THE MINMAX PRINCIPLE AND $W^{2,p}$ REGULARITY FOR SOLUTIONS OF THE SIMPLEST ISAACS EQUATIONS

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ABSTRACT. In this paper, we consider the simplest uniformly elliptic Isaacs equations and prove that when the control matrix is appropriately separable, C^2 solutions satisfy an interior $W^{2,p}$ estimate for all $0 < p < \infty$.

0. INTRODUCTION

In this paper, we consider viscosity solutions of the simplest uniformly elliptic Isaacs equation of the form

$$(0.1) \quad F(u_{xx}) := \max_{y \in Y} \min_{z \in Z} \operatorname{tr}[a(y, z)u_{xx}] = 0 \quad \text{in a domain } D \subset \mathbb{R}^d,$$

where Y, Z are compact sets in Euclidean space, and $\forall y \in Y, z \in Z$, the control matrix $a(y, z)$ is diagonal, satisfies $\lambda I_d \leq a(y, z) \leq \Lambda I_d$ for some fixed positive constants $0 < \lambda \leq \Lambda$, and is Lipschitz continuous in both y and z . Here, $u_{xx} = (u_{x^i x^j})$ denotes the Hessian matrix of the function u , and tr denotes the trace. Other than denoting matrices in lower case, the notation we use is as in [CC1]. In particular, we recall that an operator F is *uniformly elliptic* if there exist constants $0 < \lambda \leq \Lambda$ such that $\forall m \in \mathcal{S} = \mathcal{S}_d$ (the space of real symmetric $d \times d$ matrices)

$$\lambda \|n\| \leq F(m + n) - F(m) \leq \Lambda \|n\| \quad \forall 0 \leq n \in \mathcal{S},$$

where for $m \in \mathcal{S}$, $\|m\| := \sup_{e \in \mathbb{R}^d, |e|=1} |me|$. From the elementary inequalities

$$\inf_{y,z} (f_{y,z} - g_{y,z}) \leq \sup_y \inf_z f_{y,z} - \sup_y \inf_z g_{y,z} \leq \sup_{y,z} (f_{y,z} - g_{y,z}),$$

and the fact that the matrix $a(y, z)$ in (0.1) satisfies $\lambda I_d \leq a(y, z) \leq \Lambda I_d$, it follows that Isaacs equation (0.1) is uniformly elliptic with ellipticity constants $\lambda, \Lambda d$.

We restrict our attention to the case where the operator $F = F(m)$ in (0.1) satisfies the *minmax* principle: for $m \in \mathcal{S}$,

$$(0.2) \quad \max_{y \in Y} \min_{z \in Z} \operatorname{tr}[a(y, z)m] = F(m) = \min_{z \in Z} \max_{y \in Y} \operatorname{tr}[a(y, z)m].$$

A necessary and sufficient condition for this to occur is that there exists a saddle point for F . That is, if we define $p(y, z, m) = \operatorname{tr}[a(y, z)m]$, then $\exists (\bar{y}, \bar{z}) = (\bar{y}(m), \bar{z}(m)) \in Y \times Z$ such that $\forall y \in Y, z \in Z, m \in \mathcal{S}$,

$$(0.3) \quad p(y, \bar{z}, m) \leq p(\bar{y}, \bar{z}, m) \leq p(\bar{y}, z, m).$$

Saddle points are known to exist in a number of cases. A saddle point will occur, for example, if $p(y, z)$ is convex in y , concave in z and Y and Z are convex (see

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Corollary 37.3.2 in [R]). It will also occur when the controls y, z in $a(y, z)$ can be *separated*, i.e. $a(y, z) = a^1(y) + a^2(z)$. If the matrices $a^1(y), a^2(z)$ are nondegenerate, the Isaacs operator can be written as a sum of two uniformly elliptic Bellman operators, one convex and the other concave.

It is well known (see [I], [CC1]) that if F is uniformly elliptic, and D is a smooth domain, then for arbitrary $g \in C(\bar{D})$, there exists a unique viscosity solution $u \in C(\bar{D})$ of the Dirichlet problem

$$(0.4) \quad \begin{cases} F(u_{xx}) = 0 & \text{in } D, \\ u = g & \text{on } \partial D. \end{cases}$$

For general bounded domains $D \subset \mathbb{R}^d$, it is well known that if $u \in C(D)$ is a viscosity solution of $F(u_{xx}) = 0$ in D , then the second derivatives of u exist almost everywhere in D (see [T]), and belong to $L_{loc}^\delta(D)$ for some small $\delta = \delta(d, \lambda, \Lambda) > 0$ (see [L] and (1.3) below). In particular, viscosity solutions of Isaacs equation (0.1) belong to $W_{loc}^{2,\delta}(D)$. The question is: under what conditions do they belong to $W_{loc}^{2,p}(D)$ for $1 < p < \infty$? Our interest in $W^{2,p}$ regularity (with p finite) is justified in light of the recent result of Nadirashvili and Vlăduț in [NV2], wherein the authors construct an example in $d = 24$ of a uniformly elliptic equation $F(u_{xx}) = 0$ and a viscosity solution u whose second derivative blows up at an interior point, i.e. $u \notin W_{loc}^{2,\infty}$. Although the solution constructed in [NV2] was not $W_{loc}^{2,p}$ for p sufficiently large, the pde itself was a Hessian equation, which Isaacs equations, in general, are not (see Example 1 below). The significance of their example was that it proved the optimality of the $C^{1+\alpha}$ regularity theory for general uniformly elliptic equations of the form $F(u_{xx}) = 0$. Individual special cases of such equations, of course, may fare better. Isaacs equations are of interest, not only because of their lack of concavity, but because any uniformly elliptic operator $F(m)$ can be shown to be of Isaacs *type* (though not necessarily satisfying the conditions following (0.1); see [CC2]). In particular, although the singular solution in the recent paper [NV3] was shown to satisfy a uniformly elliptic Isaacs equation in $B_1 \subset \mathbb{R}^{12}$, it was not an Isaacs equation satisfying the conditions of (0.1).

In this paper, we examine C^2 solutions of Isaacs equations which satisfy the min-max principle and are separable in a nondegenerate way (see §2). Since the matrix $a(y, z)$ is diagonal, the operator F depends only on the *pure* second derivatives of the solution; i.e., the pde is of the form $F(u_{x^1 x^1}, u_{x^2 x^2}, \dots, u_{x^d x^d}) = 0$. In §1, we prove the interior $W^{2,p}$ regularity ($0 < p < \infty$) of C^2 solutions of a particular Isaacs equation of the form (0.1). The method of proof illustrates the technique used in the more general setting (see Theorem 2 in §2). Specifically, the proof uses the subsolution estimate in conjunction with the $W_{loc}^{2,\delta}$ estimate, as was done in [CC2]. It is unclear exactly what role the saddle point structure of the Isaacs operator plays in the regularity of solutions (compare examples 1 and 2 below). However, in the special case where the matrix $a(y, z)$ is separable in y, z , and the control set for either y or z is a closed interval in $\mathbb{R}$, we prove that C^2 solutions of (0.1) satisfy an interior $W^{2,p}$ estimate for all $0 < p < \infty$ (see §2). The difficulty in applying our method of proof to non- C^2 viscosity solutions is discussed at the end of §1 (see Remark 3). We close this introduction with some examples of Isaacs equations of the form (0.1).

Example 1. The simplest nontrivial Isaacs equation is undoubtedly

$$(0.5) \quad F(u_{xx}) := \Delta u + (u_{x^1 x^1})_+ - (u_{x^2 x^2})_- = 0,$$

where for $a \in \mathbb{R}$, $a_+ := \max\{a, 0\}$, $a_- := -\min\{a, 0\}$. This is a uniformly elliptic Isaacs equation (with ellipticity constants $1, 2d$) since it can be written as

$$\max_{1 \leq y \leq 2} \min_{1 \leq z \leq 2} \operatorname{tr}[a(y, z)u_{xx}] = 0, \quad \text{where } a(y, z) = \begin{pmatrix} y & & & \\ & z & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, \quad y, z \in [1, 2].$$

Of course, these extrema over $[1, 2]$ are really extrema over $\{1, 2\}$. Despite the simplicity of equation (0.5), it is not of the form to which some recent regularity results (see [CC2], [CY], [NV3]) apply. First, the operator is of the form

$$F(m) := \operatorname{tr} m + (m_{11})_+ - (m_{22})_- = \min \{ \operatorname{tr} m + (m_{11})_+, \operatorname{tr} m + (m_{11})_+ + m_{22} \} \\ := \min \{ G^1(m), G^2(m) \},$$

where both G^1, G^2 are convex, and not the minimum of two operators where one is convex and the other concave (see (1.9) in [CC2]).

Secondly, the operator $F(m)$ does not satisfy the property that $\forall t \in \mathbb{R}$, the set $C_t := \{m \in \mathcal{S}_d : F(m) = 0, \operatorname{tr} m = t\}$ is convex (see [CY]). For example, in $d = 3$, and taking $t = 1$, both $m = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 3 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & & \\ & -2 & \\ & & 2 \end{pmatrix} \in C_1$, but $\frac{m+n}{2} = \begin{pmatrix} 0 & & \\ & -\frac{3}{2} & \\ & & \frac{5}{2} \end{pmatrix} \notin C_1$. Hence, C_1 is not a convex subset of $\mathcal{S}_3$.

We also observe that the operator $F(m) = \operatorname{tr} m + (m_{11})_+ - (m_{22})_-$ is not of *Hessian* type (see [NV3]). That is, it is not invariant under the mapping $m \mapsto o^t m o$, where $o \in \mathcal{O}_d$ (the space of $d \times d$ orthogonal matrices). An example where $F(m) \neq F(o^t m o)$ is given in $d = 3$ by $m = \begin{pmatrix} 2 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ and $o = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}_3$.

Thus $o^t m o = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$; hence $F(o^t m o) = \frac{3}{2}$, while $F(m) = 2$.

Nonetheless, C^2 solutions of (0.5) satisfy an interior $W^{2,p}$ estimate for all $0 < p < \infty$, which we prove in §1 below. Notice that F can be decomposed as

$$F(u_{xx}) = \frac{1}{2} \Delta u + (u_{x^1 x^1})_+ - \frac{1}{2} \Delta u - (u_{x^2 x^2})_- := F^1(u_{xx}) + F^2(u_{xx}),$$

where F^1 is convex, and F^2 is concave, since they can be written as

$$F^1(m) = \max_{1 \leq y \leq 2} \operatorname{tr}[a^1(y)m], \quad \text{where } a^1(y) = \begin{pmatrix} y - \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2} \end{pmatrix}, \\ F^2(m) = \min_{1 \leq z \leq 2} \operatorname{tr}[a^2(z)m], \quad \text{where } a^2(z) = \begin{pmatrix} \frac{1}{2} & & & \\ & z - \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2} \end{pmatrix}.$$

The saddle point for $F = F(m)$ occurs at $\bar{y}(m) = 1 + I_{[0, \infty)}(m_{11})$, $\bar{z}(m) = 1 + I_{(-\infty, 0]}(m_{22})$, and the inequalities in (0.3) are equivalent to $\operatorname{tr} m + y m_{11} - (m_{22})_- \leq \operatorname{tr} m + (m_{11})_+ - (m_{22})_- \leq \operatorname{tr} m + (m_{11})_+ + z m_{22}$, which holds $\forall y, z \in [1, 2]$, $m \in \mathcal{S}$.

Notation: I_A denotes the indicator function of the set A , and we also write $I_{m_{22} \leq 0}$ in place of $I_{(-\infty, 0]}(m_{22})$. Note that both F^1 and F^2 are uniformly elliptic with ellipticity constants $\frac{1}{2}, \frac{3}{2}d$ since $\frac{1}{2}I_d \leq a^1(y), a^2(z) \leq \frac{3}{2}I_d$.

Example 2. A more complicated pde is given by

$$(0.6) \quad F(u_{xx}) := \Delta u + [u_{x^1 x^1} + u_{x^2 x^2} - (u_{x^2 x^2})_-]_+ - (u_{x^2 x^2})_- = 0,$$

which is a uniformly elliptic Isaacs equation (with ellipticity constants $1, 4d$) since it can be written as

$$\max_{1 \leq y \leq 2} \min_{1 \leq z \leq 2} \operatorname{tr}[a(y, z)u_{xx}] = 0, \text{ where } a(y, z) = \begin{pmatrix} y & & & \\ & yz & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}.$$

Despite the complexity of the pde, this Isaacs operator satisfies the minmax principle. A direct computation shows that the upper and lower Isaacs operators

$$\begin{aligned} F^+(m) &:= \min_{1 \leq z \leq 2} \max_{1 \leq y \leq 2} \operatorname{tr}[a(y, z)m] \\ &= \operatorname{tr} m + \min\{m_{22} + (m_{11} + 2m_{22})_+, (m_{11} + m_{22})_+\}, \\ F^-(m) &:= \max_{1 \leq y \leq 2} \min_{1 \leq z \leq 2} \operatorname{tr}[a(y, z)m] = \operatorname{tr} m + [m_{11} + m_{22} - (m_{22})_-]_+ - (m_{22})_- \end{aligned}$$

are equal. The saddle point $(\bar{y}, \bar{z}) = (\bar{y}(m), \bar{z}(m))$ is given explicitly by $\bar{y}(m) = 1 + I_{m_{11} + m_{22} \geq (m_{22})_-}$, $\bar{z}(m) = 1 + I_{m_{22} \leq 0}$. In this example, however, the matrix $a(y, z)$ cannot be *separated* and the Isaacs operator cannot be written as the sum of two uniformly elliptic Bellman operators, one convex and the other concave. In $d = 3$, a $C^{2,1} \setminus C^3$ solution is given by $u(x_1, x_2, x_3) = x_2^3(3 - \operatorname{sgn}(x_2)) + x_2(12x_1^2 - 36x_3^2)$. Using the same technique as in Theorem 1 below, it can be shown that the Laplacian of any C^2 solution u of (0.6) satisfies the following one-sided interior estimate: $\forall x_0 \in D$ such that $B_r(x_0) \subset D$,

$$\sup_{B_{r/4}(x_0)} \Delta u \leq N_d r^{-2} \|u\|_{L^\infty(B_r(x_0))}.$$

Example 3. Nondegenerate equations are often the result of several mins and maxs combined. In the equation

$$(0.7) \quad F(u_{xx}) := \Delta u + (u_{x^1 x^1})_+ - (u_{x^2 x^2})_- + (u_{x^3 x^3})_+ = 0,$$

the operator $F(m)$, for $m \in \mathcal{S}$, can be written as

$$\begin{aligned} F(m) &= \max_{1 \leq y \leq 2} \min_{1 \leq z \leq 2} \max_{1 \leq w \leq 2} \operatorname{tr}[a(y, z, w)m], \\ \text{where } a(y, z, w) &= \begin{pmatrix} y & & & \\ & z & & \\ & & w & \\ & & & 1 \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \quad y, z, w \in [1, 2]. \end{aligned}$$

Since the function $p(y, z, w, m) := \operatorname{tr}[a(y, z, w)m]$ is an affine function of z, w , it is trivially convex in w and concave in z ; hence the innermost min and max can be interchanged and F can be expressed as a single max/min of the form $\max_w \min_z$ of the form (0.1):

$$F(m) = \max_{1 \leq y \leq 2} \max_{1 \leq w \leq 2} \min_{1 \leq z \leq 2} \operatorname{tr}[a(y, z, w)m] = \max_{(y, w) \in [1, 2] \times [1, 2]} \min_{z \in [1, 2]} \operatorname{tr}[a(y, z, w)m].$$

Since $p(\mathbf{y}, z, m)$ is trivially convex in $\mathbf{y} := (y, w)$ and concave in z , F will satisfy the minmax principle. Hence there exists a saddlepoint $(\bar{\mathbf{y}}, \bar{z}) = (\bar{\mathbf{y}}(m), \bar{z}(m))$ such that $\forall \mathbf{y} \in [1, 2] \times [1, 2], \forall z \in [1, 2], \forall m \in \mathcal{S}$,

$$(0.8) \quad p(\mathbf{y}, \bar{z}, m) \leq p(\bar{\mathbf{y}}, \bar{z}, m) \leq p(\bar{\mathbf{y}}, z, m).$$

The saddle point $(\bar{\mathbf{y}}, \bar{z})$ is given explicitly by $\bar{\mathbf{y}}(m) = (\bar{y}(m), \bar{w}(m)) = (1 + I_{m_{11} \geq 0}, 1 + I_{m_{33} \geq 0})$, $\bar{z}(m) = 1 + I_{m_{22} \leq 0}$ and it is easily verified that $p(\bar{\mathbf{y}}(m), \bar{z}(m), m) = \text{tr } m + (m_{11})_+ - (m_{22})_- + (m_{33})_+$ and that (0.8) holds. As in Example 1, the coefficient matrix $a(y, z, w)$ can be separated into two nondegenerate matrices and the pde can be written as the sum of two nondegenerate Bellman operators, one convex and the other concave. This, along with the fact that the control set for the concave operator is a closed interval, ensures that C^2 solutions of (0.7) satisfy an interior $W^{2,p}$ estimate for all $0 < p < \infty$ (see Theorem 2 in §2). In $d = 4$, a $C^{2,1} \setminus C^3$ solution is given by $u(x_1, x_2, x_3, x_4) = -x_1^3 (\text{sgn}(x_1) + 3) + x_2^3 (\text{sgn}(x_2) + 3) - x_3^3 (\text{sgn}(x_3) + 3) + 12x_4^2(x_1 - x_2 + x_3)$.

Example 4. The following uniformly elliptic Isaacs equation fails the minmax principle:

$$F(u_{xx}) := \max \left\{ \Delta u - \left(u_{x^2 x^2} - \frac{1}{2} u_{x^1 x^1} \right)_-, \Delta u - \left(u_{x^1 x^1} - \frac{1}{2} u_{x^2 x^2} \right)_- \right\} = 0.$$

This is a uniformly elliptic Isaacs equation (with ellipticity constants $\frac{1}{2}, 2d$) since it can be written as

$$\max_{1 \leq y \leq 2} \min_{1 \leq z \leq 2} \text{tr}[a(y, z)u_{xx}] = 0, \quad \text{where } a(y, z) = \begin{pmatrix} \frac{y}{z} & & & \\ & \frac{z}{y} & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, \quad y, z \in [1, 2].$$

Note that $\frac{1}{2}I_d \leq a(y, z) \leq 2I_d$. A direct computation shows that the upper and lower Isaacs operators

$$\begin{aligned} F^+(m) &:= \min_{1 \leq z \leq 2} \max_{1 \leq y \leq 2} \text{tr}[a(y, z)m] \\ &= \min \left\{ \text{tr } m + \left(m_{11} - \frac{m_{22}}{2} \right)_+, \text{tr } m + \left(m_{22} - \frac{m_{11}}{2} \right)_+ \right\}, \\ F^-(m) &:= \max_{1 \leq y \leq 2} \min_{1 \leq z \leq 2} \text{tr}[a(y, z)m] \\ &= \max \left\{ \text{tr } m - \left(m_{22} - \frac{m_{11}}{2} \right)_-, \text{tr } m - \left(m_{11} - \frac{m_{22}}{2} \right)_- \right\}, \end{aligned}$$

are not equal. Of course, $F^- \leq F^+$ always holds, but for $m_{11} \neq 0$, $F^-(m_{11}, m_{11}, 0') < F^+(m_{11}, m_{11}, 0')$, since

$$\begin{aligned} F^-(m_{11}, m_{11}, 0') &= \frac{1}{2} \left(3m_{11} + \frac{3m_{11}}{2} - \frac{|m_{11}|}{2} \right), \\ F^+(m_{11}, m_{11}, 0') &= \frac{1}{2} \left(3m_{11} + \frac{3m_{11}}{2} + \frac{|m_{11}|}{2} \right). \end{aligned}$$

Hence the minmax condition $F^- = F^+$ fails. We do not study the regularity of solutions to such equations in this paper.

1. AN EXAMPLE OF $W^{2,p}$ REGULARITY IN THE SIMPLEST CASE

In this section, we prove that $C^2(D)$ solutions of (0.5) satisfy an interior $W^{2,p}$ estimate for all $0 < p < \infty$. Before doing so, we recall some notation which will be used throughout this paper. For $0 < \lambda \leq \Lambda$ and $m \in \mathcal{S}_d$, Pucci's extremal operators $\mathcal{M}^+$ and $\mathcal{M}^-$ are defined by

$$\mathcal{M}^+(m, \lambda, \Lambda) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \quad \mathcal{M}^-(m, \lambda, \Lambda) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

where $e_i = e_i(m)$ are the eigenvalues of m . For an open set $D \subset \mathbb{R}^d$, $\underline{S}(\lambda, \Lambda, 0)$ is defined to be the set of $u \in C(D)$ such that $\mathcal{M}^+(u_{xx}, \lambda, \Lambda) \geq 0$ in the viscosity sense in D . Similarly, $\bar{S}(\lambda, \Lambda, 0)$ denotes the set of $u \in C(D)$ such that $\mathcal{M}^-(u_{xx}, \lambda, \Lambda) \leq 0$ in the viscosity sense in D . Functions in $\underline{S}$ or $\bar{S}$ are called *subsolutions* or *supersolutions*, respectively. (For more details, see §2.2 of [CC1].)

Theorem 1. *Let D be a bounded domain in $\mathbb{R}^d$ and let $u \in C^2(D)$ be a solution of $F(u_{xx}) := \Delta u + (u_{x^1 x^1})_+ - (u_{x^2 x^2})_- = 0$ in D . Then $\forall x_0 \in D$ such that $B_r(x_0) \subset D$, and any $0 < p < \infty$,*

$$(1.2) \quad \|u_{xx}\|_{L^p(B_{r/8}(x_0))} \leq N(d, p) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r(x_0))}.$$

The first step is to establish the following subsolution lemma.

Lemma. *Under the above assumptions, both $(u_{x^1 x^1})_+$ and $(u_{x^2 x^2})_-$ belong to the subsolution class $\underline{S}(0) := \underline{S}(\frac{1}{d}, 2d, 0)$ in D .*

Proof. Define $D^+ = \{x \in D : u_{x^1 x^1}(x) > 0\}$ and for $h > 0$, $D_h^+ = \{x \in D^+ : \text{dist}(x, \partial D^+) > h\}$. Note that in D^+ , u satisfies $F^+[u](x) := \Delta u(x) + u_{x^1 x^1}(x) - (u_{x^2 x^2}(x))_- = 0$. Fix $x \in D_h^+$ and set $w^h(x) = u(x + h e_1) + u(x - h e_1) - 2u(x)$. Since $F^+[u](x \pm h e_1) = 0$ in D_h^+ , setting these 3 equations = 0 and using $a_- + b_- - 2c_- \geq -(a + b - 2c)_+$ yields

$$\begin{aligned} 0 &= \Delta w^h(x) + w_{x^1 x^1}^h(x) - \{[u_{x^2 x^2}(x + h e_1)]_- + [u_{x^2 x^2}(x - h e_1)]_- - 2[u_{x^2 x^2}(x)]_-\} \\ &\leq \Delta w^h(x) + w_{x^1 x^1}^h(x) + (w_{x^2 x^2}^h(x))_+. \end{aligned}$$

That is, in D_h^+ , w^h is a viscosity subsolution of the uniformly elliptic equation $\Delta v + v_{x^1 x^1} + (v_{x^2 x^2})_+ = 0$, with the same ellipticity constants $1, 2d$. Thus w^h and hence $\Delta_{h, e_1}^2 u = \frac{w^h}{h^2} \in \underline{S}(0)$ in D_h^+ . Since $u \in C^2(D)$ and $\underline{S}(0)$ is closed, $u_{x^1 x^1} \in \underline{S}(0)$ in D^+ , from which it follows that $(u_{x^1 x^1})_+ \in \underline{S}(0)$ in D (see Remark 1 below).

To see that $(u_{x^2 x^2})_- \in \underline{S}(0)$ in D , define $D^- = \{x \in D : u_{x^2 x^2}(x) < 0\}$ and for $h > 0$, $D_h^- = \{x \in D^- : \text{dist}(x, \partial D^-) > h\}$. In D^- , u satisfies $F^-[u](x) := \Delta u(x) + (u_{x^1 x^1}(x))_+ + u_{x^2 x^2}(x) = 0$. Fix $x \in D_h^-$ and set $w^h(x) = u(x + h e_2) + u(x - h e_2) - 2u(x)$. Since $F^-[u](x \pm h e_2) = 0$ in D_h^- , setting these 3 equations = 0 and using $a_+ + b_+ - 2c_+ \geq -(a + b - 2c)_-$ yields

$$\begin{aligned} 0 &= \Delta w^h(x) + [u_{x^1 x^1}(x + h e_2)]_+ + [u_{x^1 x^1}(x - h e_2)]_+ - 2[u_{x^2 x^2}(x)]_+ + w_{x^2 x^2}^h(x) \\ &\geq \Delta w^h(x) - (w_{x^1 x^1}^h(x))_- + w_{x^2 x^2}^h(x). \end{aligned}$$

Thus, in D_h^- , w^h is a viscosity supersolution of the uniformly elliptic equation $\Delta v - (v_{x^1 x^1})_- + v_{x^2 x^2} = 0$, with the same ellipticity constants $1, 2d$. Thus w^h and hence $\Delta_{h, e_2}^2 u \in \bar{S}(0)$ in D_h^- . Again, $u \in C^2(D)$ and $\bar{S}(0)$ is closed implies that $u_{x^2 x^2} \in \bar{S}(0)$ in D^- , and hence $(u_{x^2 x^2})_- \in \underline{S}(0)$ in D (also by the following remark). $\square$

Remark 1. If $v \in C(D)$, and $v \in \underline{S}(0) = \underline{S}(\lambda, \Lambda, 0)$ in $D^+ := \{x \in D : v(x) > 0\}$, then $v_+ \in \underline{S}(0)$ in D . To see this, fix $x_0 \in D$, and say $\exists \epsilon > 0$ and $\phi \in C^2(B_\epsilon(x_0))$ with $v_+ \leq \phi$ in $B_\epsilon(x_0)$, $v_+(x_0) = \phi(x_0)$. We claim $\mathcal{M}^+(\phi_{xx}(x_0), \lambda, \Lambda) \geq 0$. If $x_0 \in D^+$ (open), then $\exists r > 0$ with $B_r(x_0) \subset D^+$, and we may suppose that r is so small that $B_r(x_0) \subset B_\epsilon(x_0)$. But from $v \leq v_+ \leq \phi$ in $B_\epsilon(x_0)$, we have $v \leq v_+ \leq \phi$ in $B_r(x_0)$, and $v(x_0) = v_+(x_0) = \phi(x_0)$. Since $v \in \underline{S}(0)$ in D^+ , we have $\mathcal{M}^+(\phi_{xx}(x_0), \lambda, \Lambda) \geq 0$. If $x_0 \notin D^+$, then $0 \leq v_+ \leq \phi$ in $B_\epsilon(x_0)$, $0 = v_+(x_0) = \phi(x_0)$, and $0 \in \underline{S}(0)$ in D implies that $\mathcal{M}^+(\phi_{xx}(x_0), \lambda, \Lambda) \geq 0$.

From this it also follows that if $w \in C(D)$, and $w \in \bar{S}(0)$ in $D^- := \{x \in D : w(x) < 0\}$, then $w_- \in \underline{S}(0)$ in D . Indeed, $-w \in \underline{S}(0)$ in $\{x \in D : -w(x) > 0\}$ implies $w_- = (-w)_+ \in \underline{S}(0)$ in D .

Proof of Theorem 1. We recall the $W^{2,\delta}$ interior estimate, due to Lin (see [L], and Proposition 7.4 in [CC1]: if $u \in S(\lambda, \Lambda, 0)$ in B_r , there exists a universal $\delta > 0$ such that $u \in W^{2,\delta}(B_{r/2})$ and

$$(1.3) \quad \|u\|_{W^{2,\delta}(B_{r/2})} \leq C(d, \lambda, \Lambda) r^{-2+\frac{\delta}{2}} \|u\|_{L^\infty(B_r)}.$$

This result, along with the well-known subsolution estimate (see Theorem 4.8 in [CC1], Theorem 9.20 in [GT]) (with $p = \delta$), applied to our subsolution $(u_{x^1 x^1})_+$ gives: $\forall x_0 \in D$ such that $B_r(x_0) \subset D$,

$$(1.4) \quad \begin{aligned} \sup_{B_{r/4}(x_0)} (u_{x^1 x^1})_+ &\leq N_d \left\{ \int_{B_{r/2}(x_0)} [(u_{x^1 x^1})_+]^\delta dx \right\}^{\frac{1}{\delta}} \\ &= N'_d r^{-\frac{\delta}{2}} \|(u_{x^1 x^1})_+\|_{L^\delta(B_{r/2}(x_0))} \leq N'_d C r^{-2} \|u\|_{L^\infty(B_r(x_0))}, \end{aligned}$$

with an identical estimate holding for $(u_{x^2 x^2})_-$. This, and the inequality $|\Delta u(x)| \leq (u_{x^2 x^2}(x))_- + (u_{x^1 x^1}(x))_+$ immediately yield: $\forall x_0 \in D$ such that $B_r(x_0) \subset D$,

$$\sup_{B_{r/4}(x_0)} |\Delta u| \leq C'_d r^{-2} \|u\|_{L^\infty(B_r(x_0))},$$

and hence $\forall 1 < p < \infty$, the $W^{2,p}$ estimate in the form $\|u_{xx}\|_{L^p(B_{r/8})} \leq C(d, p) \left\{ r^{-2} \|u\|_{L^p(B_{r/4})} + \|\Delta u\|_{L^p(B_{r/4})} \right\}$ yields: $\forall x_0 \in D$ such that $B_r(x_0) \subset D$,

$$\|u_{xx}\|_{L^p(B_{r/8}(x_0))} \leq C''(d, p) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r(x_0))}.$$

The estimate for $0 < p \leq 1$ follows from the estimate for $p = 2$ and Hölder's inequality, since

$$\begin{aligned} \|u_{xx}\|_{L^p(B_{r/8})} &\leq \|u_{xx}\|_{L^2(B_{r/8})} |B_{r/8}|^{\frac{2-p}{2p}} \\ &\leq C''_d r^{-2+\frac{d}{2}} \|u\|_{L^\infty(B_r)} |B_{r/8}|^{\frac{2-p}{2p}} = C'''(d, p) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r)}. \end{aligned}$$

Thus (1.2) holds $\forall 0 < p < \infty$. $\square$

Remark 2. In equation (0.5), both the minimum and the maximum were taken over the interval $[1, 2]$. In the case when controls y, z each appear only once in the matrix $a(y, z)$, the choice of interval(s) is immaterial to the regularity issue, so long as the matrix $a(y, z)$ is nondegenerate. For example, if we consider

$$G(u_{xx}) := \max_{y \in [y_1, y_2]} \min_{z \in [z_1, z_2]} \text{tr}[a(y, z)u_{xx}] = 0, \quad \text{where } a(y, z) = \begin{pmatrix} y & z & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

for fixed positive constants y_1, z_1 , the pde can be expressed as

$$G(u_{xx}) := \Delta u + (y_1 - 1)u_{x_1 x_1} + (z_1 - 1)u_{x_2 x_2} + (y_2 - y_1)(u_{x_1 x_1})_+ - (z_2 - z_1)(u_{x_2 x_2})_- = 0$$

which is uniformly elliptic with ellipticity constants $\lambda := \min\{y_1, z_1, 1\}$ and $\Lambda d := \max\{y_2, z_2, 1\}d$. If u satisfies $G(u_{xx}) = 0$ in D , then the function $v(x) = u(Ax)$, $Ax = (\sqrt{y_1}x_1, \sqrt{z_1}x_2, x')$ satisfies

$$\Delta v + \left(\frac{y_2 - y_1}{y_1}\right)(v_{x_1 x_1})_+ - \left(\frac{z_2 - z_1}{z_1}\right)(v_{x_2 x_2})_- = 0, \quad \text{in } D' := A^{-1}(D),$$

a nondegenerate equation with control matrix $a(\alpha, \beta) = \begin{pmatrix} \alpha + \frac{y_2}{y_1} & & \\ & \beta + \frac{z_2}{z_1} & \\ & & \mathbf{1} \end{pmatrix}$, where $\alpha, \beta \in [0, 1]$. Since $I_d \leq a(\alpha, \beta) \leq KI_d$, where $K := 1 + \max\{\frac{y_2}{y_1}, \frac{z_2}{z_1}\}$, the above equation is uniformly elliptic with ellipticity constants $1, Kd$. The exact argument as before yields $(v_{x_1 x_1})_+, (v_{x_2 x_2})_- \in \underline{S}(\frac{1}{d}, Kd, 0)$ in D' , which in turn implies the $W^{2,p}(D')$ interior estimate for v : $\forall x_0 \in D'$ such that $B_s(x_0) \subset D'$, and any $0 < p < \infty$,

$$(1.5) \quad \|v_{xx}\|_{L^p(B_{s/8}(x_0))} \leq C'(p, d, K)s^{-2+\frac{d}{p}}\|v\|_{L^\infty(B_s(x_0))}.$$

Now take any $y_0 \in D$, such that $B_r(y_0) \subset D$. Then $x_0 := A^{-1}y_0 \in D'$. From the inequalities $\frac{|x|}{M_C} \leq |C^{-1}x| \leq \frac{|x|}{m_C}$, which hold for any positive definite matrix C (here m_C, M_C denote, respectively, the minimum and maximum eigenvalues of C), we have $B_s(x_0) \subset D'$ for $s = \frac{r}{M_A}$, and it follows from (1.5) that for $k := \frac{m_A}{M_A}$,

$$\|u_{xx}\|_{L^p(B_{kr/8}(y_0))} \leq C''(p, d, y_1, y_2, z_1, z_2)r^{-2+\frac{d}{p}}\|u\|_{L^\infty(B_r(y_0))}.$$

That is, we arrive at the $W^{2,p}$ interior estimate for C^2 solutions of $G(u_{xx}) = 0$, where the constant on the right-hand side now depends on y_1, y_2, z_1, z_2 , in addition to dimension, and p .

Remark 3. The proof of Theorem 1 relies heavily on the fact that $u \in C^2(D)$. Specifically, it relies on the fact that the set $D^+ = \{x \in D : u_{x^1 x^1}(x) > 0\}$ is open, and that $\lim_{h \rightarrow 0} \Delta_{h, e_1}^2 = u_{x^1 x^1} \in C(D)$ (with a corresponding statement holding for $u_{x^2 x^2}$). The obvious difficulty in attempting to apply our above technique to the viscosity solution case is that if $u \in C(D)$ is a viscosity solution of (0.5), all that is known *a priori* is that $u_{x^1 x^1}$ exists almost everywhere in D , belongs to $L_{loc}^\delta(D)$, and that the set D^+ is measurable. Due to the “separability” of (0.5), the hypothesis that $u \in C^2(D)$ can be removed and estimate (1.2) will hold for continuous viscosity solutions u of (0.5) if we make the additional assumption that either of $(u_{x^1 x^1})_+$ or $(u_{x^2 x^2})_-$ satisfies the interior estimate (1.2). This follows from the $W^{2,p}$ regularity theory ($d < p < \infty$) for convex (or concave) equations and is discussed in greater detail in §2. (See the discussion following (2.0) below.)

2. $W^{2,p}$ REGULARITY FOR MORE GENERAL EQUATIONS

In the pde (0.5) from Theorem 1, y and z each appeared only once in the matrix $a(y, z)$ and each was restricted to a closed interval. This is not the case for the equation in Example 3, or in the Isaacs equation

$$(2.0) \quad \Delta u + (u_{x^1 x^1})_+ + \cdots + (u_{x^{d-1} x^{d-1}})_+ - (u_{x^d x^d})_- = 0.$$

Nonetheless, C^2 solutions of these equations will satisfy an interior $W^{2,p}$ estimate $\forall 0 < p < \infty$. The reason is that if the control matrix $a(y, z)$ is *separable* and either the maximizing or minimizing control variable appears only once and is restricted to a closed interval, the L^∞ norm of the second derivative “corresponding” to that variable will be appropriately bounded by the L^∞ norm of u . In (2.0), $a(\mathbf{y}, z) = a(y_1, \dots, y_{d-1}, z)$, where $\mathbf{y} \in Y := [1, 2]^{d-1}$, while $z \in Z := [1, 2]$ appears alone. One can show that $(u_{x^d x^d})_-$ (the derivative “corresponding” to z) is locally L^∞ and can be estimated in terms of distance to the boundary of D and the L^∞ norm of u . If $(u_{x^d x^d})_-$ is then brought to the other side of (2.0), the new left-hand side is convex and uniformly elliptic (with the same ellipticity constants as the original equation). Since the right-hand side belongs to L^∞ , the $W^{2,p}$ regularity theory ($d < p < \infty$) for such equations applies. More generally, we have the following theorem.

Theorem 2. *Let D be a bounded domain in $\mathbb{R}^d$ and let $u \in C^2(D)$ be a solution of*

$$(2.1) \quad F(u_{xx}) := \max_{y \in Y} \min_{z \in Z} \operatorname{tr}[a(y, z)u_{xx}] = 0 \quad \text{in } D,$$

where $a(y, z) = a^1(y) + a^2(z)$ is diagonal, with both a^1, a^2 nondegenerate. Assume that one of Y or Z is an interval in $\mathbb{R}$ and that the corresponding control, y or z , appears only once in the matrix $a(y, z)$. Assume that F is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$. Then $\forall 0 < p < \infty$, $u \in W_{loc}^{2,p}(D)$ and $\forall x_0 \in D$ such that $B_r(x_0) \subset D$

$$\|u_{xx}\|_{L^p(B_{r/8}(x_0))} \leq N(d, p, \lambda, \Lambda) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r(x_0))}.$$

Proof. We first assume that the control set Z is an interval, say $[z_1, z_2]$, with $z_1 > 0$ and that in $a^2(z)$, z appears only once, say in the (k, k) position. That is, we assume that $a^2(z)$ is diagonal with $a_{ij}^2(z) = [(z - \lambda_i)\delta_{ik} + \lambda_i]\delta_{ij}$,

$$a^2(z) = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_{k-1} & & \\ & & & z & \\ & & & \lambda_{k+1} & \\ & & & & \ddots \\ & & & & & \lambda_d \end{pmatrix} \quad \lambda_i > 0, z \in [z_1, z_2].$$

Thus $\lambda'' I_d \leq a^2(z) \leq \Lambda'' I_d$, where $\lambda'' = \min\{\min_{i \neq k} \lambda_i, z_1\}$, $\Lambda'' = \max\{\max_{i \neq k} \lambda_i, z_2\}$, and (2.1) can be rewritten as

$$\begin{aligned} 0 = F(u_{xx}) &= \max_{y \in Y} \min_{z \in [z_1, z_2]} \operatorname{tr}[a(y, z)u_{xx}] = \max_{y \in Y} \operatorname{tr}[a^1(y)u_{xx}] + \min_{z \in [z_1, z_2]} \operatorname{tr}[a^2(z)u_{xx}] \\ (2.2) \quad &= \max_{y \in Y} \operatorname{tr}[a^1(y)u_{xx}] + \sum_{i \neq k}^d \lambda_i u_{x^i x^i} + z_1 u_{x^k x^k} - (z_2 - z_1) (u_{x^k x^k})_- \\ &=: F^1(u_{xx}) + F^2(u_{xx}), \end{aligned}$$

where F^1 is convex, F^2 is concave, and F^2 is uniformly elliptic with ellipticity constants $\lambda'', \Lambda'' d$. The optimal minimizer for the operator

$$F^2(m) := \min_{z \in [z_1, z_2]} \operatorname{tr}[a^2(z)m] = \sum_{i \neq k}^d \lambda_i m_{ii} + z_1 m_{kk} - (z_2 - z_1) (m_{kk})_-$$

is the point $\bar{z}(m) = z_1 + (z_2 - z_1)I_{m_{kk} \leq 0} \in [z_1, z_2]$. So define $D^- := \{x \in D : u_{x^k x^k}(x) < 0\}$ and for $h > 0$, $D_h^- = \{x \in D^- : \text{dist}(x, \partial D^-) > h\}$. Since $\bar{z}(u_{xx}) \equiv z_2$ (constant) in D^- , it follows that if $x \in D^-$, then $F^2(u_{xx}(x)) = \text{tr}[a^2(z_2)u_{xx}(x)]$ and u satisfies $G^-[u](x) := F^1(u_{xx}(x)) + \text{tr}[a^2(z_2)u_{xx}(x)] = 0$. Fix $x \in D_h^-$ and set $w^h(x) = u(x + he_k) + u(x - he_k) - 2u(x)$. Since $G^-[u](x \pm he_k) = 0$ in D_h^- , setting these 3 equations = 0 and using $\max_y f_y + \max_y g_y - 2 \max_y h_y \geq \min_y (f_y + g_y - 2h_y)$ gives

$$\begin{aligned} 0 &= \max_y \text{tr}[a^1(y)u_{xx}(x + he_k)] + \max_y \text{tr}[a^1(y)u_{xx}(x - he_k)] \\ &\quad - 2 \max_y \text{tr}[a^1(y)u_{xx}(x)] + \text{tr}[a^2(z_2)w_{xx}^h(x)] \\ &\geq \min_y \text{tr}[a^1(y)w_{xx}^h(x)] + \text{tr}[a^2(z_2)w_{xx}^h(x)]. \end{aligned}$$

That is, in D_h^- , w^h is a viscosity supersolution of the uniformly elliptic equation $\min_y \text{tr}[a^1(y)v_{xx}] + \text{tr}[a^2(z_2)v_{xx}] = 0$, with the same ellipticity constants as F , say λ, Λ . Thus w^h and hence $\Delta_{h, e_k}^2 u = \frac{w^h}{h^2} \in \bar{S}(\frac{\lambda}{d}, \Lambda, 0) := \bar{S}(0)$ in D_h^- . Since $u \in C^2(D)$ and $\bar{S}(0)$ is closed, $u_{x^k x^k} \in \bar{S}(0)$ in D^- , from which it follows that $(u_{x^k x^k})_- \in \underline{S}(0)$ in D . The subsolution estimate with $p = \delta$, where δ is from the interior $W^{2, \delta}$ estimate, now yields (see (1.4)): $\forall x_0 \in D$ such that $B_r(x_0) \subset D$,

$$\sup_{B_{r/4}(x_0)} (u_{x^k x^k})_- \leq N(d, \lambda, \Lambda) r^{-2} \|u\|_{L^\infty(B_r(x_0))}.$$

Now rewrite equation (2.2) as

$$(2.3) \quad F^1(u_{xx}) + \sum_{i \neq k}^d \lambda_i u_{x^i x^i} + z_1 u_{x^k x^k} = (z_2 - z_1) (u_{x^k x^k})_- \quad \text{in } B_{r/4}(x_0).$$

Since F^1 is convex and the sum is affine in $u_{x^i x^i}$, the left-hand side of (2.3) is convex and uniformly elliptic (with the same ellipticity constants λ, Λ as F), and the right-hand side is in $L^\infty(B_{r/4}(x_0))$. The $W^{2, p}$ estimate for convex, uniformly elliptic equations $G(u_{xx}) = g$ in $B_{r/4}$, where $g \in L^p(B_{r/4})$ and $d < p < \infty$ is the estimate $\|u_{xx}\|_{L^p(B_{r/8})} \leq C(d, p, \lambda, \Lambda) \left\{ r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_{r/4})} + \|g\|_{L^p(B_{r/4})} \right\}$. This estimate, (2.3), and the L^∞ estimate for $(u_{x^k x^k})_-$ immediately imply that $\forall x_0 \in D$ such that $B_r(x_0) \subset D$, $u \in W^{2, p}(B_{r/8}(x_0))$ for $d < p < \infty$ and

$$\begin{aligned} \|u_{xx}\|_{L^p(B_{r/8}(x_0))} &\leq C(d, p, \lambda, \Lambda) \left\{ r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_{r/4}(x_0))} \right. \\ &\quad \left. + (z_2 - z_1) \|(u_{x^k x^k})_-\|_{L^p(B_{r/4}(x_0))} \right\} \\ (2.4) \quad &\leq C'(d, p, \lambda, \Lambda, z_2) \left\{ r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_{r/4}(x_0))} \right. \\ &\quad \left. + r^{\frac{d}{p}} \|(u_{x^k x^k})_-\|_{L^\infty(B_{r/4}(x_0))} \right\} \\ &\leq C''(d, p, \lambda, \Lambda) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r(x_0))}. \end{aligned}$$

The explicit dependence of the constant C'' on z_2 has been dropped, since this dependence can be absorbed by its dependence on Λ . Estimate (2.4) holds $\forall p$ with $0 < p < \infty$. Indeed, the estimate for $0 < p \leq d$ follows from Hölder's inequality

and the above inequality for $p = d + 1$,

$$\begin{aligned} \|u_{xx}\|_{L^p(B_{r/8})} &\leq \|u_{xx}\|_{L^{d+1}(B_{r/8})} |B_{r/8}|^{\frac{d+1-p}{p(d+1)}} \\ &\leq C''(d, \lambda, \Lambda) r^{-2+\frac{d}{d+1}} \|u\|_{L^\infty(B_r)} |B_{r/8}|^{\frac{d+1-p}{p(d+1)}} \\ &= C'''(d, p, \lambda, \Lambda) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r)}. \end{aligned}$$

On the other hand, if the control set Y is an interval, say $[y_1, y_2]$, with $y_1 > 0$ and y appears only in the (l, l) position in the diagonal matrix $a^1(y)$, then just as before, we can rewrite (2.1) as

$$\begin{aligned} 0 = F(u_{xx}) &= \max_{y \in [y_1, y_2]} \min_{z \in Z} \operatorname{tr}[a(y, z)u_{xx}] = \max_{y \in [y_1, y_2]} \operatorname{tr}[a^1(y)u_{xx}] + \min_{z \in Z} \operatorname{tr}[a^2(z)u_{xx}] \\ (2.5) \quad &= \sum_{i \neq l}^d \lambda_i u_{x^i x^i} + y_1 u_{x^l x^l} + (y_2 - y_1) (u_{x^l x^l})_+ + \min_{z \in Z} \operatorname{tr}[a^2(z)u_{xx}] \\ &=: F^1(u_{xx}) + F^2(u_{xx}), \end{aligned}$$

where F^2 is concave, and F^1 is convex and uniformly elliptic with ellipticity constants $\lambda'', \Lambda''d$ (exactly as before but with y_i in place of z_i). The optimal maximizer for the operator $F^1(m)$ is $\bar{y}(m) = y_1 + (y_2 - y_1)I_{m_{ll} \geq 0} \in [y_1, y_2]$. Define $D^+ := \{x \in D : u_{x^l x^l}(x) > 0\}$ and for $h > 0$, $D_h^+ = \{x \in D^+ : \operatorname{dist}(x, \partial D^+) > h\}$. Since $\bar{y}(u_{xx}) \equiv y_2$ in D^+ , if $x \in D^-$, then u satisfies $G^+[u](x) := \operatorname{tr}[a^1(y_2)u_{xx}(x)] + F^2(u_{xx}(x)) = 0$. So fix $x \in D_h^+$ and set $w^h(x) = u(x + he_l) + u(x - he_l) - 2u(x)$. Since $G^+[u](x \pm he_l) = 0$ in D_h^+ , setting these 3 equations = 0 and using $\min_z f_z + \min_z g_z - 2 \min_z h_z \leq \max_z (f_z + g_z - 2h_z)$ yields

$$0 \leq \operatorname{tr}[a^1(y_2)w_{xx}^h(x)] + \max_z \operatorname{tr}[a^2(z)w_{xx}^h(x)].$$

That is, in D_h^+ , w^h is a viscosity subsolution of the uniformly elliptic equation $\operatorname{tr}[a^1(y_2)v_{xx}] + \max_z \operatorname{tr}[a^2(z)v_{xx}] = 0$, with the same ellipticity constants as F . Exactly as before, this implies that $u_{x^l x^l} \in \underline{S}(\frac{\lambda}{d}, \Lambda, 0) := \underline{S}(0)$ in D^+ , and hence $(u_{x^l x^l})_+ \in \underline{S}(0)$ in D . The subsolution estimate and the interior $W^{2,\delta}$ estimate then yield the interior L^∞ estimate for $(u_{x^l x^l})_+ : \forall x_0 \in D$ such that $B_r(x_0) \subset D$,

$$\sup_{B_{r/4}(x_0)} (u_{x^l x^l})_+ \leq N''(d, \lambda, \Lambda) r^{-2} \|u\|_{L^\infty(B_r(x_0))}.$$

All that remains is to rewrite (2.5) as

$$\sum_{i \neq l}^d \lambda_i u_{x^i x^i} + y_1 u_{x^l x^l} + F^2(u_{xx}) = (y_1 - y_2) (u_{x^l x^l})_+ \quad \text{in } B_{r/4}(x_0),$$

observe that the left-hand side is concave and uniformly elliptic, and then apply the $W^{2,p}$ estimate, $d < p < \infty$ (which also holds for concave, constant coefficient, uniformly elliptic equations). Exactly as before, this yields: $\forall x_0 \in D$ such that $B_r(x_0) \subset D$, and for $0 < p < \infty$,

$$\|u_{xx}\|_{L^p(B_{r/8}(x_0))} \leq C''(d, p, \lambda, \Lambda) r^{-2+\frac{d}{p}} \|u\|_{L^\infty(B_r(x_0))}. \quad \square$$

3. CONCLUDING REMARKS

Theorem 2 was proved under the assumption that the control matrix $a(y, z)$ in (0.1) was *separable* and that either one of the controls y or z appeared only once along the diagonal of $a(y, z)$, and was restricted to a closed interval. This enabled us to isolate the derivative $u_{x^i x^i}$ corresponding to that control, show that it belonged to an appropriate class of sub(super)solutions, and then bound its L^∞ norm in terms of the L^∞ norm of u . This technique yields a $W^{2,p}$ estimate for C^2 solutions of

$$(3.1) \quad \Delta u + (u_{x^1 x^1})_+ + (u_{x^2 x^2})_+ - (u_{x^3 x^3})_- = 0$$

(since $(u_{x^3 x^3})_- \in \underline{S}(0)$), but it is not clear how to do this with the equation

$$(3.2) \quad F(u_{xx}) := \Delta u + (u_{x^1 x^1})_+ + (u_{x^2 x^2})_+ - (u_{x^3 x^3})_- - (u_{x^4 x^4})_- = 0.$$

In $d = 4$, an analytic solution of (3.2) is given by $u = f(x_1 - x_3, x_2 - x_4) = f(z_1, z_2)$, where $\Delta_z f = 0$, e.g. $u(x) = \ln[(x_1 - x_3)^2 + (x_2 - x_4)^2]$, while for $d = 5$, a $C^{2,1} \setminus C^3$ solution is given by $u = -x_1^3(\operatorname{sgn}(x_1) + 3) - x_2^3(\operatorname{sgn}(x_2) + 3) + x_3^3(\operatorname{sgn}(x_3) + 3) + x_4^3(\operatorname{sgn}(x_4) + 3) + 12(x_1 + x_2 - x_3 - x_4)x_5^2$.

For $m \in \mathcal{S}$, $F(m)$ can be written as

$$\begin{aligned} F(m) &= \operatorname{tr} m + (m_{11})_+ + (m_{22})_+ - (m_{33})_- - (m_{44})_- \\ &= \max_{y_1} \max_{y_2} \min_{z_1} \min_{z_2} \operatorname{tr}[a(y_1, y_2, z_1, z_2)m], \end{aligned}$$

$$\text{where } a(y_1, y_2, z_1, z_2) = \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & z_1 & \\ & & & z_2 \\ & & & & 1 \end{pmatrix}, \quad y_i, z_i \in [1, 2],$$

which, exactly as in Example 3, can be rewritten as a single max/min Isaacs operator which satisfies the minmax principle. The saddle point is the pair $(\bar{\mathbf{y}}(m), \bar{\mathbf{z}}(m)) \in [1, 2]^2 \times [1, 2]^2$ given by $\bar{\mathbf{y}}(m) = (\bar{y}^1(m), \bar{y}^2(m)) = (1 + I_{m_{11} \geq 0}, 1 + I_{m_{22} \geq 0})$, $\bar{\mathbf{z}}(m) = (\bar{z}^1(m), \bar{z}^2(m)) = (1 + I_{m_{33} \leq 0}, 1 + I_{m_{44} \leq 0})$. The problem is that *both* the max involving $\mathbf{y}$ and the min involving $\mathbf{z}$ involve two derivatives of u , and not just one, unlike the situation in equation (3.1). The technical difficulty is getting the L^∞ estimate for $(u_{x^3 x^3})_- + (u_{x^4 x^4})_-$.

As in the proofs of Theorems 1 and 2, we suppose $u \in C^2(D)$ is a solution of (3.2) and consider the open set $D^- = \{x \in D : \max\{u_{x^3 x^3}, u_{x^4 x^4}\}(x) < 0\}$. In D^- , u satisfies $\Delta u(x) + (u_{x^1 x^1}(x))_+ + (u_{x^2 x^2}(x))_+ + u_{x^3 x^3}(x) + u_{x^4 x^4}(x) = 0$, from which it follows that $\forall |e| = 1$, $u_{ee} \in \tilde{S}(0)$ in D^- , and hence $\max\{(u_{x^3 x^3})_-, (u_{x^4 x^4})_-\} \in \underline{S}(\frac{1}{d}, 2d, 0) := \underline{S}(0)$ in D^- . But is $\max\{(u_{x^3 x^3})_-, (u_{x^4 x^4})_-\} \in \underline{S}(0)$ in D ? If so, we could apply the subsolution estimate (followed by the $W^{2,\delta}$ estimate) and locally bound $\max\{(u_{x^3 x^3})_-, (u_{x^4 x^4})_-\}$ in terms of the L^∞ norm of u . That, and the elementary inequality $a_- + b_- \leq 2 \max\{a_-, b_-\}$, would allow us to bound the L^∞ norm of $(u_{x^3 x^3})_- + (u_{x^4 x^4})_-$ in terms of the L^∞ norm of u . Then, as in Theorem 2, we rewrite (3.2) as a uniformly elliptic, nonhomogeneous equation, with convex left-hand side and L^∞ right-hand side, and conclude, as before, the $W^{2,p}$ estimate for $0 < p < \infty$. The problem is that it is not clear that $\max\{(u_{x^3 x^3})_-, (u_{x^4 x^4})_-\} \in \underline{S}(0)$ in D . A refinement of the basic methods shown here must be employed in order to handle the case of a simple Isaacs equation whose control matrix is merely separable in a nondegenerate way (without the additional hypothesis that one of the controls be restricted to an interval), as in the case of (3.2). We anticipate interior $W^{2,p}$ estimates ($0 < p < \infty$) holding for these equations as well.

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