SHARP PHASE TRANSITION THRESHOLDS FOR THE PARIS HARRINGTON RAMSEY NUMBERS FOR A FIXED DIMENSION

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ABSTRACT. This article is concerned with investigations on a phase transition which is related to the (finite) Ramsey theorem and the Paris-Harrington theorem. For a given number-theoretic function g , let $R_c^d(g)(k)$ be the least natural number R such that for all colourings P of the d -element subsets of $\{0, \dots, R-1\}$ with at most c colours there exists a subset H of $\{0, \dots, R-1\}$ such that P has constant value on all d -element subsets of H and such that the cardinality of H is not smaller than $\max\{k, g(\min(H))\}$. If g is a constant function with value e , then $R_c^d(g)(k)$ is equal to the usual Ramsey number $R_c^d(\max\{e, k\})$; and if g is the identity function, then $R_c^d(g)(k)$ is the corresponding Paris-Harrington number, which typically is much larger than $R_c^d(k)$. In this article we give for all $d \geq 2$ a sharp classification of the functions g for which the function $m \mapsto R_m^d(g)(m)$ grows so quickly that it is no longer provably total in the subsystem of Peano arithmetic, where the induction scheme is restricted to formulas with at most $(d-1)$ -quantifiers. Such a quick growth will in particular happen for any function g growing at least as fast as $i \mapsto \varepsilon \cdot \underbrace{\log(\dots \log(i) \dots)}_{(d-1)\text{-times}}$ (where $\varepsilon > 0$ is fixed) but not for the function

$$g(i) = \frac{1}{\log^*(i)} \cdot \underbrace{\log(\dots \log(i) \dots)}_{(d-1)\text{-times}}. \quad (\text{Here } \log^* \text{ denotes the functional inverse}$$

of the tower function.) To obtain such results and even sharper bounds we employ certain suitable transfinite iterations of *nonconstructive* lower bound functions for Ramsey numbers. Thereby we improve certain results from the article *A classification of rapidly growing Ramsey numbers* (PAMS 132 (2004), 553–561) of the first author, which were obtained by employing *constructive* ordinal partitions.

1. INTRODUCTION

This article is part of a general investigation of phase transitions in logic and combinatorics. The underlying idea is to consider true mathematical assertions A about the natural numbers and to introduce an appropriate order parameter g provided by a number-theoretic function in a meaningful way so that the resulting assertion $A(g)$ is still true for all choices of g . In quite a few situations it occurs that, for g of very slow growth, the assertion $A(g)$ follows from a simple set S

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of axioms, e.g. (slight extensions of) $I\Delta_0 + (\exp)$ (formalizing bounded quantifier induction over the natural numbers plus the totality of the exponential function), and that moreover $A(g)$ does not follow from S when g exceeds in growth functions from a certain threshold region. In general we expect that classifying such thresholds is interesting in itself. Moreover, such investigations may shed light on the fundamental question: *What makes a given true assertion unprovable from S or even unprovable from stronger systems than S ?*

In [9] the first author classified the phase transition emerging from the Paris-Harrington (PH) function for unbounded dimensions. That paper also provided upper bounds for the Paris-Harrington numbers of a fixed dimension, but these upper bounds did not match with the lower bounds provided by the Erdős-Rado bounds on the classical Ramsey numbers. The gap for dimension equal to two has been closed in the meantime in [6] via an elementary proof which avoids ordinals completely. In this article we will close the remaining gap for all higher dimensions, and we re-prove the result for dimension two. For this purpose we replace the Ramsey bounds resulting from constructively given ordinal partitions by suitable transfinitely iterated functions for nonconstructive Ramsey bounds refining techniques of Ketonen and Solovay [5]. Intuitively, our proof resembles the behaviour of iterations of a discrete dynamical system at an unstable fixed point (in our case the threshold region). This proof technique is rather far reaching and can be applied (in combination with other techniques) to phase transitions emerging from various combinatorial statements in Ramsey theory. Examples in this regard are provided by the Kanamori-McAloon-Ramsey theorem (regressive Ramsey theorem) [2] and the canonical Ramsey theorem (with largeness condition).

2. RAMSEY FUNCTIONS: SOME BASIC RESULTS

In this section we recall some basic terminology and results. We denote the set of natural numbers with $\mathbb{N}$, and we identify a natural number $n \in \mathbb{N}$ with its set of predecessors, so that $n = \{m \in \mathbb{N} : m < n\}$. We also identify n with the half open interval $[0, n[$. In the sequel small Latin letters like $c, d, e, l, m, n, x, y, z$ denote natural numbers. The cardinality of a given set X is denoted by $|X|$, and the set of all d -element subsets of a set X is denoted by $[X]^d$. For a function $P : [X]^d \rightarrow \mathbb{N}$ with $X \subseteq \mathbb{N}$, we write $P(x_0, \dots, x_{d-1})$ for $P(\{x_0, \dots, x_{d-1}\})$ under the tacit assumption that $x_0 < x_1 < \dots < x_{d-1}$. We write $X \rightarrow (m)_c^d$ if for all functions (partitions/labourings) $P : [X]^d \rightarrow c$ there exists a set $H \subseteq X$ such that the restriction of P to $[H]^d$ is a constant function and $|H| \geq m$. If the restriction of P to $[H]^d$ is a constant function, then H is called *P -homogeneous* or *P -monochromatic*.

The classical Ramsey function is then defined as follows:

$$R_c^d(m) := \min\{R : R \rightarrow (m)_c^d\}.$$

In this article we investigate a variant of $R_c^d(m)$ which is defined with respect to the notion of (parameterized) largeness. For a given function $g : \mathbb{N} \rightarrow \mathbb{N}$ we call a nonempty set $H \subseteq \mathbb{N}$ *g -large* if $|H| \geq g(\min(H))$. We write $X \rightarrow_g^* (m)_c^d$ if for all partitions $P : [X]^d \rightarrow c$ there exists a g -large set $H \subseteq X$ such that the restriction of P to $[H]^d$ is a constant function and $|H| \geq m$. The resulting Paris-Harrington-Ramsey function is then defined as follows:

$$R_c^d(g)(m) := \min\{R : R \rightarrow_g^* (m)_c^d\}.$$

The following lemmas will be very useful further on. (Let $\log_2$ denote the binary log function with the tacit assumption used throughout in this paper that $\log_2(0) := 0$.)

Lemma 2.1. *For all $d > 0$ there exists a (positive integer) constant $\Gamma > 0$ such that $R_c^{d+1}(m) \leq 2_d(\Gamma \cdot c \cdot \log_2(c) \cdot m)$ for all $c, m > 0$.*

Proof. See Theorem 26.6 and its proof in [3]. One can take, e.g., $\Gamma := d! \cdot 2^d$. $\square$

Lemma 2.2. *There exists a (positive integer) constant Λ such that $R_c^2(m) \geq 2^{\frac{1}{\Lambda} \cdot c \cdot m}$ holds for all natural numbers c, m .*

Proof. See [1] or [7]. One can take $\Lambda := 14$. $\square$

Lemma 2.3. *For all $d > 0$ there exists a (positive integer) constant Δ such that $R_c^{d+1}(m) \geq 2_d(\frac{1}{\Delta} \cdot c \cdot m)$ for all c, m .*

Proof. This follows from [3] and Lemma 2.2. In fact we have for $n, c > 0$ by Lemma 24.2 of [3] that $n \not\vdash (m)_c^{r-1}$ implies $2^n \not\vdash (m+1)_{2c}^r$ for $m \geq r = 3$. Moreover, we have for $n, c > 0$ by Lemma 24.2 of [3] that $n \not\vdash (m)_c^{r-1}$ implies $2^n \not\vdash (2 \cdot m)_c^r$ for $m \geq r \geq 4$. These bounds together with Lemma 2.2 yield the desired estimate. $\square$

Let PH_g denote the assertion

$$(\forall d, c, m)(\exists R)[R \rightarrow_g^* (m)_c^d],$$

and let PH_g^d denote the assertion

$$(\forall c, m)(\exists R)[R \rightarrow_g^* (m)_c^d].$$

Let id denote the identity function on $\mathbb{N}$.

Theorem 2.4 (Paris-Harrington Theorem).

- (1) *The assertion PH_{id} is true but not provable from the Peano axioms.*
- (2) *For all $d \geq 2$, the assertion PH_{id}^d is true but not provable from the fragment of the Peano axioms where the induction scheme is restricted to formulas having at most $(d-1)$ -quantifiers.*

Proof. This follows, for example, from [5] or [8]. $\square$

Lemma 2.1 yields that for fixed $d > 1$ and a constant function g the assertion

$$\text{PH}_g^d$$

is provable even in $I\Delta_0 + (\text{exp})$. So in between constant functions and the identity function there will be a region where there is a transition from provability to unprovability. For first-order Peano Arithmetic (PA), the phase transition threshold for the assertion PH_g has been determined in [9]. In this article we determine the corresponding phase transition threshold for the systems of $(d-1)$ -quantifier induction and PH_g^d . To this end we classify the growth rate of the associated function $m \mapsto R_m^d(g)(m)$ for the relevant number-theoretic functions g . The crucial point of this paper is the use of nonconstructive lower bounds from Ramsey theory (cf. Lemma 2.3) which have not been needed in the case of PA.

3. SOME FACTS ABOUT PARTITIONS AND SUITABLE SETS

In this section we develop (using ideas and terminology of Ketonen and Solovay [5]) some basic properties of partitions and suitable sets. We call a partition $P : [\mathbb{N}]^d \rightarrow c$ a (d, c) -algebra. A partition P is called an *algebra* if there exist c, d such that P is a (d, c) -algebra. For a given function $f : \mathbb{N} \rightarrow \mathbb{N}$, we call a set $S \subseteq \mathbb{N}$ *f-suitable* for a (d, c) -algebra P if the following three conditions are satisfied: S is P -homogeneous, $|S| \geq d + 1$, and $|S| \geq f(\min(S))$.

We say that an algebra P *f-captures* a function $g : \mathbb{N} \rightarrow \mathbb{N}$ if for every f -suitable S and all $s, s' \in S$ with $s < s'$ we have

$$g(s) \leq s'.$$

An algebra P *f-simulates* an algebra P' if every f -suitable set S for P is also f -suitable for P' .

If $P_1 : [\mathbb{N}]^d \rightarrow c_1$ and $P_2 : [\mathbb{N}]^d \rightarrow c_2$ are (d, c_1) and (d, c_2) algebras, then we define the product algebra $P : [\mathbb{N}]^d \rightarrow c_1 \cdot c_2$ by $P(x_1, \dots, x_d) := c_1 \cdot P_1(x_1, \dots, x_d) + P_2(x_1, \dots, x_d)$. Then for a given $f : \mathbb{N} \rightarrow \mathbb{N}$, P is a $(d, c_1 \cdot c_2)$ -algebra which f -simulates P_1 and P_2 .

Lemma 3.1. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ $c \leq c_1 \cdot c_2 \cdot \dots \cdot c_k$. Then any (d, c) -algebra can be f -simulated by a $(d + 1, c_1 + \dots + c_k + 1)$ -algebra.*

Proof. Cf. the proof of Corollary 1.2.3 in [5]. $\square$

Let $\log_2^d$ be the d th iterate of the binary logarithm function $\log_2$ (recall that we put $\log_2(0) := 0$), and let $m \mapsto 2_d(m)$ be the d th iterate of the function $m \mapsto 2^m$.

The following lemma is of *crucial* importance for our further results. The proof of it is already a first example of constructing algebras which capture sufficiently fast growing functions.

Lemma 3.2. *Let $d \in \mathbb{N}$ be fixed with $d > 0$. For a given $l > 0$, let $f_l(i) := \max\{1, \lceil \frac{1}{l} \log_2^d(i) \rceil\}$. There exists a constant c_l (depending elementarily recursively on l) and a $(d + 1, (c_l + 1) \cdot (d + 1)^2)$ -algebra that f_l -captures the map $m \mapsto 2_{d+1}(m)$.*

Proof. We keep l fixed and write f for f_l . Let

$$c_l = l \cdot \Delta \cdot 2^{(d+1) \cdot 2 \cdot l}$$

with Δ from Lemma 2.3. Put $u_0 := 0$, $u_1 := R_{c_l}^{d+1}(d + 2) - 1$ and for $i > 0$

$$u_{i+1} := u_i + R_{c_l}^{d+1}(f(u_i)) - 1.$$

We claim that

$$u_i \geq 2_d(2^{(d+1) \cdot 2 \cdot l \cdot i})$$

for all $i \geq 1$. Lemma 2.3 yields $u_1 \geq 2_d(\frac{1}{\Delta} \cdot c_l \cdot (d + 2)) - 1 \geq 2_d(2^{(d+1) \cdot l \cdot 1})$. Moreover, for the induction step the induction hypothesis yields

$$\begin{aligned} u_{i+1} &\geq u_i + R_{c_l}^{d+1}(f(u_i)) - 1 \geq R_{c_l}^{d+1}(f(u_i)) \\ &\geq 2_d\left(\frac{1}{\Delta} \cdot c_l \cdot f(u_i)\right) \\ &\geq 2_d\left(\frac{1}{\Delta} \cdot c_l \cdot \frac{1}{l} \cdot 2^{(d+1) \cdot 2 \cdot l \cdot i}\right) = 2_d(2^{(d+1) \cdot 2 \cdot l \cdot (i+1)}). \end{aligned}$$

By the choice of u_1 we can find a partition $Q_0 : [[0, u_1]]^{d+1} \rightarrow c_l$ such that $|H| < d + 2$ for every Q_0 -homogeneous H . By choice of u_i , we can find for $i > 0$ a

partition $Q_i : [[u_i, u_{i+1}[]^{d+1} \rightarrow c_l$ such that $|H| < f(u_i)$ for every Q_i -homogeneous H . Let us consider the following three algebras:

$$\begin{aligned} P_1(x_0, \dots, x_d) &= \begin{cases} Q_i(x_0, \dots, x_d) + 1 & \text{if } u_i \leq x_0 < \dots < x_d < u_{i+1}, \\ 0 & \text{otherwise;} \end{cases} \\ P_2(x, y) &= \begin{cases} 0 & \text{if } 2^x \leq y, \\ 1 & \text{otherwise;} \end{cases} \\ P_3(x, y) &= \begin{cases} 0 & \text{if } 2_{d+1}(x) \leq y, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Let P be a $(d+1, (c_l+1) \cdot (d+1)^2)$ -algebra which f -simulates P_1, P_2 , and P_3 (by first applying Lemma 3.1 to P_2 and P_3 and then applying the product algebra construction). Let $S = \{x_0, \dots, x_{m-1}\}$ be f -suitable for P . Assume that the restriction of P_1 to $[S]^{d+1}$ is not equal to 0. Then there exists an index i such that $S \subseteq [u_i, u_{i+1}[$, and the definition of Q_i yields $|S| < d+2$ for $i = 0$ or

$$|S| < f(u_i) \leq f(\min(S))$$

for $i > 0$, contradicting that S is f -suitable for P . So the restriction of P_1 to $[S]^{d+1}$ equals the constant function with value 0. This yields that for any j at most d elements of S are contained in $[u_j, u_{j+1}[$. This means that $[u_0, u_{j+1}[$ contains at most $d \cdot (j+1)$ elements of S . Thus $x_{m-1} \geq u_{\lfloor \frac{m-1}{d} \rfloor}$.

We know that $m = |S| \geq f(\min S) \geq \frac{1}{l} \cdot \log_2^d(x_0)$ and thus

$$\begin{aligned} x_{m-1} &\geq u_{\lfloor \frac{m-1}{d} \rfloor} \geq 2_d(2^{(d+1) \cdot 2 \cdot l \cdot \lfloor \frac{m-1}{d} \rfloor}) \\ &\geq 2_{d+1}(m \cdot l) \\ &\geq 2_{d+1}(\log_2^d(x_0)) \geq 2^{x_0}, \end{aligned}$$

since $d > 0$ and $|S| = m \geq d+2$. This yields $P_2(x_0, x_{m-1}) = 0$; thus

$$P_2(x_0, x_1) = P_2(x_1, x_2) = \dots = P_2(x_{m-2}, x_{m-1}) = 0.$$

Since P is f -suitable we obtain $m \geq d+2$; hence

$$x_{m-1} \geq 2^{x_{m-2}} \geq 2^{2^{x_{m-3}}} \geq \dots \geq 2_{m-1}(x_0) \geq 2_{d+1}(x_0).$$

This yields $P_3(x_0, x_{m-1}) = 0$ and then

$$P_3(x_0, x_1) = P_3(x_1, x_2) = \dots = P_3(x_{m-2}, x_{m-1}) = 0,$$

proving that P f -captures the map $m \mapsto 2_{d+1}(m)$. $\square$

The following lemma shows that just by using enough colours in partitions we can guarantee that the minima of suitable sets can be forced to be arbitrarily large.

Lemma 3.3. *Let $c, d, e > 0$ and P be a (d, c) -algebra. Then there exists a $(d, c+e)$ -algebra P^* f -simulating P such that $\min S \geq e$ for every f -suitable S for P^* .*

Proof. See Lemma 1.4 in [5]. For convenience we recall the proof. Let $P^* : [\mathbb{N}]^d \rightarrow e+c$ be defined by $P^*(X) = \min(X)$ if $\min(X) < e$ and $P^*(X) = e + P(X)$ otherwise. Let $S = \{x_0, \dots, x_{m-1}\}$ be f -suitable for P^* . This yields $m \geq d+1$. We claim that $x_0 \geq e$. Otherwise $P^*(\{x_0, \dots, x_{d-1}\}) = x_0$. But then $P^*(\{x_1, \dots, x_d\}) = x_1$, contradicting the f -suitability of S . $\square$

4. RAPIDLY GROWING FUNCTIONS

Recall that for any limit ordinal $\lambda < \varepsilon_0$ there is a canonical fundamental sequence for λ which can be defined recursively as follows:

- (1) If $\lambda = \omega^{\alpha+1}(\beta + 1)$, then we put $\lambda[x] := \omega^{\alpha+1} \cdot \beta + \omega^\alpha \cdot x$.
- (2) If $\lambda = \omega^\gamma(\beta + 1)$ and $\gamma < \lambda$ is a limit, then we define $\lambda[x] := \omega^\gamma \cdot \beta + \omega^{\gamma[x]}$.

For convenience we further put $(\alpha + 1)[x] := \alpha$ and $0[x] := 0$. For each $\alpha < \varepsilon_0$, we define the (rapidly growing) function $F_\alpha : \mathbb{N} \rightarrow \mathbb{N}$ recursively as follows:

- (1) $F_0(x) = x + 1$.
- (2) $F_{\beta+1}(x) = F_\beta^x(x)$ (where the upper index denotes the number of iterations).
- (3) $F_\lambda(x) = F_{\lambda[x]}(x)$ if λ is a limit.

For a given unbounded weakly increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$, we define a second hierarchy of (typically also rapidly growing) function $B(f)_\alpha$ for each $\alpha < \varepsilon_0$ as follows:

- (1) $B(f)_0(x) = 2^x$.
- (2) $B(f)_{\beta+1}(x) = B(f)_\beta^{f(x)}(x)$.
- (3) $B(f)_\lambda(x) = B(f)_{\lambda[f(x)]}(x)$ if λ is a limit.

For the specific choices of f in which we are interested, we can establish a connection between the two hierarchies of functions in the following lemma.

Lemma 4.1. *Let $d, l > 1$ and $f_l(i) := \max\{1, \lceil \frac{1}{l} \cdot \log_2^d(i) \rceil\}$. If $x \geq 1$, then*

$$B(f_l)_\alpha(2_d(l \cdot x)) \geq 2_d(l \cdot F_\alpha(x)).$$

Proof. We proceed by induction on α . We write f for f_l and B_α for $B(f)_\alpha$. If $\alpha = 0$, then $B_0(x) = 2_{d+1}(l \cdot x) \geq 2_d(l \cdot (x + 1)) = 2_d(l \cdot F_0(x))$ since $x \geq 1$ and $l > 1$. For $\alpha > 0$ the induction hypothesis yields:

$$\begin{aligned} B_{\alpha+1}(2_d(l \cdot x)) &= B_\alpha^{x-1}(B_\alpha(2_d(l \cdot x))) \geq B_\alpha^{x-1}(2_d(l \cdot F_\alpha(x))) \\ &= B_\alpha^{x-2}(B_\alpha(2_d(l \cdot F_\alpha(x)))) \\ &\geq B_\alpha^{x-2}(2_d(l \cdot F_\alpha^2(x))) \\ &\geq \cdots \geq 2_d(l \cdot F_{\alpha+1}(x)). \end{aligned}$$

If α is a limit ordinal, then the induction hypothesis yields:

$$\begin{aligned} B_\alpha(2_d(l \cdot x)) &= B_{\alpha[\lceil \frac{1}{l} \cdot \log_2^d(2_d(l \cdot x)) \rceil]}(2_d(l \cdot x)) \\ &\geq B_{\alpha[x]}(2_d(l \cdot x)) \\ &\geq 2_d(l \cdot F_{\alpha[x]}(x)) \\ &= 2_d(l \cdot F_\alpha(x)). \end{aligned} \quad \square$$

For a given unbounded weakly increasing function f , we define its functional inverse function by

$$f^{-1}(i) := \min\{j : f(j) \geq i\}.$$

The following proposition is very useful for further results.

Proposition 4.2. *Let $d > 0$ and f be a weakly monotone unbounded function, and let f^{-1} be its functional inverse. Then there exists a $(d, 1 + f^{-1}(2d + 3))$ -algebra G such that if S is f -suitable for G , then $f(\min(S)) \geq 2d + 3$.*

Proof. This follows from Lemma 3.3 applied to the trivial algebra $P : [\mathbb{N}]^d \rightarrow 1$. Note that the number of colours is here increased by $f^{-1}(2d + 3)$. $\square$

Lemma 4.3. *Let $c, e > 0$ and $d \geq 0$. Let $h : [\mathbb{N}]^e \rightarrow \mathbb{N}$, and assume $h(x_0, \dots, x_{e-1}) < 2_d(f(x_0)^c)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a weakly increasing unbounded function. Then there exists an $(e + d + 1, (18 + 6 \cdot f^{-1}(2e + 3))^c \cdot (d^2 + 5d + 6)^d)$ -algebra G such that for every f -suitable S for G there exists a function $g_S : S \rightarrow \mathbb{N}$ such that $h(X) = g_S(\min X)$ for every $X \in [S]^e$.*

Proof. This is proved by induction on d .

Case 1. $d = 0$ and $c = 1$.

- (1) We define $G_1 : [\mathbb{N}]^e \rightarrow 2$ and $G_2 : [\mathbb{N}]^{e+1} \rightarrow 3$ and $G_3 : [\mathbb{N}]^{e+1} \rightarrow 2$ by

$$\begin{aligned} G_1(X) &= \begin{cases} 0 & h(X) < \lfloor \frac{f(\min X)}{2} \rfloor, \\ 1 & \text{otherwise;} \end{cases} \\ G_2(x_0, \dots, x_e) &= \begin{cases} 0 & h(x_0, \dots, x_{e-1}) = h(x_0, x_2, \dots, x_n), \\ 1 & h(x_0, \dots, x_{e-1}) > h(x_0, x_2, \dots, x_e), \\ 2 & h(x_0, \dots, x_{e-1}) < h(x_0, x_2, \dots, x_e); \end{cases} \\ G_3(X) &= \begin{cases} 0 & \text{if } h \text{ is min-homogeneous on } [X]^e, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

By Lemmas 3.3 and 3.1 we can find an $(e + 1, 3 + (f^{-1}(2e + 3)))$ -algebra G_4 f -simulating G_1 such that $f(\min(S)) \geq 2e + 3$ for every f -suitable S for G_1 . By the product construction we can construct the desired $(e + 1, (3 + f^{-1}(2d + 3)) \cdot 2 \cdot 3)$ -algebra G via f -simulating G_2, G_3 , and G_4 . Let $S = \{x_0, \dots, x_k\}$ be f -suitable for G . We show that G_4 is identical zero on $[S]^{e+1}$. If this is shown, then h depends only on its first coordinate $\min X$.

- (2) Define $W = \{m : m < \lfloor \frac{f(x_0)}{2} \rfloor\}$ when the restriction of G_1 to $[S]^e$ is zero and $W = \{m : \lfloor \frac{f(x_0)}{2} \rfloor \leq m < f(x_0)\}$ otherwise. Then $|W| \leq \lfloor \frac{f(x_0)}{2} \rfloor + 1$. Since $d = 0$, $c = 1$ and $h(x_0, \dots, x_{e-1}) < 2_0(f(x_0)) = f(x_0)$, it follows that $h(A) \in W$ if $A \in [S]^e$ and $\min A = x_0$.
- (3) Further in this proof, we shall need the following inequality: $\lfloor \frac{f(x_0)}{2} \rfloor + 1 < f(x_0) - e$. Since $f(x_0) - \lfloor \frac{f(x_0)}{2} \rfloor = \lfloor \frac{f(x_0)+1}{2} \rfloor$, it suffices to show $e + 2 \leq \lfloor \frac{f(x_0)+1}{2} \rfloor$. But $e + 2 \in \mathbb{N}$, so it is enough to show $e + 2 \leq \frac{f(x_0)+1}{2}$. Since S is f -suitable for G_4 , this implies $f(x_0) \geq 2e + 3$, hence the desired inequality.
- (4) For $1 \leq i \leq f(x_0) + 1 - e$ we define (note that $k + 1 \geq f(x_0)$ since S is f -large)

$$A_i = \{x_0, x_i, \dots, x_{i+e-2}\},$$

so $A_i \in [S]^e$. Further we put $w_i := h(A_i)$, such that $w_i \in W$. From (3) and the pigeonhole principle we obtain i, j such that $1 \leq i < j \leq f(x_0) + 1 - e$ with $w_i = w_j$.

- (5) That G simulates G_2 yields that G_2 is constant on $[S]^{e+1}$. Suppose that this value equals 1; then $w_1 > w_2 > \dots > w_{f(x_0)+1-e}$, which contradicts (4). For similar reasons, the constant value of G_2 is not 2, such that G_2 is identical 0 on $[S]^{e+1}$.
- (6) Since $|S| \geq f(\min S) \geq 2e + 3$, we can talk about x_{2e+2} . We shall prove that $h(Y) = h(x_0, x_{e+1}, \dots, x_{2e-1})$ when $Y \in [\{x_0, \dots, x_e\}]^e$ and $\min Y = x_0$. For this we set $\{t_0, \dots, x_{2e-2}\} = Y \cup \{x_{e+1}, \dots, x_{2e-1}\}$. Since the restriction

of G_2 to $[S]^{e+1}$ is zero, we get:

$$\begin{aligned} h(Y) &= h(t_0, \dots, t_{e-1}) = h(t_0, t_2, \dots, t_e) = \dots = h(t_0, t_i, \dots, t_{i+e-2}) \\ &= \dots = h(t_0, t_e, \dots, t_{2e-2}) = h(x_0, x_{e+1}, \dots, x_{2e-1}). \end{aligned}$$

- (7) From (6) it follows that h is min-homogeneous if the first element is x_0 . But there exists just one $Y \in [\{x_0, \dots, x_e\}]^d$ with the property that $\min Y \neq x_0$; hence h is min-homogeneous on $[\{x_0, \dots, x_n\}]^d$, which implies $G_4(x_0, \dots, x_e) = 0$.

Case 2. $d = 0$ and $c > 1$.

We can write $h(x_0, \dots, x_{e-1})$ in $f(x_0)$ -ary notation $h(x_0, \dots, x_{e-1}) = h_1(x_0, \dots, x_{e-1}) \cdot f(x_0)^{a_1} + \dots + h_c(x_0, \dots, x_{e-1}) \cdot f(x_0)^{a_c}$, with $a_1 < \dots < a_c$ and $h_i(x_0, \dots, x_{e-1}) < f(x)$. For $1 \leq i \leq c$ let G'_i be a $(e+1, 18+6 \cdot f^{-1}(2e+3))$ -algebra such that if S is suitable for G'_i , h_i depends only on its first coordinate on $[S]^e$. Let G be a $(e+1, (18+6 \cdot f^{-1}(2d+3))^c)$ -algebra f -simulating $G'_1, \dots, G'_c$. If S is f -suitable for G , then h depends only on its first coordinate on $[S]^e$.

Case 3. $d = d' + 1$.

Let $Y = \{y_0, \dots, y_e\}$. If $h(y_0, \dots, y_{e-1}) = h(y_0, y_2, \dots, y_e)$, let $g(Y) = 0$. Otherwise $g(Y)$ is the largest j at which the binary expansion of the two values of h differ. This yields $g(Y) < 2^{d'}(f(\min Y)^c)$. Let $d'' := (18+6 \cdot f^{-1}(2e+3))^c \cdot (d'^2 + 5d' + 6)^{d'}$ and let G_1 be the $(e+d'+1, d'')$ -algebra from the induction hypothesis, applied to g , and let G_2 be a $(e+d'+1, (18+6 \cdot f^{-1}(2e+3))^c \cdot (d'^2 + 5d' + 6)^{d'} + f^{-1}(2e+3))$ -algebra simulating G_1 such that $f(\min S) \geq 2d+3$ for every S which is f -suitable for G_1 . Further we define $G_3 : [\mathbb{N}]^{e+1} \rightarrow 3$ and $G_4 : [\mathbb{N}]^{e+1} \rightarrow 2$ by

$$\begin{aligned} G_3(x_0, \dots, x_e) &= \begin{cases} 0 & h(x_0, \dots, x_{e-1}) = h(x_0, x_2, \dots, x_e), \\ 1 & h(x_0, \dots, x_{e-1}) > h(x_0, x_2, \dots, x_e), \\ 2 & h(x_0, \dots, x_{e-1}) < h(x_0, x_2, \dots, x_e); \end{cases} \\ G_4(X) &= \begin{cases} 0 & \text{if } h \text{ is min-homogeneous on } [X]^e, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

We have

$$\begin{aligned} &(3+d) \cdot (2+d) \cdot (f^{-1}(2d+3) + (18+6 \cdot f^{-1}(2e+3))^c \cdot (d'^2 + 5d' + 6)^{d'}) \\ &\leq (18+6 \cdot f^{-1}(2e+3))^c \cdot (d^2 + 5d + 6)^d. \end{aligned}$$

Let G be a $(e+d+1, (18+6 \cdot f^{-1}(2e+3))^c \cdot (d^2 + 5d + 6)^d)$ -dimensional algebra f -simulating $G_2, \dots, G_4$ using Lemma 3.1 and the product construction. Let S be suitable for G . We shall show that the restriction of G_3 to $[S]^{e+1}$ is equal to zero, which will prove the lemma since then the value of $h(X)$ depends only on $\min(X)$ (following (6) and (7) from the very first case). Aiming at a contradiction, assume that this restriction has value one and let $S = \{x_0, \dots, x_r\}$. Since G f -simulates G_1 , it follows $r \geq 2d+2$. For $1 \leq i \leq 3$, we define $z_i := h(x_0, x_i, \dots, x_{e-2+i})$. Our assumption yields $z_1 > z_2 > z_3$. Further we let $j_0 = g(x_0, \dots, x_d)$ and $j_1 = g(x_0, x_2, \dots, x_{d+1})$. Since G f -simulates G_2 , there exists a certain j such that $j_0 = j_1 = j$.

From the definition of g it follows that the j th digit is the largest binary digit for which z_1 and z_2 differ. The inequality $z_1 > z_2$ yields that the j th digit of z_2 is

zero, but the other inequality $z_2 > z_3$ yields that the j th digit of z_2 equals one, a contradiction. $\square$

Lemma 4.4. *Let $g : \mathbb{N}^e \rightarrow \mathbb{N}$ and assume $g(x_0, \dots, x_{e-1}) \leq f(x_0)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a weakly monotone increasing function. Then there exists a $(e+1, (18+7 \cdot f^{-1}(2e+3) \cdot 18))$ -algebra G such that if S is f -suitable for G , the following are true:*

- (1) *On $[S]^e$, g depends only on its first coordinate.*
- (2) *If $X, Y \in [S]^e$ and $\min X < \min Y$, then $g(X) \leq g(Y)$.*

Proof. First we define the $(e, 2)$ -algebra G_1 as follows:

$$G_1(X) = \begin{cases} 0 & g(X) < f(\min X), \\ 1 & \text{otherwise.} \end{cases}$$

Further we consider the $(e+1, 18+6 \cdot f^{-1}(2e+3))$ -algebra G_2 provided by Lemma 4.3 and the $(e, 2)$ -algebra G_3 defined by

$$G_3(X) = \begin{cases} 0 & g(X) < \lfloor \frac{f(\min X)}{2} \rfloor, \\ 1 & \text{otherwise.} \end{cases}$$

Next we define the $(e+1, 2)$ -algebra G_4 ,

$$G_4(x_0, \dots, x_e) = \begin{cases} 0 & g(x_0, \dots, x_{e-1}) \leq g(x_1, \dots, x_e), \\ 1 & \text{otherwise,} \end{cases}$$

and the $(e+1, 18+7 \cdot f^{-1}(2e+3))$ -algebra G_5 f simulating G_2 such that for every f -suitable S for G_2 we have $f(\min(S)) \geq 2e+3$.

Let G be the $(e+1, (18+7 \cdot f^{-1}(2e+3)) \cdot 3 \cdot 3 \cdot 2)$ -algebra simulating G_1, G_3, G_4 , and G_5 using Lemma 3.1 and the product construction. Let S be an f -suitable set for G . We shall now show that g satisfies the conditions of the lemma.

Since G f -simulates G_1 and S is f -suitable for G , we know that G_1 is constant on $[S]^e$. Assume this value is 1; then

$$f(\min(X)) \geq g(X) \geq f(\min(X)),$$

such that $g(X) = f(\min(X))$. The claim would follow since f is weakly increasing.

From now on we assume that G_1 equals zero on $[S]^e$. Since G f -simulates G_2 , the first item of the lemma follows trivially. Let $S = \{x_0, \dots, x_r\}$ and

$$z_i := g(x_i, \dots, x_{i+e-1}) \text{ for } 1 \leq i \leq r-e+1.$$

When G_4 has (constant) value 0 on $[S]^e$, then $z_0 \leq \dots \leq z_{r-e+1}$ from which the result follows. To derive a contradiction, assume that G_4 has constant value 1 which implies $f(x_0) > z_0 > \dots > z_{r-e+1}$. Let $W := \{i : i < \lfloor \frac{f(x_0)}{2} \rfloor\}$ when the restriction of G_3 to $S^{[n]}$ is 0 and $W := \{i : \lfloor \frac{f(x_0)}{2} \rfloor \leq i < f(x_0)\}$ otherwise. Then (by the choice of G_1 and G_3) $z_i \in W$ for $1 \leq i \leq r-e+1$. Since $i < j$ yields $z_i > z_j$ we see that $|W| \geq r-e+1$. Since S is f -suitable for G we obtain $r+1 = |S| \geq f(x_0) \geq 2e+3$. Hence $|W| \leq \lfloor \frac{f(x_0)}{2} \rfloor + 1 < f(x_0) - e \leq r+1-e$, a contradiction. $\square$

5. CONTROLLING AND CAPTURING FUNCTIONS WITH ALGEBRAS

In this section we apply the Ketonen-Solovay toolkit for our purposes.

Definition 5.1. Let $\alpha < \beta \leq \varepsilon_0$. We write $\beta \rightarrow_x \alpha$ if there exists a sequence of ordinals $\gamma_0, \dots, \gamma_r$ for which the following is true:

$$\gamma_0 = \beta, \gamma_{i+1} = \gamma_i[x] \text{ for } 0 \leq i < r, \gamma_r = \alpha.$$

Then as usual $\beta \rightarrow_x \alpha$ yields $F_\beta(x) \geq F_\alpha(x)$. Let

$$T_{d,c,x} = \{\alpha : \omega_d(c) \rightarrow_x \alpha\},$$

with

$$\omega_0(c) = c \text{ and } \omega_{k+1}(c) = \omega^{\omega_k(c)}.$$

Definition 5.2. Let $e > 0$, and let $g : [\mathbb{N}]^e \rightarrow \varepsilon_0$. We say that g is *weakly f -controlled* by the algebra G if for every f -suitable S for G , g depends only on its first coordinate on $[S]^e$.

When g is weakly f -controlled by G and S is f -suitable for G , then we define $g_S(x_0) = g(x_0, \dots, x_{e-1})$ for any choice of strictly increasing numbers $x_0, \dots, x_{e-1} \in S$.

We say that g is *f -controlled* by G if g is weakly f -controlled by G and whenever S is f -suitable for G and $x, y \in S$ with $x < y$, then $g_S(x) \leq g_S(y)$ (if $g_S(x)$ and $g_S(y)$ are defined).

Lemma 5.3. Let $d \geq 0$, $e \geq 1$, and $g : [\mathbb{N}]^e \rightarrow \varepsilon_0$, and suppose $g(x_0, \dots, x_{e-1}) \in T_{d,c,f(x_0)}$ for all strictly increasing sequences $x_0 < \dots < x_{e-1}$. Then g is weakly f -controlled by a $(e + d, (d^2 + 5d + 6)^d \cdot (18 + 7f^{-1}(2d + 3)^{6c})$ -algebra.

Proof. We may assume $d > 0$. By Lemma 4.3 we can choose an $(e + d, (d^2 + 5d + 6)^d \cdot (18 + 6f^{-1}(2e + 3)^{6c})$ -algebra (actually a smaller value for d suffices) that insures that $g(x_0, \dots, x_{e-1})$ depends only on x_0 provided $x_0 \geq 2$, since $|T_{d,c,x}| \leq 2_{d-1}(x^{6c})$ holds for $x \geq 2$ according to Proposition 2.10 in [5]. Since

$$\begin{aligned} & (d^2 + 5d + 6)^d \cdot (18 + 6f^{-1}(2e + 3)^{6c}) + f^{-1}(2) \\ & \leq (d^2 + 5d + 6)^d \cdot (18 + 7f^{-1}(2e + 3)^{6c}), \end{aligned}$$

the assertion follows from Lemma 3.3. $\square$

Proposition 5.4. Let $c > 0$ and $g : [\mathbb{N}]^e \rightarrow \omega^c$ and suppose $g(x_0, \dots, x_{e-1}) \in T_{1,c,f(x_0)}$ for all $x_0 < \dots < x_{e-1}$. Then g can be f -controlled by a $(e + 1, (18 + 7f^{-1}(2e + 3) \cdot 18)^c)$ -algebra.

Proof. Consider the Cantor normal form of g :

$$g(x_0, \dots, x_{e-1}) = \omega^{c-1} \cdot g_1(x_0, \dots, x_{e-1}) + \dots + \omega^0 \cdot g_c(x_0, \dots, x_{e-1}).$$

Since $g(x_0, \dots, x_{e-1}) \in T_{1,c,f(x_0)}$, we know for $g_i : [\mathbb{N}]^e \rightarrow \mathbb{N}$ that $g_i(x_0, \dots, x_{e-1}) \leq f(x_0)$. Lemma 4.4 yields the existence of an $(e + 1, (18 + 7f^{-1}(2e + 3)) \cdot 18)$ -algebra G_i f -controlling g_i . Let G be the $(e + 1, (18 + 7f^{-1}(2e + 3) \cdot 18)^c)$ -algebra f -simulating $G_1, \dots, G_c$. Then G f -controls g . $\square$

If λ has the Cantor normal form $\omega^{\alpha_1} \cdot x_1 + \dots + \omega^{\alpha_k} \cdot n_k$, we call $\{\alpha_1, \dots, \alpha_k\}$ the support of λ . If moreover λ' has the Cantor normal form $\omega^{\beta_1} \cdot m_1 + \dots + \omega^{\beta_l} \cdot m_l$ and if $\lambda > \lambda'$, then for some α_i in the support of λ we have that for all $j < i$ $\alpha_j = \beta_j$ and $m_j = n_j$ and $\alpha_i \geq \beta_i$ or β_i is undefined. Moreover, if $\alpha_i = \beta_i$, then $n_i > m_i$.

Let $u_{c,d,e} := ((18 + 7f^{-1}(2e + 3))^{6c+1} \cdot (d^2 + 5d + 6)^d \cdot 18 \cdot (d + 2))^d$.

Lemma 5.5. *Let $d, e \geq 1$. Suppose that $g : [\mathbb{N}]^e \rightarrow \varepsilon_0$ satisfies $g(x_0, \dots, x_{e-1}) \in T_{d,c,f(x_0)}$. Then g can be f -controlled by a $(e + d, u_{c,d,e})$ -algebra.*

Proof. We proceed by induction on d . If $d = 1$, then the result follows from Proposition 5.4. So assume $d \geq 2$. From Lemma 5.3 we obtain a $(e + d, (d^2 + 5d + 6)^d \cdot (18 + 7f^{-1}(2e + 3)^{6c}))$ -algebra G_0 weakly controlling g . Further we define the $(e + 1, 2)$ -algebra G_1 :

$$G_1(x_0, \dots, x_e) = \begin{cases} 0 & \text{if } g(x_0, \dots, x_{e-1}) \leq g(x_1, \dots, x_e), \\ 1 & \text{otherwise.} \end{cases}$$

We introduce an auxiliary function $h : [\mathbb{N}]^{e+1} \rightarrow \varepsilon_0$ as follows:

- When $g(x_0, \dots, x_{e-1}) \leq g(x_1, \dots, x_e)$, then we set $h(x_0, \dots, x_e) = 0$.
- When $g(x_0, \dots, x_{e-1}) > g(x_1, \dots, x_e)$, then we let $h(x_0, \dots, x_e)$ be equal to the largest element of the support of $g(x_0, \dots, x_{e-1})$ at which $g(x_0, \dots, x_{e-1})$ is different from $g(x_1, \dots, x_e)$.

For $r := d - 1$ we obtain $h(x_0, \dots, x_e) \in T_{r,c,f(x_0)}$. The induction hypothesis implies the existence of an $(e + r, u_{c,r,e})$ -algebra G_2 controlling h .

We define a second auxiliary function $h' : [\mathbb{N}]^{e+1} \rightarrow \omega$ as follows. When the coefficient of $\omega^{h(x_0, \dots, x_e)}$ in the Cantor normal form of $g(x_0, \dots, x_{e-1})$ exists, let $h'(x_0, \dots, x_e)$ be equal to this coefficient. If not, let $h'(x_0, \dots, x_e) = 0$. Since $g(x_0, \dots, x_{e-1}) \in T_{d,c,f(x_0)}$, we know $h'(x_0, \dots, x_e) \leq f(x_0)$. Lemma 4.4 yields an $(e + 2, (18 + 7 \cdot f^{-1}(2e + 3)) \cdot 18)$ -algebra G_3 controlling h' .

We know $e + 2 \leq e + d$, but further also $(d^2 + 5d + 6)^d \cdot (18 + 7f^{-1}(2e + 3)^{6c}) \cdot (d + 2) \cdot u_{c,r,e} \cdot (18 + 7 \cdot f^{-1}(2e + 3)) \cdot 18 \leq u_{c,d,e}$. So let G be the $(e + d, u_{c,d,e})$ -product algebra f -simulating $G_0, \dots, G_3$. We shall now show that G f -controls g . Let $S = \{x_0, \dots, x_m\}$ be suitable for G . Further, we set $S' = \{x_i : i \leq m - e + 1\}$ and define $g_S : S' \rightarrow \varepsilon_0$ by $g_S(x_i) = g(x_i, \dots, x_{i+e-1})$. Since G f -simulates G_0 , we know $g(x_0, \dots, x_{e-1}) = g_S(x_0)$, for all $x_0 < \dots < x_{e-1} \in S$.

Since G_1 is f -simulated by G , G_1 takes a constant value $v \leq 1$ on $[S]^{e+1}$. When v equals zero, the lemma is proven. Assume that $v = 1$. From our assumption we know that there exist x_0, x_1 (with $x_0 < x_1$) such that $g_S(x_0) > g_S(x_1)$. Further, there exists an ordinal $\eta_S(x_0) = h(x_0, \dots, x_e) \in \text{support}(g_S(x_0))$ with coefficient $m_S(x_0) = h'(x_0, \dots, x_e)$ such that the Cantor normal forms of $g_S(x_0)$ and $g_S(x_1)$ are equal when the exponents of these forms are bigger than $\eta_S(x_0)$, but differ when these exponents are smaller than or equal to $\eta_S(x_0)$.¹ Since G f -simulates G_2 and G_3 , which f -control h and h' respectively, the following is true: when $x, y \in S'$ with $x < y$, then $\eta_S(x) \leq \eta_S(y)$ and $m_S(x) \leq m_S(y)$. Since $m + 1 \geq e + 2$, we know $\{x_0, x_1, x_2\} \subseteq S'$. $g_S(x_0)$ and $g_S(x_1)$ are equal in the part where the exponents are greater than $\eta_S(x_0)$, so this is also true for $g_S(x_0)$ and $g_S(x_2)$. Consequently, $g_S(x_1)$ and $g_S(x_2)$ are equal in the part where the exponents are greater than $\eta_S(x_0)$, i.e. $\eta_S(x_1) \leq \eta_S(x_0)$. From the previous, we get $\eta_S(x_1) \geq \eta_S(x_0)$, such that $\eta_S(x_1) = \eta_S(x_0)$. But this yields $m_S(x_1) < m_S(x_0)$, a contradiction. $\square$

Lemma 5.6. *Let $d > 1$ and $c > 0$ be fixed. For a given $l \in \mathbb{N}$, let $f_l(i) := \max\{1, \lceil \frac{1}{l} \cdot \log_2^d(i) \rceil\}$. Then there exists a natural number $t_{l,d,c}$ (which depends elementarily recursively on l, d, c) and a $(d + 1, t_{l,d,c})$ -algebra f_l -capturing the function $B(f_l)_{\omega_{d-1}(c)}$.*

¹The fact that $\eta_S(x_0)$ does not depend on x_1 is a consequence of G f -simulating G_2 , which weakly f -controls h . Since G_3 is f -simulated by G , we know that $m_S(x_0)$ does not depend on x_1 .

Proof. We abbreviate f_l by f , and keep l fixed. Define the $(2, 2)$ -algebra G_0 as follows:

$$G_0(x, y) = \begin{cases} 0 & \text{when } y \geq B(f)_{\omega_{d-1}(c)}(x), \\ 1 & \text{otherwise.} \end{cases}$$

Further, let G_1 be a $(2, 2 + f^{-1}(2_d(3 \cdot l)))$ -algebra simulating G_0 and such that whenever S is f -suitable for G_1 , $\min(S) \geq 2_d(3 \cdot l)$ (according to Lemma 3.3). Next we define $h : [\mathbb{N}]^2 \rightarrow \varepsilon_0$ as follows. When there exists a $\xi \in T_{d-1, c, f(x)}$ such that $B_\xi(x) \geq y$, then let $h(x, y)$ be the smallest ξ with this property. Otherwise, we set $h(x, y) = 0$. From Lemma 5.5 we know that there exists a $(d + 1, u_{c, d, 2})$ -algebra G_2 f -controlling h . Let $c_l := l \cdot \Delta \cdot 2^{(d+1) \cdot 2^l}$. Lemma 3.2 implies the existence of a $(d + 1, c_l)$ -algebra G_3 that f_l -captures the map $x \mapsto 2_{d+1}(x)$. Finally we define the $(2, 3)$ -algebra G_4 as follows:

$$G_4(x, y) = \begin{cases} 2 & \text{when } h(x, y) \text{ is not a successor,} \\ 0 & \text{when } h(x, y) = \delta + 1 \text{ and } B(f)_\delta^{[f(x)]-1}(x) < y, \\ 1 & \text{when } h(x, y) = \delta + 1 \text{ and } B(f)_\delta^{[f(x)]-1}(x) \geq y. \end{cases}$$

Set $t_{l, d, c} := (2 + d + f^{-1}(2_{d+1}(3 \cdot l))) \cdot c_l \cdot (d + u_{c, d, 2}) \cdot (3 + d)$ and let G be the $(d + 1, t_{l, d, c})$ -algebra f -simulating $G_0, \dots, G_4$. We shall prove that G f -captures $B(f)_{\omega_d(c)}$. Let $S = \{x_0, \dots, x_m\}$ be f -suitable for G . Since S is f -suitable for G_0 , the algebra G_0 is constant on $[S]^2$ with value v , say. We shall now show that $v = 0$, for suppose otherwise that $v = 1$. Let $\{x, y\} \in [S]^2$. Since $G_0(x, y) = 1$, we know $y < B(f)_{\omega_{d-1}(c)}(x)$. We set $\xi_0 = \omega_{d-1}(c)[f(x)]$, so $y < B_{\xi_0}(x)$ and $\xi_0 \in T_{d-1, c, f(x)}$. Let $\xi_1 := h(x, y)$. Then ξ_1 is the smallest α for which $\alpha \in T_{d-1, c, f(x)}$ and $y \leq B(f)_\alpha(x)$ is true. The ordinal ξ_1 is not a limit ordinal. Assume otherwise. Put $\xi_2 := \xi_1[f(x)]$. Then $\omega_{d-1}(c) \rightarrow_{f(x)} \xi_1 \rightarrow_{f(x)} \xi_2$ and $\xi_2 < \xi_1 \in T_{d-1, c, f(x)}$ and $y \leq B(f)_{\xi_2}(x) = B(f)_{\xi_1}(x)$. But this contradicts the minimality of ξ_1 . We claim that $\xi_1 > 0$. Assume otherwise that $\xi_1 = 0$. Then G f -simulates G_3 yields

$$y \geq 2_{d+1}(x) > B(f)_0(x),$$

but this contradicts $y \leq B(f)_{\xi_1}(x) = B(f)_0(x)$.

So $\xi_1 = h(x, y)$ has the form $\delta + 1$. On $[S]^2$, the function h depends only on its first coordinate, say $h(x, y) = \delta(x) + 1$ for $x, y \in S$ with $x < y$. When $\{x, y\} \in [S]^2$ and $\delta(y)$ is defined, then $\delta(x) \leq \delta(y)$ since G_2 controls h . Since $h(x, y) \rightarrow_{f(x)} \delta(x)$, we know $B(f)_{\delta(x)}(x) < y$. For $0 \leq i < m - 1$ we shall now prove that $x_{i+1} > B(f)_{\delta(x_0)}(x_i)$. By the preceding paragraph $x_{i+1} > B(f)_{\delta(x_i)}(x_i)$ and $\delta(x_0) \leq \delta(x_i)$. When $\delta(x_0) = \delta(x_i)$, our claim is trivial. So suppose $\delta(x_0) < \delta(x_i)$. Since $\omega_{d-1}(c) \rightarrow_{f(x)} h(x, y) \rightarrow_{f(x)} \delta(x)$, we get for $x = x_0$: $\omega_{d-1}(c) \rightarrow_{f(x_0)} \delta(x_0)$, and this implies $\omega_{d-1}(c) \rightarrow_{f(x_i)} \delta(x_0)$ (see [5, p. 282]). Further $\omega_k(c) \rightarrow_{f(x_i)} \delta(x_i)$ is true, so $\delta(x_i) \rightarrow_{f(x_i)} \delta(x_0)$ (see [5, p. 281]) and then

$$B(f)_{\delta(x_0)}(x_i) \leq B(f)_{\delta(x_i)}(x_i) < x_{i+1}.$$

Using iteration we get

$$x_{i+1} > B(f)_{\delta(x_0)}(x_i) > B(f)_{\delta(x_0)}(B(f)_{\delta(x_0)}(x_{i-1})) > \dots > B(f)_{\delta(x_0)}^{i+1}(x_0),$$

which implies $x_m > B(f)_{\delta(x_0)}^m(x_0)$. The inequality $|S| \geq f(\min S)$ yields $m \geq \lceil f(x_0) \rceil - 1$, hence $x_m > B(f)_{\delta(x_0)}^{\lceil f(x_0) \rceil - 1}(x_0)$, and thus $G_4(x_0, x_m) = 0$. S is homogeneous for G_4 . Thus $G_4(x_0, x_1) = G_4(x_1, x_2) = 0$, and so

$$x_2 > B(f)_{\delta(x_1)}^{\lceil f(x_1) \rceil - 1}(x_1) \geq B(f)_{\delta(x_0)}^{\lceil f(x_0) \rceil - 1}(x_1).$$

$x_0 \geq 2_d(3 \cdot l)$ yields $f(x_0) \geq 3$, and since $G_4(x_0, x_1) = 0$, we obtain

$$\begin{aligned} B(f)_{\delta(x_0)}^{\lceil f(x_0) \rceil - 1}(x_1) &\geq B(f)_{\delta(x_0)}^{\lceil f(x_0) \rceil - 1}(B(f)_{\delta(x_0)}^{\lceil f(x_0) \rceil}(x_0)) \\ &\geq B(f)_{\delta(x_0)}^{2 \cdot \lceil f(x_0) \rceil - 2}(x_0) \geq B(f)_{\delta(x_0)}^{\lceil f(x_0) \rceil}(x_0) = B(f)_{\delta(x_0)+1}(x_0). \end{aligned}$$

This implies $x_2 > B(f)_{\delta(x_0)+1}(x_0)$, but this contradicts $\delta(x_0) + 1 = h(x_0, x_2)$. $\square$

6. RAMSEY NUMBERS: FINAL RESULT

Recall that for a given unbounded weakly increasing function f , we defined its functional inverse function by

$$f^{-1}(i) := \min\{j : f(j) \geq i\}.$$

Moreover, recall that we put $\omega_0(c) := c$ and $\omega_{k+1}(c) := \omega^{\omega_k(c)}$ and $\omega_k := \omega_k(1)$. For $\alpha < \varepsilon_0$, let us introduce the following functions f_α which play a crucial role for classifying the phase transition:

$$f_\alpha^d(i) := \max\left\{1, \frac{\log_2^d(i)}{F_\alpha^{-1}(i)}\right\}.$$

Theorem 6.1. *Let $d > 0$ and $\alpha < \varepsilon_0$ be fixed. For every $c, k \in \mathbb{N}$ the following is true:*

$$R_c^{d+1}(t) + F_\alpha(\Gamma \cdot c) \rightarrow_{f_\alpha^d}^* (k)_c^{d+1},$$

with $t := \max(k, F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)))$ with Γ from Lemma 2.1.

Proof. Let $c, k \in \mathbb{N}$ be arbitrary and set $N := R_c^{d+1}(t)$. Lemma 2.1 yields

$$\begin{aligned} N + F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)) &\leq 2_d(\Gamma \cdot c \cdot \log_2(c) \cdot t) + F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)) \\ &\leq 2_d(\Gamma \cdot c(\log_2(c) + 1) \cdot t). \end{aligned}$$

Let $P : [0, N + F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1))]^{d+1} \rightarrow c$. Since

$$N + F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)) - F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)) = N,$$

there exists an $H \subseteq [F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)), N + F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1))]$ which is homogeneous for P and such that $|H| \geq t$. This implies

$$\frac{\log_2^d(\min(H))}{F_\alpha^{-1}(\min(H))} \leq \frac{\log_2^d(2_d(\Gamma \cdot c \cdot (\log_2(c) + 1) \cdot t))}{F_\alpha^{-1}(F_\alpha(\Gamma \cdot c \cdot (\log_2(c) + 1)))} \leq t \leq |H|.$$

From this it follows that H is f_α^d -large and that $|H| \geq k$. $\square$

Recall that $t_{l,d,c} := (2 + d + f^{-1}(2_{d+1}(3 \cdot l))) \cdot c_l \cdot (d + u_{c,d,2}) \cdot (3 + d)$ using the notation from Lemma 5.6.

Theorem 6.2. *Let $d > 1$ be fixed. Let $t(m) := t_{m,d+1,m} + 2_d(2_{d+1}(m+2) \cdot m)$. We have*

$$R_{t(m)}^{d+1}(f_{\omega_d}^d)(d+2) \geq F_{\omega_d}(m)$$

for all $m \geq 1$.

Proof. Let $l > 0$ be fixed and let $c \geq 0$. Let $t(l, m) := t_{l,d+1,m} + 2_d(2_{d+1}(m+2) \cdot l)$ and $f_l(i) := \max\{1, \lceil \frac{1}{l} \cdot \log_2^d(i) \rceil\}$ for $i \geq 0$. We claim first that

$$R_{t(l,m)}^{d+1}(f_l)(d+2) \geq F_{\omega_d}(m).$$

Assume otherwise. Choose a $(d+1, t_{l,d+1,m})$ -algebra G which f_l -captures $B(f_l)_{\omega_{d-1}(m)}$. Choose a $(d+1, t(l, m))$ -algebra G^* which f_l -simulates G and such that every S f -suitable for G^* has $\min(S) \geq 2_{d+1}(m+2)$. By assumption there will exist a G^* homogeneous $S \subseteq [0, F_{\omega_d}(m)[$ such that $|S| \geq d+2$ and such that S is f_l -large. In this situation S is f_l -suitable for G^* ; hence $\min(S) \geq 2_{d+1}(m+2)$. S has at least two elements, say $x_0 < x_1$. Lemmas 5.6 and 4.1 yield $x_1 \geq B(f_l)_{\omega_{d-1}(m)}(x_0) \geq B(f_l)_{\omega_{d-1}(m)}(2_{d+1}(m+2)) \geq F_{\omega_d}(m)$. This is not possible, and thus the claim above is shown. Now we show for $t(m) := t(m, m)$ that

$$R_{t(m)}^{d+1}(f_{\omega_d})(d+2) \geq F_{\omega_d}(m).$$

Suppose for a contradiction that there exists an $m \geq 1$ for which

$$R_{t(m)}^{d+1}(f_{\omega_d})(d+2) < F_{\omega_d}(m)$$

is true. For all $j \leq R_{t(m)}^{d+1}(f_{\omega_d})(d+2)$, we have $F_{\omega_d}^{-1}(j) \leq F_{\omega_d}^{-1}(F_{\omega_d}(m)) = m$; hence $\frac{1}{F_{\omega_d}^{-1}(j)} \geq \frac{1}{m}$. It is easy to see that in general $f(i) \leq g(i)$ for all $i \leq R_c^{d+1}(g)(m)$ yields $R_c^{d+1}(f)(m) \leq R_c^{d+1}(g)(m)$. So we obtain

$$\begin{aligned} R_{t(m)}^{d+1}(f_{\omega_d}^d)(d+2) \\ \geq R_{t(m)}^{d+1}(f_m)(d+2) \geq F_{\omega_d}(m), \end{aligned}$$

a contradiction. $\square$

Using classical results in proof theory (see, for example, [4] or [10] for a proof), we finally conclude that

$$m \mapsto R_m^{d+1}(f_{\alpha}^d)(d+1)$$

(hence $m \mapsto R_m^{d+1}(f_{\alpha}^d)(m)$) is provable from the axioms of PA^- plus d -quantifier induction iff $\alpha < \omega_d$.

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