THE ZEROS OF CERTAIN LOMMEL FUNCTIONS

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ABSTRACT. Lommel's function $s_{\mu,\nu}(z)$ is a particular solution of the differential equation $z^2 y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}$. Here we present estimates and monotonicity properties of the positive zeros of $s_{\mu-1/2,1/2}(z)$ when $\mu \in (0, 1)$. The positivity of a closely related integral is also considered.

1. INTRODUCTION

Lommel's function [10, Sec. 10.7]

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}z)^{2n} \Gamma((\mu - \nu + 1)/2) \Gamma((\mu + \nu + 1)/2)}{\Gamma((\mu - \nu + 2n + 3)/2) \Gamma((\mu + \nu + 2n + 3)/2)}$$

is a particular solution of the inhomogeneous Bessel equation

$$z^2 y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}$$

and occurs in several places in physics and engineering (see [3] for a list of references). In [9] Steinig examined the sign of $s_{\mu,\nu}(z)$ for positive z and $\mu, \nu \in \mathbb{R}$. Among other things, he showed that for $\mu < \frac{1}{2}$ the function $s_{\mu,\nu}(z)$ has infinitely many changes of sign on $(0, \infty)$. Here we present the following refinement of this result. In this paper the zeros of analytic functions will always be counted according to their multiplicities and $\mathbb{N}$ will denote the set of positive integers.

Theorem 1.1. *For $\mu \in (0, 1)$ and $n \in \mathbb{N}$ set*

$$I_{2n-1}(\mu) := \left(\left(2n - 1 + \frac{\mu}{2} \right) \pi, (2n - 1 + \mu) \pi \right]$$

and

$$I_{2n}(\mu) := \left(2n\pi, \left(2n + \frac{\mu}{2} \right) \pi \right).$$

Then, for $\mu \in (0, 1)$ and positive z , $s_{\mu-1/2,1/2}(z) = 0$ implies $z \in I_n(\mu)$ for an $n \in \mathbb{N}$ and $s_{\mu-1/2,1/2}(z)$ has exactly one zero in every interval $I_n(\mu)$.

Hence, for each $n \in \mathbb{N}$ and $\mu \in (0, 1)$ there is a $z_n := z_n(\mu) \in I_n$ such that $s_{\mu-1/2,1/2}(z_n) = 0$, and this z_n satisfies $s'_{\mu-1/2,1/2}(z_n) \neq 0$. The implicit function theorem therefore implies that each z_n is an analytic function of $\mu \in (0, 1)$.

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Theorem 1.2. Set $w_n := w_n(\mu) := z_n - n\pi$. For $\mu \in (0, 1)$ we have $\lim_{n \rightarrow \infty} w_n = \mu\pi/2$ and

$$(1.1) \quad w_{2n} < w_{2n+2} < \frac{\mu\pi}{2} < w_{2n+1} < w_{2n-1}, \quad n \in \mathbb{N}.$$

Furthermore, for $n \in \mathbb{N}$ the functions z_{2n-1} are strictly increasing in $(0, 1)$.

Proposition 1.4 below shows that z_{2n} is not monotonic in μ for any $n \in \mathbb{N}$.

Our proofs of Theorems 1.1 and 1.2 will be based on the fact that for $\mu, \nu \in \mathbb{C}$ with $\Re(\mu \pm \nu + 1) > 0$ and $z \in \mathbb{C}^- := \mathbb{C} \setminus \{z : z \leq 0\}$, one has

$$(1.2) \quad s_{\mu, \nu}(z) = \frac{\pi}{2} \left[Y_\nu(z) \int_0^z t^\mu J_\nu(t) dt - J_\nu(z) \int_0^z t^\mu Y_\nu(t) dt \right],$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the usual Bessel functions [10, Sec. 10.7]. It is well-known that

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{and} \quad Y_{1/2}(z) = -\sqrt{\frac{2}{\pi z}} \cos z,$$

and therefore we obtain from (1.2) that for $\mu > 0$,

$$s_{\mu-1/2, 1/2}(z) = \frac{1}{z^{1/2}} \int_0^z t^{\mu-1} \sin(z-t) dt.$$

Theorem 1.1 thus takes the following equivalent form.

Theorem 1.3. For all $\mu \in (0, 1)$ the function

$$F_\mu(z) := \int_0^z t^{\mu-1} \sin(z-t) dt$$

has exactly one zero in each interval $I_n(\mu)$, $n \in \mathbb{N}$, and vanishes nowhere else on the positive real axis.

Obviously, $F_\mu(z)$ is analytic for $\Re \mu > 0$ and $z \in \mathbb{C}^-$. Partial integration shows that

$$(1.3) \quad F_\mu(z) = \frac{1}{\mu} \int_0^z t^\mu \cos(z-t) dt =: \frac{1}{\mu} G_\mu(z).$$

The following proposition is thus clear.

Proposition 1.4. For $n \in \mathbb{N}$ we have $z_n \rightarrow n\pi$ as $\mu \rightarrow 0$ and $z_{2n-1}(1) = z_{2n}(1) = 2n\pi$.

Our interest in the Lommel functions $s_{\mu-1/2, 1/2}(z)$ stems from the fact that the function $F_\mu(z)$ plays an important role in the theory of positive trigonometric sums.

In [11, V.2.29] it is shown that there is exactly one $\alpha^* \in (0, 1)$ such that $F_{1-\alpha^*}(\frac{3\pi}{2})$ vanishes. In [2] it is proven that α^* is such that for all $\alpha \geq \alpha^*$, $n \in \mathbb{N}$, and $x \in (0, \pi)$,

$$\tau_n(\alpha, x) := 1 + \sum_{k=1}^n \frac{\cos(kx)}{k^\alpha} > 0,$$

while for $\alpha < \alpha^*$ the cosine polynomials $\tau_n(\alpha, x)$ do not have a uniform lower bound on $(0, \pi)$. Nowadays, $\alpha^* = 0.30844\dots$ is called the Littlewood-Salem-Izumi constant [1]. See also [4] and [6] for related results and considerations.

In [7] it is claimed that for each $\rho \in (0, 1)$ there is exactly one $\mu^*(\rho) \in (0, 1)$ such that

$$(1.4) \quad F_{\mu^*(\rho)}((\rho + 1)\pi) = 0.$$

It is then shown there that the thus defined function $\mu^*(\rho)$ plays an important role in a conjecture concerning the mapping properties of partial sums of certain univalent functions. In [5] it is proven that this conjecture is equivalent to the fact that for $n \in \mathbb{N}$, $\rho \in (0, 1)$, and $\theta \in (0, \pi)$ one has

$$\sum_{k=0}^n \frac{(\mu)_k}{k!} \sin[(2k + \rho)\theta] > 0$$

for all $\mu \in (0, \mu^*(\rho)]$ but for no $\mu \in (\mu^*(\rho), 1)$. Here, $(\mu)_k := \mu(\mu + 1) \cdots (\mu + k - 1)$ is the Pochhammer symbol.

It seems that in [7] the proof of the well-definition of $\mu^*(\rho)$ is not completely correct. In order to prove that for each $\rho \in (0, 1)$ there is exactly one $\mu^*(\rho) \in (0, 1)$ such that (1.4) holds, it is claimed in the proof of [7, Lem. 1] that for all $\rho, \mu \in (0, 1)$,

$$I(\rho, \mu) := \frac{d}{d\mu} F_{\mu}((\rho + 1)\pi) = \int_0^{(\rho+1)\pi} t^{\mu-1} \sin((\rho + 1)\pi - t) \log(t) dt > 0.$$

But for $\rho = 0$ it is easy to see that

$$I(0, \mu) = \int_0^{\pi} t^{\mu-1} \sin(t) \log(t) dt$$

takes a negative value when $\mu = 0$. Hence, because of continuity, there is an $\epsilon > 0$ such that for all $\mu \in (0, \epsilon)$ there is a $\rho > 0$ for which $I(\rho, \mu) < 0$.

Despite this, the function $\mu^*(\rho)$ is well-defined, as we can show here. It follows readily from Theorem 1.2 and Proposition 1.4 that z_1 is a strictly increasing analytic function from $(0, 1)$ onto $(\pi, 2\pi)$. Hence, $\mu^*(\rho)$ is well-defined, and we have $\mu^*(\rho) = z_1^{(-1)}((\rho + 1)\pi)$. The next proposition is thus clear.

Proposition 1.5. *For each $\rho \in (0, 1]$ there is exactly one $\mu^*(\rho) \in (0, 1]$ such that $F_{\mu^*(\rho)}((\rho + 1)\pi) = 0$. $\mu^*(\rho)$ is strictly increasing and analytic in $(0, 1)$.*

In order to prove our results we will first show a weak form of Theorem 1.3 in the next section. In Section 3 we will then present the proofs of Theorems 1.2 and 1.3. For the proof of Theorem 1.2 we will need some information concerning the set of $(\rho, \mu) \in (0, 1)^2$ for which $I(\rho, \mu)$ is positive. The relevant results will be presented in Section 4.

2. A WEAK FORM OF THEOREM 1.3

In this section we will prove the following lemma.

Lemma 2.1. *For $\mu \in (0, 1)$ the function $F_{\mu}(z)$ has exactly one zero in every interval $(n\pi, (n + 1)\pi)$, $n \in \mathbb{N}$, and vanishes nowhere else on the positive axis.*

First, note that, since $t^{\mu-1}$ is decreasing on $(0, \infty)$ when $\mu \in [0, 1)$, obviously

$$(2.1) \quad \int_0^a t^{\mu-1} \sin(t) dt > 0 \quad \text{when} \quad a > 0 \text{ and } \mu \in [0, 1).$$

The next lemma is the key result for the proof of Lemma 2.1.

Lemma 2.2. Suppose $\mu \in (0, 1)$ and $z^* > 0$ are such that $F_\mu(z^*) = 0$. Then

$$F'_\mu(z^*) = \int_0^{z^*} t^{\mu-1} \cos(z^* - t) dt \neq 0.$$

Proof. If both $F_\mu(z^*)$ and $F'_\mu(z^*)$ would vanish, then so would

$$F_\mu(z^*) + iF'_\mu(z^*) = e^{iz^*} \int_0^{z^*} t^{\mu-1} e^{-it} dt.$$

In particular,

$$\int_0^{z^*} t^{\mu-1} \sin(t) dt = 0$$

would hold. Since this contradicts (2.1), the proof is complete. $\square$

It follows from (2.1) that $F_\mu(n\pi) \neq 0$ and

$$(2.2) \quad \operatorname{sgn} F_\mu(n\pi) = (-1)^{n+1} \quad \text{for } \mu \in (0, 1), n \in \mathbb{N}.$$

Furthermore, it is easy to see that $F_\mu(z)$ is positive for $z \in (0, \pi)$ and $\mu \in (0, 1)$.

Now, in order to verify the remaining statements of Lemma 2.1, let $n \in \mathbb{N}$ and $\mu^* \in (0, 1)$ and suppose that F_{μ^*} has exactly $m \in \{1, 2, 3, \dots\}$ zeros in the interval $I := (n\pi, (n+1)\pi)$. Then, because of Lemma 2.2, F_{μ^*} has m simple zeros $x_1, \dots, x_m$ in I and therefore there are m open subintervals $J_1, \dots, J_m$ of $(0, 1)$, all containing μ^* , and m differentiable functions $x_k(\mu)$, $k \in \{1, \dots, m\}$, that satisfy $x_k(\mu^*) = x_k$ and $F_\mu(x_k(\mu)) = 0$ for $\mu \in J_k$. Since $F_\mu(n\pi) \neq 0$ for all $\mu \in (0, 1)$ and $n \in \mathbb{N}$, $x_k(\mu)$ lies in I for all $\mu \in J_k$ and $k \in \{1, \dots, m\}$.

If $k \in \{1, \dots, m\}$ and $\sigma \in (0, 1)$ is a boundary point of J_k , then $\lim_{\mu \rightarrow \sigma} x_k(\mu)$ exists. Otherwise, because $x_k(\mu)$ is continuous and satisfies $x_k(\mu) \in I$ for $\mu \in J_k$, there is an open non-empty subinterval I^* of I such that for every $z \in I^*$ the set $x_k^{(-1)}(z)$ contains an infinite number of points that accumulate at σ . Consequently, $F_\sigma(z) = 0$ for all $z \in I^*$, a contradiction.

We can therefore assume that $J_k = (0, 1)$ for all $k \in \{1, \dots, m\}$. Otherwise, because of what we have just shown, one interval J_k would have a boundary point $\sigma \in (0, 1)$ for which

$$F_\sigma \left(\lim_{\mu \rightarrow \sigma} x_k(\mu) \right) = 0 = F'_\sigma \left(\lim_{\mu \rightarrow \sigma} x_k(\mu) \right).$$

This would contradict Lemma 2.2.

We have thus proven that if F_μ has exactly m zeros in $(0, 1)$ for one $\mu \in (0, 1)$, then F_μ has at least m zeros in $(0, 1)$ for all $\mu \in (0, 1)$. This readily implies that if F_μ has exactly m zeros in $(0, 1)$ for one $\mu \in (0, 1)$, then F_μ has exactly m zeros in $(0, 1)$ for all $\mu \in (0, 1)$.

In order to complete the proof of Lemma 2.1 we will now determine a $\mu \in (0, 1)$ for which F_μ has exactly one zero in I .

To that end, observe that, because of (1.3), F_μ vanishes if, and only if, G_μ vanishes. For positive z the function $G_0(z) = \sin(z)$ vanishes exactly at the points $n\pi$, $n \in \mathbb{N}$, and satisfies $G'_0(n\pi) = (-1)^n$. Therefore for every $n \in \mathbb{N}$ there is a differentiable function $z_n(\mu)$, defined in an open real neighborhood U_n of the origin, such that $z_n(0) = n\pi$, $G_\mu(z_n(\mu)) = 0$ for all $\mu \in U_n$, and

$$(2.3) \quad z'_n(0) = -\frac{\frac{d}{d\mu} G_\mu(z_n(0))|_{\mu=0}}{G'_0(z_n(0))} = (-1)^{n+1} \int_0^{n\pi} \cos(n\pi - t) \log(t) dt.$$

Partial integration gives

$$\int_0^{n\pi} \cos(n\pi - t) \log(t) dt = (-1)^{n+1} \int_0^{n\pi} \frac{\sin(t)}{t} dt.$$

The integral on the right-hand side of this equation is positive because of (2.1), and thus it follows from (2.3) and Hurwitz's theorem that for every $n \in \mathbb{N}$ there is a $\mu_n > 0$ such that for $0 < \mu < \mu_n$ the functions G_μ and F_μ have exactly one zero in $(n\pi, (n+1)\pi)$. The proof of Lemma 2.1 is complete.

3. PROOF OF THEOREMS 1.2 AND 1.3

The following result is an easy consequence of Lemma 2.1 and (2.2).

Proposition 3.1. *For a $z \in (n\pi, (n+1)\pi)$ the relations*

$$F_\mu(z) > 0 \quad \text{and} \quad F_\mu(z) < 0$$

hold if, and only if, $z < z_n$ and $z > z_n$, respectively, in the case where n is odd, and if, and only if, $z > z_n$ and $z < z_n$, respectively, in the case where n is even.

Now, let $n \in \mathbb{N}$, $n \geq 3$. Then

$$\begin{aligned} 0 &= \int_0^{z_n} t^{\mu-1} \sin(z_n - t) dt \\ &= \int_0^{z_n-2\pi} t^{\mu-1} \sin(z_n - t) dt + \int_{z_n-2\pi}^{z_n} t^{\mu-1} \sin(z_n - t) dt. \end{aligned}$$

Since $t^{\mu-1}$ is decreasing on $(0, \infty)$, it is clear that the integral on the right-hand side of the sum is negative, and therefore

$$\int_0^{z_n-2\pi} t^{\mu-1} \sin(z_n - 2\pi - t) dt > 0.$$

Hence, it follows from Proposition 3.1 that

$$z_n - 2\pi < z_{n-2} \quad \text{or} \quad z_n - 2\pi > z_{n-2},$$

depending on whether n is odd or even.

We have thus shown that, for fixed $\mu \in (0, 1)$, the sequence w_{2n-1} is strictly decreasing, while w_{2n} is strictly increasing. Consequently, since $w_n \in (0, \pi)$ for all $n \in \mathbb{N}$, $w_e = \lim_{n \rightarrow \infty} w_{2n}$ and $w_o = \lim_{n \rightarrow \infty} w_{2n-1}$ exist.

In order to prove that $w_e = w_o = \mu\pi/2$, recall that the generalized sine and cosine integrals are defined by

$$\text{Si}(z, \mu) := \int_0^z t^{\mu-1} \sin(t) dt \quad \text{and} \quad \text{Ci}(z, \mu) := \int_0^z t^{\mu-1} \cos(t) dt,$$

respectively, and satisfy

$$(3.1) \quad \lim_{z \rightarrow \infty} \text{Si}(z, \mu) = \sin\left(\frac{\mu\pi}{2}\right) \Gamma(\mu) \quad \text{and} \quad \lim_{z \rightarrow \infty} \text{Ci}(z, \mu) = \cos\left(\frac{\mu\pi}{2}\right) \Gamma(\mu)$$

for z tending to ∞ on the real axis and $\mu \in (0, 1)$ [8]. For all $n \in \mathbb{N}$ we have

$$0 = (-1)^n F_\mu(z_n) = \sin(w_n) \text{Ci}(z_n, \mu) - \cos(w_n) \text{Si}(z_n, \mu).$$

Letting $n \rightarrow \infty$, we obtain from (3.1) that both for $w = w_e$ and $w = w_o$,

$$0 = \Gamma(\mu) \sin\left(w - \frac{\mu\pi}{2}\right),$$

and thus that $w_e = w_o = \mu\pi/2$.

Next, observe that, according to the implicit function theorem,

$$z'_n = -\frac{\frac{d}{d\sigma}F_\sigma(z_n)|_{\sigma=\mu}}{F'_\mu(z_n)}.$$

Since $F'_\mu(z_n)$ does not change sign in $(0, 1)$, it follows from Proposition 3.1 that in order to prove that z_{2n-1} is increasing in $(0, 1)$, it will suffice to show

$$(3.2) \quad \left. \frac{d}{d\sigma}F_\sigma(z_{2n-1}) \right|_{\sigma=\mu} = \int_0^{z_{2n-1}} t^{\mu-1} \log(t) \sin(t - w_{2n-1}) dt > 0$$

for $\mu \in (0, 1)$.

To that end, note first that

$$\begin{aligned} \int_0^{z_1} t^{\mu-1} \log(t) \sin(t - w_1) dt &= \int_0^{w_1} t^{\mu-1} \log(t) \sin(t - w_1) dt \\ &\quad + \int_{w_1}^{z_1} t^{\mu-1} \log(t) \sin(t - w_1) dt. \end{aligned}$$

Since $t^{\mu-1} \sin(t - w_1) < 0$ and $t^{\mu-1} \sin(t - w_1) > 0$ for $t \in (0, w_1)$ and $t \in (w_1, z_1)$, respectively, we therefore obtain

$$\int_0^{z_1} t^{\mu-1} \log(t) \sin(t - w_1) dt > \log(w_1) F_\mu(z_1) = 0.$$

We have thus shown that z_1 is a strictly increasing and analytic function in $(0, 1)$. As explained in the paragraph preceding Proposition 1.5, this implies that the function $\mu^*(\rho)$ is well-defined. We can therefore make use of the inequality

$$(3.3) \quad \rho \leq \mu^*(\rho), \quad \rho \in (0, 1),$$

which was established in [7].

Now, before we show (3.2) also for $n = 2, 3, \dots$, we will first complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Because of Proposition 3.1 and what we have shown so far, only the inequality $F_\mu((2n-1+\mu)\pi) \leq 0$, $n \in \mathbb{N}$, remains to be verified. In fact, since

$$\begin{aligned} F_\mu((2n-1+\mu)\pi) &= F_\mu((1+\mu)\pi) + \int_{(1+\mu)\pi}^{(2n-1+\mu)\pi} t^{\mu-1} \sin(t - \mu\pi) dt \\ &\leq F_\mu((1+\mu)\pi), \end{aligned}$$

it suffices to prove that $F_\mu((1+\mu)\pi) \leq 0$ or, equivalently, that $z_1 \leq (1+\mu)\pi$. However, since $\mu^*(\rho) = z_1^{(-1)}((\rho+1)\pi)$, the latter is equivalent to (3.3). $\square$

Proof of Theorem 1.2. It remains to verify (3.2) for $n \in \{2, 3, \dots\}$. Because of (3.1) we have

$$\int_0^\infty t^{\mu-1} \sin(t - x) dt = \Gamma(\mu) \sin\left(\frac{\mu\pi}{2} - x\right), \quad x \in (0, \pi),$$

and thus

$$\int_0^\infty t^{\mu-1} \log(t) \sin(t - x) dt = \Gamma'(\mu) \sin\left(\frac{\mu\pi}{2} - x\right) + \frac{\pi}{2} \Gamma(\mu) \cos\left(\frac{\mu\pi}{2} - x\right).$$

Since by Theorem 1.3 $\mu\pi/2 < w_{2n-1} \leq \mu\pi$, we find that

$$J_n := \int_0^\infty t^{\mu-1} \log(t) \sin(t - w_{2n-1}) dt > 0, \quad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ define the sequence

$$D_{n,k} := \int_{\max\{0, w_{2n-1} + (2k-3)\pi\}}^{w_{2n-1} + (2k-1)\pi} t^{\mu-1} \log(t) \sin(t - w_{2n-1}) dt, \quad k \in \mathbb{N}.$$

Then

$$\sum_{k=1}^{\infty} D_{n,k} = J_n > 0$$

and

$$\sum_{k=1}^n D_{n,k} = \int_0^{z_{2n-1}} t^{\mu-1} \log(t) \sin(t - w_{2n-1}) dt.$$

The proof of the theorem will therefore be complete if we can show that for each $n \in \{2, 3, \dots\}$ there is a $k(n) \in \mathbb{N}$ such that $D_{n,k} \geq 0$ for $k \in \{1, \dots, k(n)\}$ and $D_{n,k} \leq 0$ for $k > k(n)$.

It is easy to check that for every $\mu \in (0, 1)$ there is a $t_\mu > e$ such that $t^{\mu-1} \log(t)$ is positive and increasing in $(1, t_\mu)$ and positive and decreasing in (t_μ, ∞) . Let $k^* \in \{1, 2, 3, \dots\}$ be such that $t_\mu \in [w_{2n-1} + (2k^* - 3)\pi, w_{2n-1} + (2k^* - 1)\pi)$. Then it is clear that $D_{n,k} > 0$ for $k \in \{2, \dots, k^* - 1\}$ and $D_{n,k} < 0$ for $k > k^*$. We therefore set $k(n) = k^*$ if $D_{n,k^*} \geq 0$ and $k(n) = k^* - 1$ otherwise.

$D_{n,1} > 0$ remains to be verified. But since $\mu\pi/2 < w_{2n-1} \leq \mu\pi$ for $\mu \in (0, 1)$, this follows readily from Theorem 4.1 below. $\square$

We have thus shown that for $n \in \mathbb{N}$ the functions z_{2n-1} are strictly increasing in $(0, 1)$. While the functions z_{2n} are not monotonic in $(0, 1)$ for any $n \in \mathbb{N}$ (cf. Proposition 1.4), numerical computation strongly supports the conjecture that the functions z_{2n-1} and z_{2n} , $n \in \mathbb{N}$, are convex and concave, respectively, in $(0, 1)$. This, however, seems quite hard to prove.

4. ON THE SIGN OF $I(\rho, \mu)$

Recall that

$$I(\rho, \mu) := \frac{d}{d\mu} F_\mu((\rho + 1)\pi) = \int_0^{(\rho+1)\pi} t^{\mu-1} \sin(t - \rho\pi) \log(t) dt.$$

In this section we will determine a large subset of $(\rho, \mu) \in (0, 1)^2$ for which $I(\rho, \mu)$ is positive.

To that end, observe first that the function

$$p(\mu) := \int_0^\pi t^{\mu-1} \log(t) \sin(t) dt$$

is strictly increasing on $(0, 1)$ and satisfies $p(0) = -0.53\dots$ and $p(1) = 0.64\dots$. Consequently, the equation $p(\mu) = 0$ has a unique solution μ_0 in $(0, 1)$. The numerical value of μ_0 is $\mu_0 = 0.32\dots$. Further, let M be the union of the three sets

$$M_1 := \{(\rho, \mu) : (10\pi)^{-1} \leq \rho \leq 1, 0 \leq \mu \leq 1\},$$

$$M_2 := \{(\rho, \mu) : 0 < \rho < (10\pi)^{-1}, \mu_0 \leq \mu \leq 1\},$$

$$M_3 := \{(\rho, \mu) : 0 < \mu \leq 2\rho \leq (5\pi)^{-1}\}.$$

We will show the following.

Theorem 4.1. *For all $(\rho, \mu) \in M$ we have $I(\rho, \mu) > 0$. To each $\mu \in (0, \mu_0)$ there is a $\delta_\mu > 0$ such that $I(\rho, \mu) < 0$ for $\rho \in (0, \delta_\mu)$.*

The second statement of this theorem has essentially been shown in the introduction. Hence, it only remains to verify that $I(\rho, \mu) > 0$ for $(\rho, \mu) \in M$. To that end, we need some auxiliary lemmas. For $x \in [0, \pi]$ define

$$f(x) := \int_0^x \log(t) \sin(t-x) dt \quad \text{and} \quad g(x) := \int_0^\pi \frac{\log(t+x)}{t+x} \sin(t) dt.$$

Lemma 4.2. *For $x \in (0, 1)$ we have $f(x) > 0$.*

Proof. If $x \in (0, 1)$, then clearly

$$f(x) \geq \log(x) \int_0^x \sin(t-x) dt = -\log(x) (1 - \cos(x)) > 0.$$

□

Lemma 4.3. *The function $f(x)$ is concave on $(1, \pi)$.*

Proof. We recall that

$$\begin{aligned} \text{Si}(x) &= \int_0^x \frac{\sin(t)}{t} dt = \frac{\pi}{2} - \int_x^\infty \frac{\sin(t)}{t} dt, \\ \text{Ci}(x) &= -\int_x^\infty \frac{\cos(t)}{t} dt = \gamma + \log(x) - \int_0^x \frac{1 - \cos(t)}{t} dt, \end{aligned}$$

where γ is Euler's constant.

It is then easy to see that

$$(4.1) \quad \int_0^x \log(t) \cos(t) dt = \log(x) \sin(x) - \text{Si}(x)$$

and

$$(4.2) \quad \int_0^x \log(t) \sin(t) dt = -\log(x) \cos(x) + \text{Ci}(x) - \gamma.$$

From these, the relation

$$f(x) = -\log(x) + \cos(x) (\text{Ci}(x) - \gamma) + \sin(x) \text{Si}(x)$$

follows. We observe also that

$$f''(x) = -f(x) - \log(x).$$

Therefore

$$(4.3) \quad f''(x) = \cos(x) (-\text{Ci}(x) + \gamma) - \sin(x) \text{Si}(x).$$

It is clear that the function $w(x) := -\text{Ci}(x) + \gamma$ is strictly decreasing on $(0, \pi/2)$ and strictly increasing on $(\pi/2, \pi)$; hence $w(x) \geq w(\pi/2) = 0.1... > 0$. Since $\text{Si}(x) > 0$ for all $x > 0$, it readily follows from (4.3) that $f''(x) < 0$ for $\pi/2 \leq x \leq \pi$. On the other hand, for $1 \leq x < \pi/2$ we have

$$f''(x) \leq (-\text{Ci}(1) + \gamma) \cos(x) - \text{Si}(1) \sin(x) < 0.$$

□

Lemma 4.4. *The function $g(x)$ is concave on $(0, \pi)$.*

Proof. For $x \in (0, \pi)$ we have

$$\begin{aligned} g^{(4)}(x) &= \int_0^\pi \frac{24 \log(t+x) - 50}{(t+x)^5} \sin(t) dt \\ &\leq (24 \log(2\pi) - 50) \int_0^\pi \frac{\sin(t)}{(t+x)^5} dt < 0. \end{aligned}$$

This implies $g'''(x) > g'''(\pi) = 0.011 \dots > 0$, which, in turn, gives $g''(x) < g''(\pi) = -0.0012 \dots < 0$ for $x \in (0, \pi)$. $\square$

We will now prove Theorem 4.1 separately for the three sets M_1 , M_2 , and M_3 . Note that

$$(4.4) \quad I(\rho, \mu) = \int_0^{\rho\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) dt + \int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) dt.$$

Proof of Theorem 4.1 for $(\rho, \mu) \in M_1$. It follows readily from (4.4) that for $(\rho, \mu) \in M_1$,

$$I(\rho, \mu) \geq f(\rho\pi) + g(\rho\pi).$$

It will therefore be enough to prove

$$(4.5) \quad f(x) + g(x) > 0 \quad \text{for } x \in [1/10, \pi].$$

For $x \in [1/10, 1]$ we use Lemmas 4.2 and 4.4 to obtain

$$f(x) + g(x) > g(x) > \min\{g(1/10), g(1)\} = g(1/10) = 0.0819 \dots > 0.$$

For $x \in (1, \pi]$ Lemmas 4.3 and 4.4 give

$$f(x) + g(x) > \min\{f(1) + g(1), f(\pi) + g(\pi)\} = f(\pi) + g(\pi) = 0.0169 \dots > 0.$$

$\square$

Proof of Theorem 4.1 for $(\rho, \mu) \in M_2$. Observe that for $\rho \in (0, \frac{1}{10\pi})$,

$$\int_0^{\rho\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) dt > 0.$$

Equation (4.4) therefore implies

$$\begin{aligned} I(\rho, \mu) &> \int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) dt = \int_0^\pi (t + \rho\pi)^{\mu-1} \log(t + \rho\pi) \sin(t) dt \\ &\geq \int_0^\pi (t + \rho\pi)^{\mu_0-1} \log(t + \rho\pi) \sin(t) dt \geq \int_0^\pi t^{\mu_0-1} \log(t) \sin(t) dt = 0 \end{aligned}$$

for $\rho \in [0, \frac{1}{10\pi})$. Here we have made use of the fact that, for all $t \in (0, \pi)$, $(t + \rho\pi)^{\mu-1} \log(t + \rho\pi)$ is an increasing function of μ when $\rho \in [0, \frac{1}{10\pi})$ and an increasing function of ρ when $\mu = \mu_0$. $\square$

Proof of Theorem 4.1 for $(\rho, \mu) \in M_3$. Since $\rho \in [\frac{\mu}{2}, \frac{1}{10\pi}]$, we have

$$\int_0^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) dt > \int_0^1 t^{\mu-1} \log(t) \sin(t - \rho\pi) dt.$$

The integral on the right-hand side of this inequality can be written as

$$\cos(\rho\pi) \int_0^1 t^{\mu-1} \log(t) \sin(t) dt - \sin(\rho\pi) \int_0^1 t^{\mu-1} \log(t) \cos(t) dt$$

and is thus larger than

$$(4.6) \quad \cos\left(\frac{\mu\pi}{2}\right) \int_0^1 t^{\mu-1} \log(t) \sin(t) dt - \sin\left(\frac{\mu\pi}{2}\right) \int_0^1 t^{\mu-1} \log(t) \cos(t) dt.$$

Applying the estimates $\sin(t) < t$ and $\cos(t) > 1 - 2t/\pi$, $t \in (0, \pi/2)$, and calculating the resulting integrals, we find that the term in (4.6) is larger than

$$\frac{2\mu\pi \sin \frac{\mu\pi}{2} - \pi\mu^2 \cos \frac{\mu\pi}{2} + \sin \frac{\mu\pi}{2} (\mu^2(\pi - 2) + \pi)}{\pi\mu^2(\mu + 1)^2}.$$

It is easy to check that $2 \sin \frac{\mu\pi}{2} > \mu \cos \frac{\mu\pi}{2}$ for $\mu \in (0, \frac{1}{5\pi}]$, and therefore the proof is complete. $\square$

REFERENCES

1. J. Arias de Reyna and J. van de Lune, *High precision computation of a constant in the theory of trigonometric series*, Math. Comp. **78** (2009), no. 268, 2187–2191. MR2521284 (2010j:65063)
2. G. Brown, K. Y. Wang, and D. C. Wilson, *Positivity of some basic cosine sums*, Math. Proc. Cambridge Philos. Soc. **114** (1993), no. 3, 383–391. MR1235986 (94h:42013)
3. M. L. Glasser, *Integral representations for the exceptional univariate Lommel functions*, J. Phys. A **43** (2010), no. 15, 155207, 4 pp. MR2608275 (2011e:33011)
4. S. Koumandos, *An extension of Vietoris's inequalities*, Ramanujan J. **14** (2007), no. 1, 1–38. MR2298638 (2008e:42003)
5. S. Koumandos and M. Lamprecht, *Complete monotonicity and related properties of some special functions*, Math. Comp., to appear.
6. S. Koumandos and S. Ruscheweyh, *Positive Gegenbauer polynomial sums and applications to starlike functions*, Constr. Approx. **23** (2006), no. 2, 197–210. MR2186305 (2007b:42001)
7. ———, *On a conjecture for trigonometric sums and starlike functions*, J. Approx. Theory **149** (2007), no. 1, 42–58. MR2371613 (2009f:42001)
8. E. Kreyszig, *Über den allgemeinen Integralsinus* $\text{Si}(z, \alpha)$, Acta Math. **85** (1951), 117–181. MR0041287 (12:825a)
9. J. Steinig, *The sign of Lommel's function*, Trans. Amer. Math. Soc. **163** (1972), 123–129. MR0284625 (44:1849)
10. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944. MR0010746 (6:64a)
11. A. Zygmund, *Trigonometric series. Vols. I, II*, third ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002. MR1963498 (2004h:01041)

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