PRESCRIBED COMPRESSIONS OF DUAL HYPERCYCLIC OPERATORS

KIT C. CHAN

(Communicated by Richard Rochberg)

ABSTRACT. If M is a closed subspace of a separable, infinite dimensional Hilbert space H with $\dim(H/M) = \infty$, then we show that every bounded linear operator $A : M \rightarrow M$ is the compression of a dual hypercyclic operator $T : H \rightarrow H$.

On a separable, infinite dimensional Banach space X , a bounded linear operator $T : X \rightarrow X$ is said to be *hypercyclic* on X if there is a vector $x \in X$ whose orbit $\text{orb}(T, x) = \{x, Tx, T^2x, T^3x, \dots\}$ is dense in X . Such a vector x is called a *hypercyclic vector* for T . When the operator $T : X \rightarrow X$ is hypercyclic on X and its adjoint operator $T^* : X^* \rightarrow X^*$ is hypercyclic on the dual space X^* of X , then we say that T is *dual hypercyclic*. Since an orbit is a countable set, dual hypercyclicity can only take place when both X and X^* are separable. However, X is separable whenever its dual X^* is, but the converse is not always true.

Separability does not present an issue when the Banach space X is indeed a Hilbert space H , because the adjoint T^* is a bounded linear operator on H itself. In fact, it was the Hilbert space setting that the concept of dual hypercyclicity started to develop. A fundamental question is whether dual hypercyclic operators on H can ever exist. This question was originally raised by Herrero [4]. The first example of such an operator was found by Salas [6]. Later he [7] provided another example using a general result for hypercyclic bilateral weighted shift operators.

Recently generalizations of dual hypercyclic operators to a Banach space X were studied. For instance, Petersson [5] showed that any infinite dimensional Banach space X with a shrinking symmetric basis, such as c_0 and ℓ^p with $1 < p < \infty$, admits a dual hypercyclic operator $T : X \rightarrow X$. Then Salas [8] showed that any Banach space X with a separable dual space X^* admits a dual hypercyclic operator. More recently, Shkarin [10] studied dual hypercyclic tuples of operators on Banach spaces, and Salas [9] studied dual disjoint hypercyclic operators.

In the present paper we return to the setting of a separable, infinite dimensional Hilbert space H and study compressions of dual hypercyclic operators $T : H \rightarrow H$, making use of unique Hilbert space properties. Our main result is Theorem 2 below, which states that the compression of a dual hypercyclic operator T onto a closed subspace M of infinite codimension in H can coincide with any prescribed operator A on M . In other words, if $P : H \rightarrow H$ is the orthogonal projection onto a closed subspace M with $\dim(H/M) = \infty$, then for any bounded linear

Received by the editors March 21, 2011.

2010 *Mathematics Subject Classification*. Primary 47A16, 47A20; Secondary 47A05.

Key words and phrases. Adjoint operator, dual hypercyclic operator, compression.

©2012 American Mathematical Society
 Reverts to public domain 28 years from publication

operator $A : M \rightarrow M$, there is a dual hypercyclic operator $T : H \rightarrow H$ such that $PTP|_M = A$. Compressions of a hypercyclic operator were studied by Salas [7], who showed that if $T : H \rightarrow H$ is a hypercyclic operator and N is an invariant subspace of its adjoint T^* , then the compression of T to N is hypercyclic on N .

To prove Theorem 2, we additively decompose the Hilbert space H as an orthogonal sum of infinitely many copies of a nonzero closed subspace M with $\dim(H/M) = \infty$; that is, $H = \bigoplus_{j=-\infty}^{\infty} M_j$, where each M_j is a closed subspace of H that is isomorphic to M and $M_i \perp M_j$ whenever $i \neq j$. We identify the original closed subspace M with M_0 in H , and the prescribed operator $A : M \rightarrow M$ becomes an operator $A : M_0 \rightarrow M_0$. By extending A so that $A|_{M_0^\perp} = 0$, we can view A as a bounded linear operator $A : H \rightarrow H$. Assume $\dim M = \infty$, and let $\{e_{j,1}, e_{j,2}, e_{j,3}, \dots\}$ be an orthonormal basis of M_j . Hence $e_{j,i} \perp e_{m,n}$ whenever $(j,i) \neq (m,n)$. Let $S : H \rightarrow H$ be the unitary operator given by $Se_{j,i} = e_{j+1,i}$ whenever $i, j \in \mathbb{Z}$. In fact, $S|_{M_j}$ is an isomorphism between M_j and M_{j+1} . Lastly, we need a number $a > \max\{1, \|A\|\}$ to help set up the following technical lemma.

Lemma 1. *Suppose k is a positive integer, and for each integer j with $|j| \leq k$ we are given a positive weight w_j satisfying $0 < w_j \leq a$ and also vectors x_j and y_j in M_j . Let $\epsilon > 0$. Then there exists an integer $n \geq 2k+2$, and there exist weights w_j with $0 < w_j \leq a$ and also vectors $x_j \in M_j$ for all integers j with $1+k \leq |j| \leq n+k$ such that if D is defined by*

$$(1) \quad De_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n+k, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$(i) \quad \left\| (D+A)^n \left(\sum_{i=-k}^{n-k-1} x_i \right) \right\| < \epsilon,$$

$$(ii) \quad (D+A)^n \left(\sum_{i=n-k}^{n+k} x_i \right) = \sum_{i=-k}^k y_i,$$

$$(iii) \quad \sum_{i=1+k}^{n+k} \|x_i\|^2 < \epsilon, \quad \text{and}$$

$$(iv) \quad x_i = 0 \text{ whenever } -n-k \leq i \leq -1-k.$$

Proof. Before we determine a desired integer n for the definition of D in (1), we first observe that if $j \neq 0$, then

$$(D+A)|_{M_j} = D|_{M_j}.$$

We also observe that if m is an integer with $1+k \leq m \leq n+k$, then for any vector $x_m \in M_m$, (1) gives

$$(2) \quad (D+A)^m \left(x_m + \sum_{i=-k}^k x_i \right) \perp M_j \quad \text{whenever } j \geq 1.$$

Note that w_j are given positive numbers whenever $|j| \leq k$. We now define $w_j = a$ for all indices j with $1+k \leq j \leq m$. If we let $P : H \rightarrow H$ be the orthogonal projection onto M_0 , then we trivially have

$$P(D+A)^m \left(\sum_{i=-k}^k x_i \right) \in M_0.$$

Furthermore, if we specifically take the vector x_m in M_m to be

$$x_m = \frac{-1}{w_1 \cdots w_k w_{k+1} \cdots w_m} S^m P(D+A)^m \left(\sum_{i=-k}^k x_i \right),$$

then we can take $P(D+A)^m$ on both sides of the above equation to obtain

$$P(D+A)^m \left(\sum_{i=-k}^k x_i \right) = -P(D+A)^m x_m.$$

By combining with (2), we have

$$(3) \quad (D+A)^m \left(x_m + \sum_{i=-k}^k x_i \right) \perp M_j \quad \text{whenever } j \geq 0.$$

Recall that $w_{k+1} = \cdots = w_m = a > \|A\|$ and $\|S\| = 1$, and note that x_m can be rewritten as

$$x_m = \frac{-1}{w_1 \cdots w_k w_{k+1} \cdots w_m} S^m P A^{m-k} (D+A)^k \left(\sum_{i=-k}^k x_i \right).$$

Thus we can first choose a large enough integer m , even before we determine the desired integer n for the definition of D , so that the above formulation of x_m gives

$$(4) \quad \|x_m\|^2 < \frac{\epsilon}{2}.$$

Then we determine a large enough integer n with $n \geq m+k+1 \geq 2k+2$, so that if we define $w_{-1-k} = \cdots = w_{-n-k} = a^{-1} < 1$, then it follows from (1) and (3) that

$$\left\| (D+A)^n \left(x_m + \sum_{i=-k}^k x_i \right) \right\| < \epsilon.$$

Thus if we define $x_{1+k} = \cdots = x_{m-1} = 0$ and $x_{m+1} = \cdots = x_{n-k-1} = 0$, then (i) is clearly satisfied.

We now check how (ii) can be satisfied whenever $n \geq 2k+2$ by appropriately choosing $x_{n-k}, \dots, x_{n+k}$. For that, let $w_{m+1} = \cdots = w_{n+k} = a$ to complete the definitions for w_j in the whole range $1+k \leq |j| \leq n+k$ as required by the lemma. For the given vectors y_j in M_j with $|j| \leq k$ as in the statement of our lemma, we first let

$$(5) \quad x_{n-k} = \frac{1}{w_{1-k} \cdots w_{n-k}} S^n y_{-k}.$$

Next, for every index i with $1-k \leq i \leq 0$, let

$$(6) \quad x_{n+i} = \frac{1}{w_{1+i} \cdots w_{n+i}} \left(S^n y_i - \frac{1}{w_i} S^{n+i} A S^{1-i} y_{i-1} \right).$$

Lastly, for every index i with $1 \leq i \leq k$, let

$$(7) \quad x_{n+i} = \frac{1}{w_{1+i} \cdots w_{n+i}} S^n y_i.$$

One can easily verify that the above definitions give $x_i \in M_i$ whenever $n - k \leq i \leq n + k$.

To establish (ii), we first make an observation that if $1 - k \leq i \leq 0$, then

$$\begin{aligned} & (D + A)^n \left(\frac{S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i-1} w_{n+i}} \right) \\ &= (D + A)^{-i+1} \left(\frac{(D + A)^{n+i-1} S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{(D + A)^{n+i-1} S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i-1} w_{n+i}} \right). \end{aligned}$$

To continue our computations, we note that $S^n y_{i-1} \in M_{n+i-1}$ and $S^{n+i} A S^{1-i} y_{i-1} \in M_{n+i}$. Thus we can simplify the above expression as

$$(D + A)^{-i+1} \left(\frac{S^{1-i} y_{i-1}}{w_i \cdots w_0} - \frac{S A S^{1-i} y_{i-1}}{w_i \cdots w_0 w_1} \right).$$

Since $S^{1-i} y_{i-1} \in M_0$ and $S A S^{1-i} y_{i-1} \in M_1$, we can further reduce the above expression to

$$\begin{aligned} & (D + A)^{-i} \left(\left(\frac{S^{-i} y_{i-1}}{w_i \cdots w_{-1}} + \frac{A S^{1-i} y_{i-1}}{w_i \cdots w_0} \right) - \frac{A S^{1-i} y_{i-1}}{w_i \cdots w_0} \right) \\ &= (D + A)^{-i} \left(\frac{S^{-i} y_{i-1}}{w_i \cdots w_{-1}} \right) \\ &= y_{i-1}. \end{aligned}$$

Hence we conclude from our above observation that if $1 - k \leq i \leq 0$, then

$$(8) \quad (D + A)^n \left(\frac{S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i-1} w_{n+i}} \right) = y_{i-1}.$$

We are now ready to establish (ii) by first using the definitions of x_i in (5), (6), and (7):

$$\begin{aligned} & (D + A)^n \left(\sum_{i=n-k}^{n+k} x_i \right) \\ &= (D + A)^n \left(\frac{S^n y_{-k}}{w_{1-k} \cdots w_{n-k}} + \right. \\ & \quad \left. + \sum_{i=1-k}^0 \frac{1}{w_{1+i} \cdots w_{n+i}} (S^n y_i - \frac{1}{w_i} S^{n+i} A S^{1-i} y_{i-1}) + \sum_{i=1}^k x_{n+i} \right), \end{aligned}$$

which can be rearranged as

$$\begin{aligned} & (D + A)^n \left(\sum_{i=1-k}^0 \left(\frac{S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i}} \right) \right) \\ & + (D + A)^n \left(\frac{S^n y_0}{w_1 \cdots w_n} \right) + (D + A)^n \left(\sum_{i=1}^k \frac{S^n y_i}{w_{1+i} \cdots w_{n+i}} \right) \\ & = \sum_{i=1-k}^0 y_{i-1} + y_0 + \sum_{i=1}^k y_i, \text{ by (8) and the fact that } S^n y_i \in M_{n+i}. \end{aligned}$$

Hence we have established (ii).

Since we have set $w_{m+1} = \cdots = w_{n+k} = a > 1$, and S is an isometry, we can assume that n is chosen large enough so that the definitions for $x_{n-k}, \dots, x_{n+k}$ in (5), (6), (7) give

$$\sum_{i=n-k}^{n+k} \|x_i\|^2 < \frac{\epsilon}{2}.$$

This inequality, along with (4) and the fact that $x_{1+k} = \cdots = x_{m-1} = x_{m+1} = \cdots = x_{n-k-1} = 0$, yields (iii). The proof of the lemma is completed by setting $x_{-1-k} = \cdots = x_{-n-k} = 0$ to establish (iv). $\square$

For an arbitrary operator $A : M_0 \rightarrow M_0$, the above lemma provides an operator D in (1) that shifts some orthonormal basis members in the backward direction, that is, taking some $e_{j,i}$ to $w_j e_{j-1,i}$. For the adjoint operator $A^* : M_0 \rightarrow M_0$, one can easily modify the techniques in the lemma to provide an operator D' which shifts in the forward direction, taking some $e_{j,i}$ to $w_{j+1} e_{j+1,i}$. We omit the statement of the forward shifting version of the lemma.

To prove our main result in Theorem 2 below, we apply the lemma to A and then apply the forward shifting version of the lemma to A^* . The two applications to A and A^* are then repeated inductively.

Theorem 2. *Let H be a separable, infinite dimensional Hilbert space and let M be a closed subspace of H with $\dim(H/M) = \infty$. Let $P : H \rightarrow H$ be the orthogonal projection onto M . If $A : M \rightarrow M$ is a bounded linear operator, then there is an operator $T : H \rightarrow H$ such that*

- (1) T is dual hypercyclic,
- (2) $PTP|_M = A$, and
- (3) $PT^*P|_M = A^*$.

Proof. Since M has infinite codimension in H , we can use an orthonormal basis argument to additively decompose H as an orthogonal sum $H = \bigoplus_{j=-\infty}^{\infty} M_j$, where M_0 is the closed subspace M given in the statement of our theorem and each M_j is isomorphic to M .

In the rest of the proof, we assume that $\dim M = \infty$, and the same argument works if $\dim M$ is finite. For that, let $\{e_{j,1}, e_{j,2}, e_{j,3}, \dots\}$ be an orthonormal basis of M_j . Since A takes M_0 to M_0 , we can view A as an operator from H to H with $A|_{M_0^\perp} = 0$. It follows that $\overline{\text{ran } A^*} = (\ker A)^\perp \subset M_0$ and also that $M_0^\perp \subset (\text{ran } A)^\perp = \ker A^*$, and so we can also view the original adjoint operator $A^* : M_0 \rightarrow M_0$ as

an operator from H to H with $A^*|_{M_0^\perp} = 0$. That allows us to define an operator $T : H \rightarrow H$ by $T = A + B$ and so $T^* = A^* + B^*$, where $B : H \rightarrow H$ is a linear map defined by $Be_{j,i} = w_j e_{j-1,i}$ for all integers i, j and $\{w_j : j \in \mathbb{Z}\}$ is a bounded two-sided sequence of positive numbers. Thus B takes each M_j to M_{j-1} and $B|_{M_j}$ is in the form of “ w_j times a Hilbert space isomorphism”. In fact, if $S : H \rightarrow H$ is the unitary operator given by $Se_{j,i} = e_{j+1,i}$ whenever $i, j \in \mathbb{Z}$, then $S|_{M_j}$ is an isomorphism from M_j to M_{j+1} , and $B|_{M_j} = w_j S^{-1}|_{M_j}$. It is easy to verify that $B^*|_{M_j} = w_{j+1} S|_{M_j}$; that is, $B^*e_{j,i} = w_{j+1} e_{j+1,i}$. In addition, the definition of T gives $PTP|_{M_0} = A$ and $PT^*P|_{M_0} = A^*$. In fact, B is a bilateral backward shift, and $\|B\| = \sup |w_j| < \infty$. In the rest of the proof we need to choose w_j so that T and T^* are hypercyclic.

Let a be a number with $a > \max\{1, \|A\|\}$, and let $v_1, v_2, v_3, \dots$ be an enumeration of all vectors, each of which has only a finite number of nonzero coefficients $\langle v, e_{j,i} \rangle$ and all coefficients are rationals. The set of all v_i is dense in H . Let $k_1 \geq 1$ be such that $v_1 \in \bigoplus_{|j| \leq k_1} M_j$. After choosing k_1 , let $w_j = a$ and $x_j = 0$ whenever $|j| \leq k_1$, and write $v_1 = \sum_{|j| \leq k_1} y_j$ with each $y_j \in M_j$. Let $\epsilon_1 = (2a^{2k_1})^{-1} > 0$. The lemma provides an integer n_1 and weights w_j with $0 < w_j \leq a$ and vectors $x_j \in M_j$ whenever $1 + k_1 \leq |j| \leq n_1 + k_1$ such that if D_1 is given by

$$D_1 e_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n_1 + k_1, \\ 0 & \text{otherwise,} \end{cases}$$

then statements (i), (ii), (iii), and (iv) are satisfied.

With the above setting, we apply the lemma to the operator A^* in a direction opposite to the statement of the lemma, that is, to obtain a forward shift operator D' . Hence, for the same vector $v_1 = \sum_{|j| \leq k_1} y_j$ we can continue to define w_j and x_j by setting $k'_1 = n_1 + k_1$ and $\epsilon'_1 = (2a^{2k'_1})^{-1} > 0$. The lemma provides an integer n'_1 and weights w_j with $0 < w_j \leq a$, and vectors $x_j \in M_j$, for all indices j in the range $1 + k'_1 \leq |j| \leq n'_1 + k'_1$. Since we are applying the statement of the lemma in the opposite direction, statement (iv) gives $x_i = 0$ for those indices i in the range $1 + k'_1 \leq i \leq n'_1 + k'_1$, and statement (iii) gives

$$\sum_{i=-n'_1-k'_1}^{-1-k'_1} \|x_i\|^2 < \frac{1}{2a^{2k'_1}}.$$

In addition, the lemma provides a forward shift operator D'_1 by

$$D'_1 e_{j,i} = \begin{cases} w_{j+1} e_{j+1,i} & \text{if } i \in \mathbb{Z} \text{ and } -n'_1 - k'_1 \leq j \leq n'_1 + k'_1 - 1, \\ 0 & \text{otherwise,} \end{cases}$$

such that statements (i) and (ii) become

$$\left\| (D'_1 + A^*)^{n'_1} \left(\sum_{i=-n'_1+k'_1+1}^{k'_1} x_i \right) \right\| < \epsilon'_1$$

and

$$(D'_1 + A^*)^{n'_1} \left(\sum_{i=-n'_1-k'_1}^{-n'_1+k'_1} x_i \right) = v_1.$$

Note that the lemma provides the term $w_{-n'_1-k'_1}$ that we do not actually use in the above definition of D'_1 , and this term causes no consequence in our subsequent argument. In the second step, we let $k_2 = n'_1 + k'_1$, and $\epsilon_2 = (2^2 a^{2k_2})^{-1} > 0$ and assume that, without loss of generality, $v_2 \in \bigoplus_{|j| \leq k_2} M_j$. In the case that v_2 is not in the subspace $\bigoplus_{|j| \leq k_2} M_j$, we can choose v_i with the least integer i such that v_i is in that subspace. We can use the lemma to define operator D_2 and then D'_2 for the same vector v_2 , as we define D_1 and D'_1 in the previous case for v_1 .

Inductively, in the m -th step we take $k_m = n'_{m-1} + k'_{m-1}$ and $\epsilon_m = (2^m a^{2k_m})^{-1} > 0$ and we can assume, without loss of generality, that $v_m \in \bigoplus_{|j| \leq k_m} M_j$. Together with those w_j and x_j already defined for indices j with $|j| \leq n'_{m-1} + k'_{m-1} = k_m$ in the $(m-1)$ -th step, we obtain from the lemma an integer n_m and weights w_j with $0 < w_j \leq a$, and vectors x_j in M_j , where $|j| \leq n_m + k_m$ and in fact,

$$(9) \quad x_i = 0 \text{ for those indices } i \text{ in the range } -n_m - k_m \leq i \leq -1 - k_m,$$

such that if D_m is given by

$$D_m e_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n_m + k_m, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$(10) \quad \sum_{i=1+k_m}^{n_m+k_m} \|x_i\|^2 < \frac{1}{2^m a^{2k_m}}$$

and

$$(11) \quad \left\| (D_m + A)^{n_m} \left(\sum_{i=-k_m}^{n_m-k_m-1} x_i \right) \right\| < \frac{1}{2^m a^{2k_m}}$$

and also

$$(12) \quad (D_m + A)^{n_m} \left(\sum_{i=n_m-k_m}^{n_m+k_m} x_i \right) = v_m.$$

Next we apply the lemma to obtain a forward shift D'_m for the same vector v_m , but for the adjoint A^* instead of A itself. For that purpose, let $k'_m = n_m + k_m$ and $\epsilon'_m = (2^m a^{2k'_m})^{-1} > 0$. Together with those w_j and x_j that have already been defined for indices j in the range $-k'_m \leq j \leq k'_m$, we obtain from the lemma an integer n'_m and weights w_j with $0 < w_j \leq a$ and vectors x_j in M_j whenever $|j| \leq n'_m + k'_m$, and in fact

$$(13) \quad x_i = 0 \text{ for those indices } i \text{ in the range } 1 + k'_m \leq i \leq n'_m + k'_m,$$

such that

$$(14) \quad \sum_{i=-n'_m-k'_m}^{-1-k'_m} \|x_i\|^2 < \frac{1}{2^m a^{2k'_m}}.$$

In addition, the lemma provides an operator D'_m given by

$$D'_m e_{j,i} = \begin{cases} w_{j+1} e_{j+1,i} & \text{if } i \in \mathbb{Z} \text{ and } -n'_m - k'_m \leq j \leq n'_m + k'_m - 1, \\ 0 & \text{otherwise,} \end{cases}$$

such that

$$(15) \quad \left\| (D'_m + A^*)^{n'_m} \left(\sum_{i=-n'_m+k'_m+1}^{k'_m} x_i \right) \right\| < \epsilon'_m$$

and also

$$(16) \quad (D'_m + A^*)^{n'_m} \left(\sum_{i=-n'_m-k'_m}^{-n'_m+k'_m} x_i \right) = v_m.$$

With all weights w_j given by the inductive process above, we define an operator $B : H \rightarrow H$ by $B|_{M_j} = w_j S^{-1}|_{M_j}$ as we have outlined in the beginning of this proof. The operator B is bounded because $0 < w_j \leq a$. Also, with all vectors x_j given by the inductive process, we let $x = \sum_{j \in \mathbb{Z}} x_j$, which represents a vector in H because the terms x_j are mutually orthogonal and square summable. To verify that, we note that $k'_m = n_m + k_m$, and $k_{m+1} = n'_m + k'_m$ by their definitions, and hence

$$\begin{aligned} & \sum_{i=1+k_1}^{\infty} \|x_i\|^2 \\ &= \sum_{m=1}^{\infty} \left(\sum_{i=1+k_m}^{n_m+k_m} \|x_i\|^2 + \sum_{i=1+k'_m}^{n'_m+k'_m} \|x_i\|^2 \right) \\ &= \sum_{m=1}^{\infty} \left(\sum_{i=1+k_m}^{n_m+k_m} \|x_i\|^2 \right) \quad \text{by (13),} \end{aligned}$$

and hence by (10) we have

$$\sum_{i=1+k_1}^{\infty} \|x_i\|^2 < \sum_{m=1}^{\infty} \frac{1}{2^m a^{2k_m}} < \infty.$$

Similarly, using (9) and (14), we obtain

$$\sum_{i=-\infty}^{-k'_1-1} \|x_i\|^2 < \infty,$$

and so the sum $\sum_{j \in \mathbb{Z}} x_j$ defines a vector in H .

We now proceed to prove that if we write $x = \sum_{j \in \mathbb{Z}} x_j$, then x is a hypercyclic vector for both $A + B$ and $A^* + B^*$. We begin by rewriting (12) as

$$(A + B)^{n_m} \left(\sum_{i=n_m-k_m}^{n_m+k_m} x_i \right) = v_m,$$

and hence by orthogonality we have

$$(17) \quad \begin{aligned} & \| (A + B)^{n_m} x - v_m \|^2 \\ &= \left\| (A + B)^{n_m} \left(\sum_{i=-\infty}^{n_m-k_m-1} x_i \right) \right\|^2 + \left\| (A + B)^{n_m} \left(\sum_{i=n_m+k_m+1}^{\infty} x_i \right) \right\|^2. \end{aligned}$$

We continue our computations with the two summands in (17) separately. For the first summand we note that by orthogonality, we can write it as the sum of two terms:

$$(18) \quad \left\| (A+B)^{n_m} \left(\sum_{i=-n_m-k_m}^{n_m-k_m-1} x_i \right) \right\|^2 + \left\| B^{n_m} \left(\sum_{i=-\infty}^{-n_m-k_m-1} x_i \right) \right\|^2.$$

To continue our estimations, note that (9) and (11) give

$$\begin{aligned} & \left\| (A+B)^{n_m} \left(\sum_{i=-n_m-k_m}^{n_m-k_m-1} x_i \right) \right\|^2 \\ &= \left\| (A+B)^{n_m} \left(\sum_{i=-k_m}^{n_m-k_m-1} x_i \right) \right\|^2 \\ &< \left(\frac{1}{2^m a^{2k_m}} \right)^2. \end{aligned}$$

Thus we obtain an estimate for the first summand in (17) using (9) and (18) as follows:

$$\left\| (A+B)^{n_m} \left(\sum_{i=-\infty}^{n_m-k_m-1} x_i \right) \right\|^2 < \left(\frac{1}{2^m a^{2k_m}} \right)^2 + a^{2n_m} \sum_{j=m}^{\infty} \sum_{i=-k_{j+1}}^{-n_j-k_j-1} \|x_i\|^2.$$

Note that $k'_j = n_j + k_j$ and $k_{j+1} = n'_j + k'_j$, and so

$$\begin{aligned} a^{2n_m} \sum_{j=m}^{\infty} \sum_{i=-k_{j+1}}^{-n_j-k_j-1} \|x_i\|^2 &= a^{2n_m} \sum_{j=m}^{\infty} \left(\sum_{i=-n'_j-k'_j}^{-1-k'_j} \|x_i\|^2 \right) \\ &< a^{2n_m} \sum_{j=m}^{\infty} \frac{1}{2^j a^{2k'_j}}, \text{ by (14)} \\ &< \sum_{j=m}^{\infty} \frac{1}{2^j}, \text{ (because } k'_j \geq k'_m = n_m + k_m) \\ &= 2^{1-m}. \end{aligned}$$

Hence the first summand in (17) is bounded above by $(2^m a^{2k_m})^{-2} + 2^{1-m}$, which goes to 0 as m goes to ∞ .

To estimate the second summand in (17) we first use (13) to obtain that $x_i = 0$ for those indices i in the range $1 + k'_j = n_j + k_j + 1 \leq i \leq n'_j + k'_j = k_{j+1}$, and hence

$$\begin{aligned}
 & \left\| (A + B)^{n_m} \left(\sum_{i=n_m+k_m+1}^{\infty} x_i \right) \right\|^2 \\
 &= \left\| B^{n_m} \left(\sum_{j=m}^{\infty} \sum_{i=1+k_{j+1}}^{n_{j+1}+k_{j+1}} x_i \right) \right\|^2 \\
 &\leq \|B\|^{2n_m} \left(\sum_{j=m}^{\infty} \sum_{i=1+k_{j+1}}^{n_{j+1}+k_{j+1}} \|x_i\|^2 \right) \\
 &< a^{2n_m} \sum_{j=m}^{\infty} \left(\frac{1}{2^{j+1} a^{2k_{j+1}}} \right) \quad \text{by (10)} \\
 &\leq \sum_{j=m}^{\infty} \frac{1}{2^{j+1}} \quad (\text{because } k_{j+1} \geq k_{m+1} > k'_m > n_m) \\
 &= 2^{-m},
 \end{aligned}$$

which goes to 0 as m goes to ∞ . Since $T = A + B$, it follows from (17) that $\|T^{n_m}x - v_m\|^2 \rightarrow 0$ as $m \rightarrow \infty$. Recall that the set $\{v_1, v_2, v_3, \dots\}$ is dense in H , and so is the set $\{x, Tx, T^2x, \dots\}$. In other words, T is hypercyclic.

A similar argument using (9), (10), (13), (14), (15), and (16) shows that $T^* = A^* + B^*$ is also hypercyclic. Thus T is dual hypercyclic. $\square$

In the above proof, we construct a vector x that is hypercyclic for both $A + B$ and $A^* + B^*$. As a matter of fact, for any two hypercyclic operators T_1 and T_2 on H , there must be a vector that is hypercyclic for both operators because for any hypercyclic operator T , the set of all hypercyclic vectors $HC(T)$ is a dense G_δ set. By the Baire Category Theorem, $HC(T_1) \cap HC(T_2)$ is also a dense G_δ set.

In the above theorem, if we take M to be a one dimensional subspace and $A : M \rightarrow M$ to be the zero operator, then the proof provides a bilateral weighted shift operator that is dual hypercyclic. This result was obtained by Salas [7].

It may be a surprising fact that dual hypercyclic operators can exist on a separable, infinite dimensional Hilbert space H , but Theorem 2 shows that they are indeed very common in terms of prescribed behaviors of their compressions on a closed subspace M with infinite codimension in H . For such subspaces M , Chan and Turcu [2] and Grivaux [3] showed that any bounded linear operator $A : M \rightarrow M$ can be extended to a chaotic operator $T : H \rightarrow H$. Here, a *chaotic operator* is a hypercyclic operator with a dense set of periodic points x ; that is, $T^n x = x$ for some positive integer n depending on x . However, a chaotic operator cannot be dual hypercyclic, because if T has a periodic point, say $T^n x = x$, then for any vector y in H the set $\{\langle T^{*kn}y, x \rangle \mid k \geq 0\}$ cannot be dense in the scalar field, and hence T^{*n} cannot have a hypercyclic vector. That contradicts Ansari's result [1] that the n -th power of a hypercyclic operator is also hypercyclic.

REFERENCES

- [1] S. I. Ansari, Hypercyclic and cyclic vectors, *Journal of Functional Analysis*, **128** (1995), 374-383. MR1319961 (96h:47002)
- [2] K. C. Chan and G. Turcu, Chaotic extensions of operators on Hilbert subspaces, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A. Matemáticas*, **105** (2011), 415-421.
- [3] S. Grivaux, Topologically transitive extensions of bounded operators, *Math Z.*, **249** (2005), 85-96. MR2106971 (2005g:47008)
- [4] D. A. Herrero, Limits of hypercyclic and supercyclic operators, *Journal of Functional Analysis*, **99** (1991), 179-190. MR1120920 (92g:47026)
- [5] H. Petersson, Spaces that admit hypercyclic operators with hypercyclic adjoints, *Proceedings of the American Mathematical Society*, **134** (2006), 1671-1676. MR2204278 (2007b:47020)
- [6] H. Salas, A hypercyclic operator whose adjoint is also hypercyclic, *Proceedings of the American Mathematical Society*, **112** (1991), 765-770. MR1049848 (91j:47016)
- [7] H. Salas, Hypercyclic weighted shifts, *Transactions of the American Mathematical Society*, **347** (1995), 993-1004. MR1249890 (95e:47042)
- [8] H. Salas, Banach spaces with separable duals support dual hypercyclic operators, *Glasgow Mathematical Journal*, **49** (2007), 281-290. MR2347261 (2008j:47008)
- [9] H. Salas, Dual disjoint hypercyclic operators, *J. Math. Anal. Appl.*, **374** (2011), 106-117. MR2726191
- [10] S. Shkarin, A short proof of existence of disjoint hypercyclic operators, *Journal of Mathematical Analysis and Applications*, **367** (2010), 713-715, MR2607296 (2011f:47009)

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOWLING GREEN STATE UNIVERSITY,
BOWLING GREEN, OHIO 43403

E-mail address: `kchan@bgsu.edu`