ARITHMETIC NORMAL FUNCTIONS AND FILTRATIONS ON CHOW GROUPS

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ABSTRACT. Let $X/\mathbb{C}$ be a smooth projective variety, and let $\mathrm{CH}^r(X, m)$ be the higher Chow group defined by Bloch. Saito and Asakura defined a descending candidate Bloch-Beilinson filtration $\mathrm{CH}^r(X, m; \mathbb{Q}) = F^0 \supset \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots$, using the language of mixed Hodge modules. Another more geometrically defined filtration is constructed by Kerr and Lewis in terms of germs of normal functions. We show that under the assumptions (i) $X/\mathbb{C} = X_0 \times \mathbb{C}$ where X_0 is defined over $\overline{\mathbb{Q}}$, and (ii) the general Hodge conjecture, that $F^\bullet \mathrm{CH}^r(X, m; \mathbb{Q})$ coincides with the aforementioned geometric filtration. More specifically, it is characterized in terms of germs of reduced arithmetic normal functions.

1. INTRODUCTION

The notion of a (still conjectural) Bloch-Beilinson (B-B) filtration on Chow groups is an idea that originated with S. Bloch and was later fortified by Beilinson in terms of extension datum in a conjectural category of mixed motives. Over the years a number of candidate B-B filtrations have surfaced, due to S. Saito [SS], U. Jannsen, W. Raskind, J. Lewis [Lew1], M. Kerr and J. Lewis [K-L], P. Griffiths and M. Green, J. Murre, M. Saito and M. Asakura [As], to name a few. The filtration in [As] is also discussed in [K-L], where it also gives rise to an explicit geometrically defined filtration in terms of arithmetic normal functions. The natural question then is whether these two filtrations coincide. Two obvious consequences of an affirmative answer are that:

- The filtration in [As] can be described geometrically.
- The geometric filtration inherits the known properties of the one in [As], which makes use of the powerful cohomological machinery of mixed Hodge modules.

Our main result is the following:

Theorem 1.1. *Let $X/\mathbb{C}$ be a smooth projective variety. Assume that (i) $X = X_0 \times_{\overline{\mathbb{Q}}} \mathbb{C}$, where X_0 is defined over $\overline{\mathbb{Q}}$, and (ii) the general Hodge conjecture holds. Then the filtration in [As] admits a geometric description in terms of germs of (reduced) arithmetic normal functions.*

For the convenience of the reader, we review the definition of the filtration in question and along the way prove more general results.

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2. NOTATION AND CONVENTIONS

(i) Throughout this paper we will assume that $k = \bar{k} \subset \mathbb{C}$ is an algebraically closed subfield and that K/k (with $K \subset \mathbb{C}$) is a finitely generated field extension. Let $\mathcal{V}_k$ be the category of smooth projective varieties over k .

(ii) $\mathbb{Q}(r) = (2\pi i)^r \mathbb{Q}$ (Tate twist).

(iii) For a $\mathbb{Q}$ -mixed Hodge structure (MHS) $H = H_{\mathbb{Q}}$, we put

$$\Gamma(H) := \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H), \quad J(H) := \text{Ext}_{\text{MHS}}^1(\mathbb{Q}(0), H).$$

This is in the category of MHS which are not necessarily graded polarizable. Thus by [Ca], [Ja, Lemma 9.2],

$$J(H) = \frac{W_0 H_{\mathbb{C}}}{W_0 H + F^0 W_0 H_{\mathbb{C}}}.$$

(Note: With regard to the category of graded polarizable MHS, the corresponding $\text{Ext}^1(\mathbb{Q}(0), H)$ is given by $W_{-1} H_{\mathbb{C}} / \{ \{ W_0 H + F^0 W_0 H_{\mathbb{C}} \} \cap W_{-1} H_{\mathbb{C}} \} \hookrightarrow J(H)$, [As, §2.5]. A formal consequence of the right exactness of $\text{Ext}_{\text{MHS}}^1(\mathbb{Q}(0), -)$ is that the higher extension groups vanish, viz., $\text{Ext}_{\text{MHS}}^{\nu}(\mathbb{Q}(0), H) = 0$ for all $\nu \geq 2$, a fact established by Beilinson [Be, Cor. 1.10].)

(iv) Let $L \subset \mathbb{C}$ be any subfield and $X \in \mathcal{V}_L$. For singular cohomology we define $H^i(X, \mathbb{Q}) := H^i(X(\mathbb{C}), \mathbb{Q})$. The coniveau filtration is given by

$$N_L^{\nu} H^i(X, \mathbb{Q}) := \ker \left(H^i(X, \mathbb{Q}) \rightarrow \varinjlim_{Y \subset X/L, \text{cd}_X Y \geq \nu} H^i(X \setminus Y, \mathbb{Q}) \right).$$

(v) For $X \in \mathcal{V}_{\mathbb{C}}$, the general Hodge conjecture (abbreviated GHC) says that $N_{\mathbb{C}}^{\nu} H^i(X, \mathbb{Q}) = N_H^{\nu} H^i(X, \mathbb{Q})$, where $N_H^{\nu} H^i(X, \mathbb{Q}) :=$ largest sub-Hodge structure in $F^{\nu} H^i(X, \mathbb{C}) \cap H^i(X, \mathbb{Q})$.

(vi) $\text{CH}^r(X_K, m)$ is the higher Chow group introduced in [B], and $\text{CH}^r(X_K, m; \mathbb{Q}) := \text{CH}^r(X_K, m) \otimes \mathbb{Q}$.

Owing to the technical nature of this paper, the reader may find the following sources of background information useful. For a discussion of the general Hodge conjecture, see [Lew2]. Section 11 in [Ja] describes Beilinson's conjecture on a filtration on Chow groups for $X \in \mathcal{V}_K$. A good source of mixed Hodge theory is the survey article [Br-Z]. For extension classes of MHS, see [Ca] and [Ja, §9]. For mixed Hodge modules, the reader should consult [As] and the references cited therein.

3. A DESCENDING FILTRATION

Consider fields $k \subset K \subset \mathbb{C}$, where K/k is finitely generated. We consider the filtration constructed in [Lew1] in the case $m = 0$, and more generally for $m \geq 0$ in [As]. We recall:

Theorem 3.1. *Let $X \in \mathcal{V}_K$ be of dimension d . Then for all r , there is a filtration, depending on $k \subset K$,*

$$\begin{aligned} \text{CH}^r(X_K, m; \mathbb{Q}) = & F^0 \supset F^1 \supset \dots \supset F^{\nu} \supset F^{\nu+1} \\ & \supset \dots \supset F^r \supset F^{r+1} = F^{r+2} = \dots, \end{aligned}$$

which satisfies the following:

(i) $F^1 = \text{CH}_{\text{hom}}^r(X_K, m; \mathbb{Q})$.

(ii) $F^2 \subset \ker AJ \otimes \mathbb{Q} : \text{CH}_{\text{hom}}^r(X_K, m; \mathbb{Q}) \rightarrow J(H^{2r-m-1}(X_K(\mathbb{C}), \mathbb{Q}(r)))$.

(iii)¹ $F^{\nu_1} CH^{r_1}(X, m_1; \mathbb{Q}) \bullet F^{\nu_2} CH^{r_2}(X, m_2; \mathbb{Q}) \subset F^{\nu_1+\nu_2} CH^{r_1+r_2}(X, m_1+m_2; \mathbb{Q})$, where $\bullet$ is the intersection product.

(iv) F^ν is preserved under the action of correspondences between smooth projective varieties in $\mathcal{V}_K$.

(v) Let $Gr_F^\nu := F^\nu / F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \bigoplus_{p+q=2d} [\Delta_X(p, q)] \in H^{2d}(X \times X, \mathbb{Q}(d))$ are algebraic and defined over K . Then

$$\Delta_X(2d - 2r + \ell + m, 2r - \ell - m)_* \big|_{Gr_F^\nu CH^r(X, m; \mathbb{Q})} = \delta_{\ell, \nu} \cdot \text{Identity}.$$

(If we assume the conjecture that homological and numerical equivalence coincide, then (v) says that Gr_F^ν factors through the Grothendieck motive.)

(vi) Let $D^r(X) := \bigcap_\nu F^\nu$ and $k = \mathbb{Q}$. If the Bloch-Beilinson conjecture on the injectivity of the Abel-Jacobi map $(\otimes \mathbb{Q})$ holds for smooth quasi-projective varieties defined over $\mathbb{Q}$ (see §4 of [K-L]), then $D^r(X) = 0$.

It is instructive to briefly explain how this filtration comes about. For $X \in \mathcal{V}_K$, one can find $\mathcal{S}/k$ such that $k(\mathcal{S})$ is identified with K . One can then spread out X/K to a family $\rho : \mathcal{X} \rightarrow \mathcal{S}$ (called a k -spread), where ρ is a smooth and proper morphism of smooth quasi-projective varieties over k . Let η be the generic point of $\mathcal{S}$, (hence $k(\mathcal{S}) = k(\eta)$ and K corresponds to an embedding $k(\eta) \xrightarrow{\sim} K \subset \mathbb{C}$), with X_K identified with the generic fiber of ρ . Via M. Saito's notion of polarizable mixed Hodge modules (see [As], as well as §4 of [K-L]), there is a decreasing filtration $\mathcal{F}^\nu CH^r(\mathcal{X}/k, m; \mathbb{Q})$, with the property that $Gr_{\mathcal{F}}^\nu CH^r(\mathcal{X}/k, m; \mathbb{Q}) \hookrightarrow E_\infty^{\nu, 2r-\nu-m}(\rho)$, where $E_\infty^{\nu, 2r-\nu-m}(\rho)$ is the ν -th graded piece of a Leray filtration associated to ρ . More specifically, there is a cycle class map

$$CH^r(\mathcal{X}, m; \mathbb{Q}) \rightarrow \text{Ext}_{\text{MHM}(\mathcal{X})}^{2r-m}(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r)),$$

where $\text{MHM}(\mathcal{X})$ is the category of polarizable mixed Hodge modules on $\mathcal{X}$. There is the Leray spectral sequence associated to ρ :

$$E_2^{p,q} = \text{Ext}_{\text{MHM}(\mathcal{S})}^p(\mathbb{Q}_{\mathcal{S}}(0), R^q \rho_* \mathbb{Q}_{\mathcal{X}}(r)) \Rightarrow \text{Ext}_{\text{MHM}(\mathcal{X})}^{2r-m}(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r)),$$

$$p + q = 2r - m,$$

which degenerates at E_2 since ρ is smooth and proper. Then we put

$$E_\infty^{\nu, 2r-\nu-m}(\rho) := E_2^{\nu, 2r-\nu-m} = \text{Ext}_{\text{MHM}(\mathcal{S})}^\nu(\mathbb{Q}_{\mathcal{S}}(0), R^{2r-\nu-m} \rho_* \mathbb{Q}_{\mathcal{X}}(r)).$$

Applying the Leray spectral sequence to the structure map $\mathcal{S} \rightarrow \text{Spec}(k)$, together with $\text{MHM}(\text{Spec}(k)) = \text{MHS}$, we see that the term $E_\infty^{\nu, 2r-\nu-m}(\rho)$ fits in a short exact sequence:

$$0 \rightarrow \underline{E}_\infty^{\nu, 2r-\nu-m}(\rho) \rightarrow E_\infty^{\nu, 2r-\nu-m}(\rho) \rightarrow \underline{\underline{E}}_\infty^{\nu, 2r-\nu-m}(\rho) \rightarrow 0,$$

where

$$(1) \quad \begin{aligned} \underline{\underline{E}}_\infty^{\nu, 2r-\nu-m}(\rho) &= \Gamma(H^\nu(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m} \rho_* \mathbb{Q}(r))), \\ \underline{E}_\infty^{\nu, 2r-\nu-m}(\rho) &= \frac{J(W_{-1} H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m} \rho_* \mathbb{Q}(r)))}{\Gamma(Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m} \rho_* \mathbb{Q}(r)))} \\ &\subset J(H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m} \rho_* \mathbb{Q}(r))), \end{aligned}$$

¹Not specifically stated in [As], although expected to be true for the higher Chow groups.

where the latter inclusion is a result of the short exact sequence

$$\begin{aligned} W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)) &\hookrightarrow W_0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)) \\ &\twoheadrightarrow Gr_W^0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)), \end{aligned}$$

and where the image

$$(2) \quad \Gamma(Gr_W^0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))) \rightarrow J(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)))$$

is described as follows. Let $y \in \Gamma(Gr_W^0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)))$, and choose

$$x \in W^0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)), \quad x_{\mathbb{C}} \in F^0W^0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{C}),$$

which map to y under the surjection $W_0 \rightarrow Gr_W^0$. Then the image of y in (2) is given by the image of $x - x_{\mathbb{C}}$ in $J(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)))$. Under the identification of K with $k(\eta)$, one then has (by definition)

$$F^{\nu}CH^r(X_K, m; \mathbb{Q}) = \lim_{U \subset \mathcal{S}/k} \mathcal{F}^{\nu}CH^r(\mathcal{X}_U/\overline{\mathbb{Q}}, m; \mathbb{Q}), \quad \mathcal{X}_U := \rho^{-1}(U).$$

We set

$$E_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}) = \lim_{U \subset \mathcal{S}/k} E_{\infty}^{\nu, 2r-\nu-m}(\rho)$$

and the same definition for $\underline{E}_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}})$ and $\underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}})$. Specifically,

$$\begin{aligned} \underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}) &= \Gamma(H^{\nu}(\eta_{\mathcal{S}}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))), \\ \underline{E}_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}) &= J(W_{-1}H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r)))/\Gamma(Gr_W^0). \end{aligned}$$

We have a short exact sequence,

$$0 \rightarrow \underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}) \rightarrow E_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}) \rightarrow \underline{E}_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}) \rightarrow 0,$$

and an injection,

$$Gr_F^{\nu}CH^r(X_K, m; \mathbb{Q}) \hookrightarrow E_{\infty}^{\nu, 2r-\nu-m}(\eta_{\mathcal{S}}).$$

(ii) We then define

$$F^{\nu}CH^r(X_{\mathbb{C}}, m; \mathbb{Q}) = \lim_{k \subset K \subset \mathbb{C}} F^{\nu}CH^r(X_K, m; \mathbb{Q}),$$

which becomes a candidate B-B filtration on $CH^r(X_{\mathbb{C}}, m; \mathbb{Q})$ in the case $k = \overline{\mathbb{Q}}$.

4. ARITHMETIC NORMAL FUNCTIONS

Let $X \in \mathcal{V}_K$ with K/k finitely generated. We recall for the reader [K-L, §4] a geometric interpretation of the aforementioned descending filtration in terms of arithmetic normal functions, and prove some definitive results in the case where X is obtained by base change from a corresponding variety in $\mathcal{V}_k$. So we consider $X \in \mathcal{V}_K$ with $k = \overline{k} \subset K$ and the paragraph following Theorem 3.1 above. Consider the coniveau subspace $N_K^{r-\nu+1}H^{2r-\nu-m}(X, \mathbb{Q}(r)) \subset H^{2r-\nu-m}(X, \mathbb{Q}(r))$. After possibly shrinking $\mathcal{S}$, $N_K^{r-\nu+1}H^{2r-\nu-m}(X, \mathbb{Q}(r))$ determines a corresponding sub-VHS $N_K^{r-\nu+1}R^{2r-\nu-m}\rho_*\mathbb{Q}(r) \subset R^{2r-\nu}\rho_*\mathbb{Q}(r)$, which by semisimplicity considerations is a direct summand. Now let $Y \subset X/K$ be of (pure) codimension $r - \nu + 1$ such that

$$H_Y^{2r-\nu-m}(X, \mathbb{Q}(r)) \twoheadrightarrow N_K^{r-\nu+1}H^{2r-\nu-m}(X, \mathbb{Q}(r))$$

is surjective, with desingularization $\tilde{Y} \rightarrow Y$ and composite map $\sigma : \tilde{Y} \rightarrow X$. Let us assume there is a K -cycle induced map

$$\tilde{P} : H^{2r-\nu-m}(X, \mathbb{Q}) \rightarrow H^{\nu-m-2}(\tilde{Y}, \mathbb{Q}),$$

such that

$$P := \sigma_* \circ \tilde{P} : H^{2r-\nu-m}(X, \mathbb{Q}(r)) \twoheadrightarrow N_K^{r-\nu+1} H^{2r-\nu-m}(X, \mathbb{Q}) \subset H^{2r-\nu-m}(X, \mathbb{Q}(r))$$

is a projector.

Proposition 4.1. *Given the above setting*

- (i) $P_* Gr_F^\nu CH^r(X_K, m; \mathbb{Q}) = 0$.
- (ii)

$$\begin{aligned} \text{Im}(Gr_F^\nu CH^r(X_K, m; \mathbb{Q}) \rightarrow \underline{E}_\infty^{\nu, 2r-\nu-m}(\eta_S)) \\ \bigcap \Gamma(H^\nu(\eta_S, N_K^{r-\nu+1} R^{2r-\nu-m} \rho_* \mathbb{Q}(r))) = 0. \end{aligned}$$

- (iii)

$$\begin{aligned} \left\{ Gr_F^\nu CH^r(X_K, m; \mathbb{Q}) \right. \\ \left. \bigcap \text{Im}(J(W_{-1} H^{\nu-1}(\eta_S, N_K^{r-\nu+1} R^{2r-\nu-m} \rho_* \mathbb{Q}(r))) \rightarrow \underline{E}_\infty^{\nu, 2r-\nu-m}(\eta_S)) \right\} = 0. \end{aligned}$$

Proof. (i) For $[\xi] \in Gr_F^\nu CH^r(X_K, m; \mathbb{Q})$, use the fact that $P_*([\xi]) = [P_*(\xi)] = [\sigma_* \circ \tilde{P}_*(\xi)] \in Gr_F^\nu CH^r(X_K, m; \mathbb{Q})$ factors through $[\tilde{P}_*(\xi)] \in Gr_F^\nu CH^{\nu-1}(\tilde{Y}_K, m; \mathbb{Q}) = 0$. (To help the reader out here, $\tilde{P}_*$ is induced by a cycle in $CH^{d-r+\nu-1}(X_K \times \tilde{Y}_K; \mathbb{Q})$, and hence $\tilde{P}_* CH^r(X_K, m; \mathbb{Q}) \subset CH^{\nu-1}(\tilde{Y}_K, m; \mathbb{Q})$. Now use the fact that $\tilde{P}_*$ is compatible with Gr_F^ν and that $Gr_F^\nu CH^{\nu-1}(\tilde{Y}_K, m; \mathbb{Q}) = 0$, both statements due to Theorem 3.1.) For (ii) and (iii), observe that the respective intersections are preserved under P_* . Now apply (i). $\square$

Corollary 4.2. *Let us assume the Hodge conjecture (hence with cycle induced $\tilde{P}$). Then $P_* Gr_F^\nu CH^r(X_{\mathbb{C}}, m; \mathbb{Q}) = 0$.*

Remark 4.3. (i) If $X \in \mathcal{V}_k$ where we recall $k = \bar{k}$, then we can arrange for $\tilde{P}$ to be induced by a cycle over k by using a spread argument similar to that in the proof of Lemma 7.5 on page 1456 in [L-S].

(ii) Corollary 4.2 is consistent with the conjectured equivalence of the two filtrations introduced and explained in [SS, Thm. (1-1)].

We now assume that $\mathcal{S}$ is affine. Let $V \subset \mathcal{S}(\mathbb{C})$ be a smooth closed affine subvariety of dimension $\nu - 1$ with diagram of varieties, which we now view as defined over $\mathbb{C}$:

$$\begin{array}{ccc} \mathcal{X}_V & \hookrightarrow & \mathcal{X} \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & \mathcal{S} \end{array}$$

Correspondingly we have

$$\begin{array}{ccccccc}
 0 & \rightarrow & \underline{E}_{\infty}^{\nu, 2r-\nu-m}(\rho) & \rightarrow & E_{\infty}^{\nu, 2r-\nu-m}(\rho) & \rightarrow & \underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\rho) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \underline{E}_{\infty}^{\nu, 2r-\nu-m}(\rho_V) & \rightarrow & E_{\infty}^{\nu, 2r-\nu-m}(\rho_V) & \rightarrow & \underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\rho_V) \rightarrow 0
 \end{array}$$

By the weak Lefschetz theorem for affine varieties (see, for example, [A, Lemma 2.5], $\underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\rho_V) = 0$. Therefore a class $\xi \in \mathcal{F}^{\nu} \mathrm{CH}^r(\mathcal{X}/k, m; \mathbb{Q})$ determines a class $\lambda_{\xi}(V) \in \underline{E}_{\infty}^{\nu, 2r-\nu-m}(\rho_V)$. Thinking of V as part of a family $V = V_0 \in \{V_t\}_{t \in W}$, we arrive at the notion of an arithmetic normal function $\lambda_{\xi}(V_t)$. To be more precise,

Definition 4.4 ([K-L, §4]). For $\xi \in \mathcal{F}^{\nu} \mathrm{CH}^r(\mathcal{X}/k, m; \mathbb{Q})$, the corresponding λ_{ξ} is called an arithmetic normal function.

The class of $[\xi] \in \underline{\underline{E}}_{\infty}^{\nu, 2r-\nu-m}(\rho)$ is determined by λ_{ξ} (topological invariant of λ_{ξ} ; see [K-L, §4]). A reasonable expectation is that these arithmetic normal functions can be used to completely characterize $F^{\nu} \mathrm{CH}^r(X_K, m; \mathbb{Q})$. In the case where $\mathcal{X}$ is a product (viz., where $X \in \mathcal{V}_k$), we obtain such an explicit description in Theorem 4.6. To this end, consider the product situation with $\mathcal{X} = S \times X$, all defined over k , with $K = k(S)$. Set

$$H_0 = W_{-1} \left(H^{\nu-1}(\eta_S, \mathbb{Q}) \otimes \frac{H^{2r-\nu-m}(X, \mathbb{Q})}{N_H^{r-\nu+1} H^{2r-\nu-m}(X, \mathbb{Q})}(r) \right),$$

where η_S is the generic point of S/k . A key ingredient due to mixed Hodge theory is the following:

Lemma 4.5. *There is a natural map*

$$\underline{E}_{\infty}^{\nu, 2r-\nu-m}(\eta_S) \rightarrow J(H_0).$$

Proof. We assume S/k is affine and put

$$\begin{aligned}
 W_j &= W_j(H^{\nu-1}(S, \mathbb{Q}) \otimes H^{2r-\nu-m}(X, \mathbb{Q})(r)), \\
 W_j^H &= W_j(H^{\nu-1}(S, \mathbb{Q}) \otimes N_H^{r-\nu+1} H^{2r-\nu-m}(X, \mathbb{Q})(r)).
 \end{aligned}$$

The proof of the lemma boils down to showing that

$$(3) \quad \Gamma(Gr_W^0) = \Gamma(Gr_{W^H}^0),$$

together with the description of $\underline{E}_{\infty}^{\nu, 2r-\nu-m}(\rho)$ in (1). We point out that a similar argument appears in [L-S, page 1485], but for completeness, we provide the details here. First of all, it is easier to “untwist” things by observing that $\Gamma(Gr_W^0)$ can be identified with $\mathrm{hom}_{\mathrm{MHS}}(\mathbb{Q}(-r), V \otimes H^{2r-\nu-m}(X, \mathbb{Q}))$, where $V = \mathrm{Gr}_W^{\nu+m} H^{\nu-1}(S, \mathbb{Q})$. But observe that the maximum weight of $H^{\nu-1}(S, \mathbb{Q})$ is $2\nu - 2$ (see [Ja, page 85]), and that $2\nu - 2 \geq \nu + m \Leftrightarrow \nu \geq m + 2$. So we may assume that $\nu \geq m + 2$. Further, an analysis of the MHS $H^{\nu-1}(S, \mathbb{Q})$ shows that $F^{\nu-1} V_{\mathbb{C}} = V_{\mathbb{C}}$. In particular this implies that $\mathrm{hom}_{\mathrm{MHS}}(\mathbb{Q}(-r), V \otimes H^{2r-\nu-m}(X, \mathbb{Q})) \subset V \otimes N_H^{r-\nu+1} H^{2r-\nu-m}(X, \mathbb{Q})$;

hence (3) follows. Finally, the lemma follows from the commutative diagram

$$\begin{array}{ccc} \Gamma(Gr_{WH}^0) & \rightarrow & J(W_{-1}^H) \\ \parallel & & \downarrow \\ \Gamma(Gr_W^0) & \rightarrow & J(W_{-1}) \end{array} \quad \square$$

Now consider $\xi \in F^\nu CH^r(X_K, m; \mathbb{Q})$. Such a class ξ is represented by a corresponding class (still denoted by) $\xi \in CH^r(S \times_k X, m; \mathbb{Q})$ for some affine S . Let λ_ξ be the corresponding arithmetic normal function. More precisely, by λ_ξ , we mean the following. For smooth affine closed $V \subset S(\mathbb{C})$, we restrict ourselves to $\eta_V := "V \cap \eta_S"$ (with the obvious meaning), and where

$$\lambda_\xi(V) \in \frac{J(W_{-1}[H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes H^{2r-\nu-m}(X, \mathbb{Q})(r)])}{\Gamma(Gr_W^0[H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes H^{2r-\nu-m}(X, \mathbb{Q})(r)])}.$$

The reduced arithmetic normal function $\underline{\lambda}_\xi$ has the characteristic property that its value lies in

$$\underline{\lambda}_\xi(V) \in J\left(W_{-1}\left[H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes \frac{H^{2r-\nu-m}(X, \mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu-m}(X, \mathbb{Q})}(r)\right]\right),$$

which via Lemma 4.5 is the image of $\lambda_\xi(V)$. We prove:

Theorem 4.6. *In the case where $X \in \mathcal{V}_k$, and under the assumption of the GHC, the filtration $\{F^\nu CH^r(X_K, m; \mathbb{Q})\}_{\nu \geq 0}$ is characterized by the germs of reduced arithmetic normal functions.*

Proof. By Proposition 4.1 and under the assumption of the GHC, $\underline{\lambda}_\xi = 0 \Leftrightarrow \lambda_\xi = 0$. (This uses the fact that $N_k^{r-\nu+1}H^{2r-\nu-m}(X, \mathbb{Q}) = N_H^{r-\nu+1}H^{2r-\nu-m}(X, \mathbb{Q})$ (since $k = \bar{k}$) and that $\underline{\lambda}_\xi = 0 \Rightarrow \lambda_\xi = \lambda_{P_*\xi} = 0$). But recall $\lambda_\xi = 0 \Rightarrow [\xi] = 0 \in \underline{E}_\infty^{\nu, 2r-\nu-m}(\eta_S)$. Thus $[\xi] \in \underline{E}_\infty^{\nu, 2r-\nu-m}(\eta_S)$, and its value is zero in $\underline{E}_\infty^{\nu, 2r-\nu-m}(\eta_S)$ (equivalently, $\xi = 0 \in Gr_F^\nu CH^r(X_K, m; \mathbb{Q})$) if and only if its value in $J(H_0)$ of Lemma 4.5 is zero, again using Proposition 4.1. Using the affine Lefschetz theorem (injectivity part of Lemma 2.5 in [A], together with the fact that the MHS H_0 only involves negative weights), $\underline{\lambda}_\xi = 0 \Rightarrow [\xi] \mapsto 0 \in J(H_0)$. (To be more specific, if $V_1 \hookrightarrow V_2$ is an injection of MHS, then $J(W_{-1}V_1) \hookrightarrow J(W_{-1}V_2)$ is likewise an injection, the latter injection following from the short exact sequence $0 \rightarrow W_{-1}V_1 \rightarrow W_{-1}V_2 \rightarrow W_{-1}(V_2/V_1) \rightarrow 0$, together with $\Gamma(W_{-1}(V_2/V_1)) = 0$.) In summary, the image of $\xi \in F^\nu CH^r(X_K; \mathbb{Q})$ in $\underline{E}_\infty^{\nu, 2r-\nu-m}(\eta_S)$ is zero $\Leftrightarrow \underline{\lambda}_\xi = 0$; i.e., $\Leftrightarrow \xi = 0 \in Gr_F^\nu CH^r(X_K, m; \mathbb{Q})$, which is the same thing as saying $\Leftrightarrow \xi \in F^{\nu+1} CH^r(X_K, m; \mathbb{Q})$. $\square$

Theorem 1.1 now follows immediately from this.

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