^{-j}N(\mu))^n 2^{2jn\sigma} [r^{-2\delta} \chi_{B_3}(2^{-j}N(\mu)u) + \chi_{E_\mu^j}(2^{-j}N(\mu)u)].$$

For fixed $x, z \in U_{j,r}$, let $\nu = \lambda(x)$, $\mu = \lambda(z)$, $u = x - z$. By the symmetry of μ and ν and by (2.29), we obtain (2.17).

2.2.2. *Proof of (2.9). The weighted estimates of $\mathcal{R}_{\lambda(\cdot)}^j$.* Recall that

$$\mathcal{R}_{\lambda(x)}^j(f)(x) = \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} R_{j,\lambda}(y) f(x - y) dy,$$

where $R_{j,\lambda}(y) = K_j(y) - L_{j,\lambda}(y)$. Thus,

$$R_{j,\lambda}(y) = \int_{\mathbb{R}^n} [K(y)\psi_{j,\lambda}(y) - K(y - z)\psi_{j,\lambda}(y - z)] \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz.$$

By

$$(2.30) \quad \text{supp } (R_{j,\lambda}) \subseteq \left\{ \frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))} \right\}$$

and $|z| \leq \frac{2^{j(1-\sigma)-5}}{N(\lambda(x))}$, we know $|y - z| \sim |y|$. Hence, we have

$$\begin{aligned} |R_{j,\lambda}(y)| &\leq \int_{\mathbb{R}^n} |\psi_{j,\lambda}(y - z)| |K(y) - K(y - z)| \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz \\ &\quad + \int_{\mathbb{R}^n} |K(y)| |\psi_{j,\lambda}(y) - \psi_{j,\lambda}(y - z)| \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz \\ &\leq C \frac{|\Omega(y)|}{|y|^{n+1}} \int_{\mathbb{R}^n} |z| \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz \\ &\quad + \frac{C}{|y|^n} \int_{\mathbb{R}^n} |\Omega(y - z) - \Omega(y)| \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz \\ &\quad + C \frac{|\Omega(y)|}{|y|^n} \int_{\mathbb{R}^n} |2^{-j}N(\lambda(x))z| \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz \\ &\leq C 2^{-j\sigma} \frac{|\Omega(y)|}{|y|^n} + \frac{C}{|y|^n} \int_{\mathbb{R}^n} |\Omega(y - z) - \Omega(y)| \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) dz. \end{aligned}$$

Now we denote $\mathcal{R}_{\lambda(\cdot)}^j(f)$ by $\mathcal{R}_j(f)$ for short. Then by (2.30) we have

$$\begin{aligned} |\mathcal{R}_j(f)(x)| &\leq C 2^{-j\sigma} M_\Omega(f)(x) + \int_{\mathbb{R}^n} \phi_{\frac{2^j(1-\sigma)}{N(\lambda(x))}}(z) \\ &\quad \times \int_{\frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))}} \frac{|\Omega(y - z) - \Omega(y)|}{|y|^n} |f(x - y)| dy dz. \end{aligned}$$

By Hölder's inequality, the second term on the right side above is controlled by

$$\begin{aligned} & \int_{\mathbb{R}^n} \phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z) \left(\int_{\frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))}} \frac{|\Omega(y-z) - \Omega(y)|^q}{|y|^n} dy \right)^{1/q} \\ & \quad \times \left(\int_{\frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))}} \frac{|f(x-y)|^{q'}}{|y|^n} dy \right)^{1/q'} dz. \end{aligned}$$

Note the following known fact (see [1]): for $\alpha = \frac{z}{|y|}$,

$$\begin{aligned} \int_{S^{n-1}} \left| \Omega\left(\frac{y' - \alpha}{|y' - \alpha|}\right) - \Omega(y') \right|^q d\sigma(y') & \leq C \sup_{\|\rho\| < |\alpha|} \int_{S^{n-1}} |\Omega(\rho y') - \Omega(y')|^q d\sigma(y') \\ & = C\omega_q^q(|\alpha|). \end{aligned}$$

Since $|z| \leq \frac{2^{j(1-\sigma)-5}}{N(\lambda(x))}$, we get

$$\left(\int_{\frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))}} \frac{|\Omega(y-z) - \Omega(y)|^q}{|y|^n} dy \right)^{1/q} \leq C\omega_q(2^{-j\sigma-2}).$$

Hence

$$|\mathcal{R}_j(f)(x)| \leq C[2^{-j\sigma} M_\Omega(f)(x) + \omega_q(2^{-j\sigma-2}) M_{q'}(f)(x)],$$

where C is independent of the choice of $\lambda(\cdot)$. Since $p > q'$ and $w \in A_{p/q'}$, applying the weighted L^p boundedness of M_Ω and M again, we have

$$\|\mathcal{R}_j(f)\|_{L^p(w)} \leq C(2^{-j\sigma} + \omega_q(2^{-j\sigma-2}))\|f\|_{L^p(w)},$$

which is just the desired result (2.9).

REFERENCES

1. A. P. Calderón, M. Weiss and A. Zygmund, *On the existence of singular integrals*, Proc. Symp. Pure Math., **10**, AMS (1967), 56-73. MR0338709 (49:3473)
2. L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **111** (1966), 361-370. MR0199631 (33:7774)
3. L. Colzani, *Hardy spaces on spheres*, Ph.D. Thesis, Washington University, St. Louis, 1982.
4. Y. Ding and H. Liu, *L^p boundedness of Carleson type maximal operators with nonsmooth kernels*, Tohoku Math. J. (2), **63** (2011), 255-267. MR2812453
5. J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. **336** (1993), 869-880. MR1089418 (93f:42030)
6. J. García-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Mathematics Studies, **116**, North-Holland, 1985. MR807149 (87d:42023)
7. R. A. Hunt, *On the convergence of Fourier series*, Orthogonal expansions and their continuous analogues, Proceedings of the Conference held at Southern Illinois University, Edwardsville, 1967. Southern Illinois Univ. Press, Carbondale, IL. MR0238019 (38:6296)
8. R. A. Hunt and W. Young, *A weighted norm inequality for Fourier series*, Bull. Amer. Math. Soc. **80** (1974), 274-277. MR0338655 (49:3419)
9. D. Kurtz and R. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343-362. MR542885 (81j:42021)
10. E. Prestini and P. Sjölin, *A Littlewood-Paley inequality for the Carleson operator*, J. Fourier Anal. Appl. **6** (2000), 457-466. MR1781088 (2001m:42017)
11. P. Sjölin, *Convergence almost everywhere of certain singular integral and multiple Fourier series*, Ark. Mat. **9** (1971), 65-90. MR0336222 (49:998)

12. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970. MR0290095 (44:7280)
13. E. M. Stein and S. Wainger. *Oscillatory integrals related to Carleson's theorem*, Math. Res. Lett. **8** (2001), 789-800. MR1879821 (2002k:42038)
14. E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. **87** (1958), 159-172. MR0092943 (19:1184d)

SCHOOL OF MATHEMATICAL SCIENCES, LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS (BNU), BEIJING NORMAL UNIVERSITY, MINISTRY OF EDUCATION OF CHINA, BEIJING 100875, PEOPLE'S REPUBLIC OF CHINA

E-mail address: dingy@bnu.edu.cn

SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO 454000, PEOPLE'S REPUBLIC OF CHINA

E-mail address: hhliu@hpu.edu.cn