CONTINUITY OF WEIGHTED ESTIMATES IN A_p NORM

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(Communicated by Mario Bonk)

ABSTRACT. We prove that for a general Calderón-Zygmund operator T the numbers $\|T\|_{L^p(w) \rightarrow L^p(w)}$ converge to $\|T\|_{L^p(dx) \rightarrow L^p(dx)}$ as the A_p norm of w converges to 1, i.e. as $[w]_{A_p} \rightarrow 1^+$ for $1 < p < \infty$.

1. INTRODUCTION, NOTATION AND USEFUL RESULTS

The purpose of this note is to present another nice property of the A_p weights. In particular, the numbers, $\|T\|_{L^p(w) \rightarrow L^p(w)}$, converge to the norm of the operator in the usual unweighted $L^p(dx)$ space, as the A_p norm of the weight goes to 1. This means that if the A_p norm of w is close to 1, w is almost the constant weight 1. Furthermore, we can find the rate of convergence and prove that is *sharp*.

We will denote by Q a cube in $\mathcal{R}^n$ and by c an absolute constant that may depend on the dimension. Also w will denote a function from $\mathcal{R}^n \rightarrow (0, +\infty)$.

For $1 < p \leq \infty$, we can define the class A_p as follows:

1) If the norm

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w dx \right) \left(\frac{1}{|Q|} \int_Q w^{\frac{-1}{p-1}} dx \right)^{p-1}$$

is finite, we say that w belongs to the class of A_p weights.

2) If the norm

$$[w]_{A_\infty} = \sup_Q \left(\frac{\frac{1}{|Q|} \int_Q w dx}{\exp\left(\frac{1}{|Q|} \int_Q \log w dx\right)} \right)$$

is finite, we say that w belongs to the class of A_∞ weights. Note that every A_p weight is actually an A_∞ weight, with norm $[w]_{A_\infty} \leq [w]_{A_p}$. To see this, simply take the limit as $p \rightarrow \infty$ in the expression of the A_p norm. We formulate now the main result of this paper.

Theorem 1. *Suppose that T is a linear operator on $\mathcal{R}^n$ such that for any A_p weight w , $1 < p < \infty$, $\|T\|_{L^p(w) \rightarrow L^p(w)} \leq cF([w]_{A_p})$, where F is an increasing positive function. Then*

$$\lim_{[w]_{A_p} \rightarrow 1} \|T\|_{L^p(w) \rightarrow L^p(w)} = \|T\|_{L^p(dx) \rightarrow L^p(dx)}.$$

Actually by the proof of Theorem 1, it follows that

$$\|T\|_{L^p(w) \rightarrow L^p(w)} \leq \|T\|_{L^p(dx) \rightarrow L^p(dx)} (1 + c\sqrt{\delta})$$

Received by the editors December 1, 2010 and, in revised form, March 7, 2011.

2010 *Mathematics Subject Classification.* Primary 30E20, 47B37, 47B40, 30D55.

Key words and phrases. Calderón-Zygmund operators, A_2 weights, interpolation.

for all A_p weights, w , with characteristic $[w]_{A_p} \leq 1 + \delta < 2$, and c is a constant that depends on the dimension and p . We will start the proof in section 2 and finish it in section 3. In section 4, we will discuss sharpness, and we will show that the Hilbert transform behaves exactly like $1 + c\sqrt{\delta}$, for $p = 2$. As a consequence, the martingale transform is of the same order, and the Riesz projection P_+ is of the order $1 + c\delta$. In this sense, the Riesz projection goes faster to its L^2 norm than the Hilbert transform. Moreover, using the ideas of [2], one can show that $1 + C\sqrt{\delta}$ is the estimate from below for the second order Riesz transforms R_k^2 . Then automatically, it is the right estimate from below for the first order Riesz transforms as well. The detailed proof of that has certain important points of difference in comparison with the present paper, and will be published separately. In addition, Theorem 1 proves that $\|T\|_{L^p(w) \rightarrow L^p(w)}$ is Hölder continuous at $w = \text{constant}$, with exponent $\frac{1}{2}$. We need some more results before the proof of the main theorem.

The space of BMO functions in $\mathcal{R}^n$ consists of locally integrable functions f such that the norm

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| dx$$

is finite. The BMO space and the A_∞ space have many nice properties. First of all, if f is a BMO function, then for any number $\lambda \in (0, \frac{c}{\|f\|_*}]$ the function $e^{\lambda f}$ is an A_p weight, $1 < p < \infty$, where the constant c depends on p and the dimension n . Secondly, for small BMO norm, the A_p norm of the weight $e^{\lambda f}$ is bounded by the number 2 for example (see, e.g., [1]).

In the following, $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ will denote measure spaces. Suppose T is an operator of a class of functions on X into a class of functions on Y . T is called a sublinear operator if it satisfies the following properties:

- i) If $f = f_1 + f_2$ and Tf_1, Tf_2 are defined, then Tf is defined.
- ii) $|T(f_1 + f_2)| \leq |Tf_1| + |Tf_2|$, μ almost everywhere.
- iii) For any scalar k , we have $|T(kf)| = |k||Tf|$, μ almost everywhere.

Let $p, q \geq 1$ be two real numbers. We say that T is of type (p, q) if T is defined for all functions f in $L^p(X, \mathcal{M}, \mu)$ and there exists a positive number, K , independent of f such that

$$\|Tf\|_{q, \nu} \leq K \|f\|_{p, \mu},$$

where

$$\|Tf\|_{q, \nu} = \left(\int_Y |Tf|^q d\nu \right)^{\frac{1}{q}}$$

and

$$\|f\|_{p, \mu} = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

Let μ_0, μ_1 be two measures for $(X, \mathcal{M})$. If we define the measure $\mu = \mu_0 + \mu_1$, then μ_0, μ_1 are each absolutely continuous with respect to μ . Thus, by the Radon-Nikodym theorem, there exists two functions, α_0, α_1 , such that for any $E \in \mathcal{M}$,

$$\mu_j(E) = \int_E \alpha_j d\mu,$$

where $j = 0, 1$. In the following we will assume that α_0, α_1 are never zero. This is equivalent to asserting that the sets of measure zero with respect to μ_j , $j = 0, 1$, are the same as the sets of measure zero with respect to μ . Thus, in the various measure spaces that we will consider, the equivalence classes of functions will be the same.

Let $0 \leq s \leq 1$, and define the measure μ_s on X by

$$\mu_s(E) = \int_E \alpha_0^{1-s} \alpha_1^s d\mu$$

for each $E \in \mathcal{M}$. Also assume that we have two measures ν_0, ν_1 on $\mathcal{N}$, and define the measures ν_r , for $0 \leq r \leq 1$, just as we did for μ_s above.

Given any real numbers $1 \leq p_0, p_1, q_0, q_1$ and any $0 \leq t \leq 1$, we define $p_t, q_t, s(t), r(t)$ as follows:

$$\begin{aligned} \frac{(1-t)p_t}{p_0} + \frac{tp_t}{p_1} &= 1, \quad \frac{(1-t)q_t}{q_0} + \frac{tq_t}{q_1} \\ &= s(t) = \frac{(tp_t)}{p_1}, \quad r(t) = \frac{(tq_t)}{q_1}. \end{aligned}$$

We have the following theorem by [6].

Theorem 2. Suppose that T is a sublinear operator satisfying

$$\|Tf\|_{q_j, \nu_j} \leq K_j \|f\|_{p_j, \mu_j}$$

for all $f \in L^{p_j}(X, \mathcal{M}, \mu_j)$, $j = 0, 1$. Then, for $0 \leq t \leq 1$, we have

$$\|Tf\|_{q_t, \nu_{r(t)}} \leq K_0^{1-t} K_1^t \|f\|_{p_t, \mu_{s(t)}}$$

for all $f \in L^{p_t}(X, \mathcal{M}, \mu_{s(t)})$.

The following result was proved in [5].

Theorem 3. If the A_∞ norm of a weight w is small, i.e. $[w]_{A_\infty} \leq 1 + \delta < 2$, then the function $f = \log w$ and any cube Q satisfy

$$\frac{1}{|Q|} \int_Q |f - f_Q| dx \leq 32\sqrt{\delta}.$$

We will give a rough idea for A_2 , since for A_∞ it is similar. We will show that for any A_2 weight w ,

$$\|\log w\|_* \leq 2\sqrt{[w]_{A_2} - 1}.$$

Indeed, for any real number x we have that $2 + x^2 \leq e^x + e^{-x}$. Now apply it with $x = \log(\frac{w}{w_Q})$ and get

$$\frac{1}{|Q|} \int_Q \left(2 + \left(\log \frac{w}{w_Q}\right)^2\right) \leq \frac{1}{|Q|} \int_Q \frac{w}{w_Q} + \frac{1}{|Q|} \int_Q \frac{w_Q}{w}.$$

Hence,

$$\frac{1}{|Q|} \int_Q |\log w - \log(w_Q)|^2 \leq 1 + w_Q(w^{-1})_Q - 2 \leq [w]_{A_2} - 1.$$

By the Hölder inequality, we have $f: \frac{1}{|Q|} \int_Q f \leq \sqrt{\frac{1}{|Q|} \int_Q f^2}$, for any positive function f . Thus,

$$\frac{1}{|Q|} \int_Q |\log w - \log(w_Q)| \leq \sqrt{[w]_{A_2} - 1}.$$

Now using the well known inequality

$$\frac{1}{|Q|} \int_Q |\log w - (\log w)_Q| \leq 2 \inf_{r \in \mathbb{R}} \frac{1}{|Q|} \int_Q |\log w - r|,$$

we get exactly what we want. For a general A_∞ weight the proof follows the same lines. See [5] for more details. We will now use the above results to prove our main theorem.

2. MAIN THEOREM

We will denote by $\mathcal{L}$ the σ -algebra of Lebesgue measurable subsets of $\mathcal{R}^n$.

Theorem 4. *Suppose that T is a linear operator on $\mathcal{R}^n$, such that for any A_p weight w , $1 < p < \infty$,*

$$\|T\|_{L^p(w) \rightarrow L^p(w)} \leq cF([w]_{A_p}),$$

where F is an increasing positive function. Then

$$\lim_{[w]_{A_p} \rightarrow 1} \|T\|_{L^p(w) \rightarrow L^p(w)} = \|T\|_{L^p(dx) \rightarrow L^p(dx)}.$$

Proof. Let $1 > \delta > 0$ be given. Fix an A_p weight w , with $[w]_{A_p} \leq 1 + \delta$, $1 < p < \infty$. Obviously $[w]_{A_\infty} \leq 1 + \delta$. By Theorem 3 we get that the function $f = \log w$ is in BMO with very small norm. Since the number $\frac{c}{\|f\|_*}$ is very large, we can choose our $\lambda \in (0, \frac{c}{\|f\|_*}]$ also large. To be precise, we choose $\lambda = \frac{c}{32\sqrt{\delta}}$ and get that the function $e^{\lambda f} = w^{\frac{c}{32\sqrt{\delta}}}$ is an A_p weight of norm bounded by the number 2, i.e. $[w^{\frac{c}{32\sqrt{\delta}}}]_{A_p} \leq 2$. Now apply Theorem 2 for $X = Y = \mathcal{R}^n$, $\mathcal{M} = \mathcal{N} = \mathcal{L}$ and $\mu_0 = \nu_0 = dx$, $\mu_1 = \nu_1 = w^{\frac{c}{32\sqrt{\delta}}} dx$. Observe that $w = 1^{1-t}(w^{\frac{c}{32\sqrt{\delta}}})^t$ for $t = \frac{1}{\frac{c}{32\sqrt{\delta}}} \in (0, 1)$. Denote $\|T\|_{L^p(dx) \rightarrow L^p(dx)}$ by γ . By Theorem 2 we get

$$\begin{aligned} \|T\|_{L^p(w) \rightarrow L^p(w)} &\leq \gamma^{1-t} \|T\|_{L^p\left(w^{\frac{c}{32\sqrt{\delta}}}\right) \rightarrow L^p\left(w^{\frac{c}{32\sqrt{\delta}}}\right)}^t \\ &\leq \gamma^{1-t} c^t F\left([w^{\frac{c}{32\sqrt{\delta}}}]_{A_p}\right)^t \\ &\leq \gamma^{1-t} c^t F(2)^t, \end{aligned}$$

and the right hand side goes to γ as $t \rightarrow 0^+$ or equivalently as $\delta \rightarrow 0^+$. In other words,

$$\limsup_{[w]_{A_p} \rightarrow 1} \|T\|_{L^p(w) \rightarrow L^p(w)} \leq \|T\|_{L^p(dx) \rightarrow L^p(dx)}.$$

This is not the end of the proof of Theorem 4; we continue the proof in the next section. $\square$

Remark 5. It follows that

$$\limsup_{[w]_{A_p} \rightarrow 1} \|T\|_{L^p(w) \rightarrow L^p(w)} \leq \|T\|_{L^p(dx) \rightarrow L^p(dx)}$$

whenever T is a *sublinear* operator that satisfies the assumptions of Theorem 4.

Remark 6. Consider any Calderón-Zygmund operator T . By [3] we know that

$$\|T\|_{L^p(w) \rightarrow L^p(w)} \leq c[w]_{A_p}^{\max(1, \frac{1}{p-1})},$$

for any A_p weight w . This means that we can apply Theorem 4 for $1 < p < \infty$ and $F(x) = x^{\max(1, \frac{1}{p-1})}$.

3. CONTINUITY

In this section we will show

$$\|T\|_{L^p(dx) \rightarrow L^p(dx)} \leq \liminf_{[w]_{A_p} \rightarrow 1} \|T\|_{L^p(w) \rightarrow L^p(w)},$$

which finishes the proof of the main theorem. Let us assume that $p = 2$ and that $\|T\|_{L^p(dx) \rightarrow L^p(dx)} = 1$, for convenience. Note that other p 's can be treated in the same way. So far we have proven that

$$(1) \quad \limsup_{[w]_{A_2} \rightarrow 1} \|T\|_{L^2(w) \rightarrow L^2(w)} \leq 1$$

and that

$$(2) \quad [w]_{A_2} \leq 1 + \delta \Rightarrow \|T\|_{L^2(w) \rightarrow L^2(w)} \leq 1 + C\sqrt{\delta}.$$

Let M_ϕ denote the operator of multiplication by ϕ . To finish the proof of the continuity at $w = 1$, one needs to show that $\liminf_{[w]_{A_2} \rightarrow 1} \|T\|_{L^2(w) \rightarrow L^2(w)} \geq 1$. Notice that for $w \in A_2$ we have $\|T\|_{L^2(w) \rightarrow L^2(w)} = \|M_{w^{1/2}} T M_{w^{-1/2}}\|_{L^2 \rightarrow L^2}$. So we need to prove that $\|M_{w^{1/2}} T M_{w^{-1/2}}\|_{L^2 \rightarrow L^2} \leq 1 - \tau, \tau > 0$, cannot happen for a weight w whose A_2 norm is sufficiently close to 1. Suppose this is true, and we get a contradiction. There exists a sequence of weights $w_n, [w_n]_{A_2} \leq 1 + \varepsilon_n, \varepsilon_n \rightarrow 0$ such that for any bounded function with compact support f one has

$$(3) \quad \|w_n^{1/2} T(w_n^{-1/2} f)\|_2 \leq (1 - \tau) \|f\|_2.$$

Without loss of generality we fix

$$(4) \quad \langle w_n \rangle_Q = 1,$$

where Q is a cube containing $\text{supp } f$. We want to show the following:

- 1) $\|w_n^{-1/2} - 1\|_{L^2(Q)} \rightarrow 0$, and
 - 2) there exists a subsequence $\{n_k\}$ such that $w_{n_k} \rightarrow 1$ almost everywhere in Q .
- Clearly 2) is implied by 1). Suppose 1) is proved. Then noticing that 1) implies

$$\|T(w_n^{-1/2} f) - T f\|_{L^2(Q)} \rightarrow 0$$

and passing to a subsequence, we conclude that

$$w_{n_j}^{1/2} T(w_{n_j}^{-1/2} f) \rightarrow T f$$

almost everywhere in the cube Q .

Now we apply Fatou's lemma in inequality (3),

$$\|\liminf_j w_{n_j}^{1/2} T(w_{n_j}^{-1/2} f)\|_{L^2(Q)} \leq \liminf_j \|w_{n_j}^{1/2} T(w_{n_j}^{-1/2} f)\|_{L^2(Q)} \leq (1 - \tau) \|f\|_2,$$

and by virtue of $w_{n_j}^{1/2} T(w_{n_j}^{-1/2} f) \rightarrow T f$ we obtain $\|T f\|_{L^2(Q)} \leq (1 - \tau) \|f\|_2$ for $\tau > 0$. Now we let Q expand to $\mathcal{R}^n$. This is true for any f bounded with compact support, and so $\|T\|_{L^2 \rightarrow L^2} \leq 1 - \tau$. This gives a contradiction to our assumption that $\|T\|_{L^2 \rightarrow L^2} = 1$.

All that remains is showing 1), which follows from the following lemma.

Lemma 7. *Let $w \in A_2$ such that $[w]_{A_2} \leq 1 + \varepsilon$. Let us have a normalization assumption $\langle w \rangle_Q = 1$. Then $\|w^{-1/2} - 1\|_{L^2(Q)} \leq |Q|^{1/2} \varepsilon^{1/2}$.*

Proof. We start with

$$\frac{1}{|Q|} \|w^{-1/2} - 1\|_{L^2(Q)}^2 = \langle w^{-1} \rangle_Q + 1 - 2 \frac{1}{|Q|} \int_Q w^{-1/2} dx \leq 2 + \epsilon - 2 \langle w^{-1/2} \rangle_Q.$$

By the Cauchy–Schwarz inequality, $\langle w^{-1/2} \rangle_Q \geq (\langle w^{1/2} \rangle_Q)^{-1} \geq \langle w \rangle_Q^{-1/2} = 1$. Note that last inequality is by our normalization assumption. Plugging this into the previous inequality, we prove the lemma. $\square$

4. SHARPNESS

In the following we are going to consider the Hilbert transform, H , the Riesz projection P_+ , and weights in A_2 on the circle. We are going to show that Theorem 4 is *sharp* for the Hilbert transform and that *it is not sharp* for the Riesz projection. This result is interesting because these two operators, H and P_+ , are very closely related, i.e.

$$H = -iP_+ + i(I - P_+).$$

But as we shall see they do not behave in the same way.

We start with a weight $w \in A_2$ such that $[w]_{A_2} = 1 + \delta$, where $\delta > 0$ is really close to 0. We know that there exists an outer function h such that $w = |h|^2$. Outer means that $h = e^{u+i\tilde{u}}$, where $\tilde{u}$ denotes the harmonic conjugate of the function u . As we have already mentioned, $\log w = 2u$ is in BMO with norm $\|\log w\|_* \leq c\sqrt{\delta}$. This means that the conjugate function of u has small BMO norm, i.e. $\|\tilde{u}\|_* \leq c\sqrt{\delta}$. From [5] the square root of δ is *sharp*. So we can choose our function u such that $c_1\sqrt{\delta} \leq \|u\|_*$. Observe also that $\frac{\bar{h}}{h} = e^{-2i\tilde{u}}$. Let us now look at the operator $f \mapsto e^{if}$ that maps the space BMO continuously into itself (this is clear since if the oscillation of f is bounded, then the same should be true for the function e^{if}). Of course, this is not a linear operator but it has some nice properties. For example, for $\epsilon > 0$ small, it maps the ball $B(0, \epsilon) = \{f \in BMO: \|f\|_* \leq \epsilon\}$ into another ball of center 0 and radius say $c\epsilon$. We claim that the ball $B(0, \epsilon)$ is mapped *homeomorphically onto* its image and that the image contains a ball $B(0, c\epsilon)$ for some c . For this it suffices to see that the derivative of this map at the point 0 is exactly the linear map $f \mapsto if$ which is a continuous surjection from BMO onto itself. Then make use of the inverse function theorem for Banach spaces. We did all this in order to be able to claim that we can choose our function h satisfying

$$c_2\sqrt{\delta} \leq \left\| \frac{\bar{h}}{h} \right\|_* \leq c_1\sqrt{\delta}.$$

Let $f_{\pm}$ denote the analytic and anti-analytic parts of a bounded function f on the circle. Now the space BMO can be written as the direct sum of the $BMOA$ and $\overline{BMOA}$ spaces, the BMO analytic and the BMO anti-analytic spaces, respectively. Without loss of generality we can assume that

$$c_1\sqrt{\delta} \geq \left\| \left(\frac{\bar{h}}{h} \right)_- \right\|_{\overline{BMOA}} \geq \frac{c_2\sqrt{\delta}}{2}.$$

But

$$\begin{aligned} \left\| \left(\frac{\bar{h}}{h} \right)_- \right\|_{\overline{BMOA}} &= \text{dist} \left(\frac{\bar{h}}{h}, H^\infty \right) \\ &= \sup_{\|\phi\|_1 \leq 1, \phi \in H_0^1} \left| \int \frac{\bar{h}}{h} \phi \right| = \sup_{\|\phi_1\|_2, \|\phi_2\|_2 \leq 1, \phi_2(0)=0} \left| \int \frac{\bar{h}}{h} \phi_1 \phi_2 \right|, \end{aligned}$$

where $\|\cdot\|_2$ is the norm in the Hardy space H^2 . This last supremum is exactly equal to

$$\sup_{\|\phi_1\|_2, \|\phi_2\|_2 \leq 1, \phi_2(0)=0} (H_{\frac{\bar{h}}{h}} \phi_1, \overline{\phi_2}) = \|H_{\frac{\bar{h}}{h}}\|,$$

where $H_{\frac{\bar{h}}{h}} : H^2 \rightarrow H_-^2$ is the Hankel operator of symbol $\frac{\bar{h}}{h}$. Now consider the spaces

$$H_+ = \text{clos}_{L^2(w)} \{1, z, z^2, \dots\}, H_- = \text{clos}_{L^2(w)} \{\bar{z}, \bar{z}^2, \dots\}.$$

These spaces are called the future and the past spaces (the terminology comes from the probability, where w plays the role of the spectral density of a stationary stochastic process; see, e.g., [7] and the literature cited therein).

The next step is to find the angle θ of these two spaces in $L^2(w)$. This is exactly

$$\sup_{\|\phi_-\|_{L^2(w)}=1, \|\phi_+\|_{L^2(w)}=1} \left| (\phi_+, \phi_-)_{L^2(w)} \right|.$$

If we write down just one of these inner products, we see the following:

$$\left| \int \phi_+ \overline{\phi_-} |h|^2 \right| = \left| \int (\phi_+ h) (\overline{\phi_-} h) \frac{\bar{h}}{h} \right|.$$

The first two functions that appear in the integrand are analytic since they are products of analytic functions. Note that since the function ϕ_- is anti-analytic, the function $\overline{\phi_-}$ is analytic. Also their H^2 norm is ≤ 1 . This means that the supremum is exactly equal to

$$\frac{c_2 \sqrt{\delta}}{2} \leq \|H_{\frac{\bar{h}}{h}}\| = \sup_{\|\phi_-\|_{L^2(w)}=1, \|\phi_+\|_{L^2(w)}=1} \left| (\phi_+, \phi_-)_{L^2(w)} \right| \leq c_1 \sqrt{\delta}.$$

Therefore, the $\cos \theta$ is exactly of the order $\sqrt{\delta}$. This means that $\sin \theta - 1$ is of the order δ . Now, all that remains is an easy problem. We are given that the cosine of the angle of two directions is of the order $\sqrt{\delta}$, and we would like to find the order of $\sup \frac{\|u+v\|}{\|u-v\|}$ over all vectors u that have the first direction and v that have the second direction. Using the theorem of cosines, we can see that the order of this supremum must be $1 + c\sqrt{\delta}$. Thus

$$\|H\|_{L^2(w) \rightarrow L^2(w)} \geq \sup \frac{\|u+v\|}{\|u-v\|} \asymp 1 + c\sqrt{\delta}$$

and

$$\|P_+\|_{L^2(w) \rightarrow L^2(w)} = \frac{1}{\sin \theta} \asymp 1 + c\delta.$$

Remark 8. It follows that the $L^2(w)$ norm of the operator P_+ is exactly of the order $1 + c\delta$, for some weights $w \in A_2$, such that $\|w\|_{A_2} - 1 \asymp \delta$ if and only if Korey's estimate of the square root of δ in Theorem 3 is sharp for A_2 weights of A_2 norm $\leq 1 + \delta$.

Remark 9. In [7], a complete description of completely regular stationary processes was given. The setting is different, since [7] describes all matrix weights W such that the spaces $X_0 = \text{span}\{z^k \mathcal{C} : k < 0\}$, $X^n = \text{span}\{z^k \mathcal{C} : k \geq n\}$ are asymptotically (as $n \rightarrow \infty$) orthogonal to each other,

$$\sup \left\{ |(\xi, \eta)_{L^2(W)}| : \xi \in X_0, \eta \in X^n, \|\xi\|_{L^2(W)} \leq 1, \|\eta\|_{L^2(W)} \leq 1 \right\} \rightarrow 0$$

as $n \rightarrow \infty$. But we can fix n to be zero, W to be a scalar weight, and see that as the A_2 norm of the weight goes to 1, the norm of the operator P_+ goes to 1. However, the order of this convergence is worse than δ , which is the sharp order of $\|P_+\|_{L^2(w) \rightarrow L^2(w)} - 1$. If one works through the estimates of [7], one gets $\|P_+\|_{L^2(w) \rightarrow L^2(w)} \leq 1 + c\delta^{\frac{1}{2}}$. In other words, the estimate for the cosine of the angle between the past and the future is $\leq C\delta^{\frac{1}{4}}$. For the Hilbert transform this would give $\|H\|_{L^2(w) \rightarrow L^2(w)} \leq 1 + C\delta^{\frac{1}{4}}$. But we saw that actually $\|H\|_{L^2(w) \rightarrow L^2(w)} \leq 1 + C\delta^{\frac{1}{2}}$, and for certain weights, it cannot go faster to 1 than this square root.

ACKNOWLEDGMENT

The first author is grateful to Professor Joe Conlon from the University of Michigan in Ann Arbor for bringing this problem to his attention.

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