COMPACTNESS ESTIMATES FOR $\square_b$ ON A CR MANIFOLD

TRAN VU KHANH, STEFANO PINTON, AND GIUSEPPE ZAMPIERI

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ABSTRACT. This paper aims to state compactness estimates for the Kohn-Laplacian on an abstract CR manifold in full generality. The approach consists of a tangential basic estimate in the formulation given by the first author in his thesis, which refines former work by Nicoara. It has been proved by Raich that on a CR manifold of dimension $2n - 1$ which is compact pseudoconvex of hypersurface type embedded in the complex Euclidean space and orientable, the property named “ $(CR - P_q)$ ” for $1 \leq q \leq \frac{n-1}{2}$, a generalization of the one introduced by Catlin, implies compactness estimates for the Kohn-Laplacian $\square_b$ in any degree k satisfying $q \leq k \leq n - 1 - q$. The same result is stated by Straube without the assumption of orientability. We regain these results by a simplified method and extend the conclusions to CR manifolds which are not necessarily embedded nor orientable. In this general setting, we also prove compactness estimates in degree $k = 0$ and $k = n - 1$ under the assumption of $(CR - P_1)$ and, when $n = 2$, of closed range for $\bar{\partial}_b$. For $n \geq 3$, this refines former work by Raich and Straube and separately by Straube.

1. INTRODUCTION AND STATEMENTS

Let M be a compact pseudoconvex CR manifold of hypersurface type of real dimension $2n - 1$ endowed with the Cauchy-Riemann structure $T^{1,0}M$. We choose a basis $L_1, \dots, L_{n-1}$ of $T^{1,0}M$, the conjugated basis $\bar{L}_1, \dots, \bar{L}_{n-1}$ of $T^{0,1}M$, and a transversal, purely imaginary, vector field T . We also take a hermitian metric on the complexified tangent bundle in which we get an orthogonal decomposition $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T$. We denote by $\omega_1, \dots, \omega_{n-1}, \bar{\omega}_1, \dots, \bar{\omega}_{n-1}, \gamma$ the dual basis of 1-forms. We denote by $\mathcal{L}_M$ the Levi form defined by $\mathcal{L}_M(L, \bar{L}') := d\gamma(L, \bar{L}')$ for $L, L' \in T^{1,0}M$. The coefficients of the matrix (c_{ij}) of $\mathcal{L}_M$ in the above basis are described through the Cartan formula as

$$c_{ij} = \langle \gamma, [L_i, \bar{L}_j] \rangle.$$

We denote by $\mathcal{B}^k$ the space of $(0, k)$ -forms u with C^∞ coefficients. They are expressed, in the local basis, as $u = \sum'_{|J|=k} u_J \bar{\omega}_J$ for $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \dots \wedge \bar{\omega}_{j_k}$. Associated to the Riemannian metric $\langle \cdot, \cdot \rangle_z$, $z \in M$, and to the element of volume dV , there is a L^2 -inner product $(u, v) = \int_M \langle u, v \rangle_z dV$. We denote by $(L^2)^k$ the completion of $\mathcal{B}^k$ under this norm. We also use the notation $(H^s)^k$ for the completion under the Sobolev norm H^s . The de-Rham exterior derivative induces a complex $\bar{\partial}_b : \mathcal{B}^k \rightarrow \mathcal{B}^{k+1}$. We denote by $\bar{\partial}_b^* : \mathcal{B}^k \rightarrow \mathcal{B}^{k-1}$ the adjoint and set $\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$. Let φ be a smooth function, denote by (φ_{ij}) the matrix of the

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Levi form $\mathcal{L}_\varphi = \frac{1}{2}(\partial_b \bar{\partial}_b - \bar{\partial}_b \partial_b)(\varphi)$ in the basis above, and by $\lambda_1^{\varphi^\epsilon} \leq \dots \leq \lambda_{n-1}^{\varphi^\epsilon}$ the ordered eigenvalues of $\mathcal{L}_\varphi$. Let L_φ^2 be the L^2 space weighted by $e^{-\varphi}$ and, for $\varphi_j := L_j(\varphi)$, denote by $L_j^\varphi = L_j - \varphi_j$ the L_φ^2 -adjoint of $-\bar{L}_j$. The following is the tangential version of the celebrated Hörmander-Kohn-Morrey basic estimate. Here we present the refinement by Khanh [Kh10] of a former statement by Nicoara [N06]. Let $z_o \in M$. For a suitable neighborhood U of z_o and a constant $c > 0$, we have

$$\begin{aligned}
 & \|\bar{\partial}_b u\|_\varphi^2 + \|\bar{\partial}_{b,\varphi}^* u\|_\varphi^2 + c\|u\|_\varphi^2 \\
 & \geq \sum'_{|K|=k-1} \sum_{i,j} (\varphi_{ij} u_{iK}, u_{jK})_\varphi - \sum'_{|J|=k} \sum_{j=1}^{q_o} (\varphi_{jj} u_J, u_J)_\varphi \\
 (1.1) \quad & + \sum'_{|K|=k-1} \sum_{i,j} (c_{ij} T u_{iK}, u_{jK})_\varphi - \sum'_{|J|=k} \sum_{j=1}^{q_o} (c_{jj} T u_J, u_J)_\varphi \\
 & + \frac{1}{2} \left(\sum_{j=1}^{q_o} \|L_j^\varphi u\|_\varphi^2 + \sum_{j=q_o+1}^{n-1} \|\bar{L}_j u\|_\varphi^2 \right),
 \end{aligned}$$

for any $u \in \mathcal{B}_c^k(U)$ where q_o is any integer with $0 \leq q_o \leq n-1$. We now introduce a potential-theoretical condition which is a variant of the “ P -property” by Catlin [C84]. In the present version it has been introduced by Raich [R10].

Definition 1.1. Let z_o be a point of M and q an index in the range $1 \leq q \leq n-1$. We say that M satisfies property $(CR - P_q)$ at z_o if there is a family of weights $\{\varphi^\epsilon\}$ in a neighborhood U of z_o such that

$$(1.2) \quad \begin{cases} |\varphi^\epsilon(z)| \leq 1, & z \in U, \\ \sum_{j=1}^q \lambda_j^{\varphi^\epsilon}(z) \geq \epsilon^{-1}, & z \in U \text{ and } \ker \mathcal{L}_M(z) \neq \{0\}. \end{cases}$$

It is obvious that $(CR - P_q)$ implies $(CR - P_k)$ for any $k \geq q$.

Remark 1.2. Outside a neighborhood V_ϵ of $\ker d\gamma$, the sum $\sum_{j \leq q} \lambda_j^{\varphi^\epsilon}$ can get negative; let $-b_\epsilon$ be a bound from below. Now, if c_ϵ is a bound from below for $d\gamma$ outside V_ϵ , by setting $a_\epsilon := \frac{\epsilon^{-1} + b_\epsilon}{qc_\epsilon}$, we have

$$(1.3) \quad \sum_{j \leq q} \lambda_j^{\varphi^\epsilon} + a_\epsilon d\gamma = \sum_{j \leq q} \lambda_j^{\varphi^\epsilon} + qa_\epsilon c_\epsilon \geq \epsilon^{-1} \quad \text{on the whole } U.$$

Conversely, (1.3) readily yields the second line of (1.2). This equivalence was already noticed in [S10] and justifies our abuse of notation. In fact, (1.3) is named $(CR - P_q)$ by [S10] in accordance with [R10], whereas (1.2) is named “property (P_q) in the nullspace of the Levi form”.

Again, (1.3) for q implies (1.3) for any $k \geq q$.

We now state one of the two main results of the paper.

Theorem 1.3. *Let M be a compact pseudoconvex CR manifold of hypersurface type of dimension $2n - 1$. Assume that $(CR - P_q)$ holds for a fixed q with $1 \leq q \leq \frac{n-1}{2}$ over a covering $\{U\}$ of M . Then we have compactness estimates: given ϵ there is C_ϵ such that*

$$(1.4) \quad \|u\|^2 \leq \epsilon(\|\bar{\partial}_b u\|^2 + \|\bar{\partial}_b^* u\|^2) + C_\epsilon \|u\|_{-1}^2 \quad \text{for any } u \in D_{\bar{\partial}_b^*}^k \cap D_{\bar{\partial}_b}^k \text{ and } k \in [q, n - 1 - q],$$

where $D_{\bar{\partial}_b^*}^k$ and $D_{\bar{\partial}_b}^k$ are the domains of $\bar{\partial}_b^*$ and $\bar{\partial}_b$ respectively.

By Hodge duality between forms of complementary degree, we need the double constraint $k \geq q$ (for the positive microlocalization) and $k \leq n - 1 - q$ (for the negative one); this forces $q \leq \frac{n-1}{2}$. For M embedded and orientable, Theorem 1.3 is contained in [R10]. The same is proved in [S10] without the assumption of orientability. The proof of this, as well as of the theorem which follows, is given in Section 2. Let $\mathcal{H}^k = \ker \bar{\partial}_b \cap \ker \bar{\partial}_b^*$ be the space of harmonic forms of degree k . As a consequence of (1.4), we have that for $q \leq k \leq n - 1 - q$, the space $\mathcal{H}^k$ is finite-dimensional, $\square_b$ is invertible over $\mathcal{H}^{k \perp}$ (cf. [N06] Lemma 5.3) and its inverse G_k is a compact operator. When $k = 0$ and $k = n - 1$ it is no longer true that it is finite-dimensional. However, if $q = 1$, we have a result analogous to (1.4) also in the critical degrees $k = 0$ and $k = n - 1$.

Theorem 1.4. *Let M be a compact, pseudoconvex CR manifold of hypersurface type of dimension $2n - 1$. Assume that property $(CR - P_q)$ holds for $q = 1$ over a covering $\{U\}$ of M and, in case $n = 2$, make the additional hypothesis that $\bar{\partial}_b$ has closed range. Then for any ϵ there is C_ϵ such that*

$$(1.5) \quad \|u\|^2 \leq \epsilon(\|\bar{\partial}_b u\|^2 + \|\bar{\partial}_b^* u\|^2) + C_\epsilon \|u\|_{-1}^2 \quad \text{for any } u \in \mathcal{H}^{k \perp}, \quad k = 0 \text{ and } k = n - 1.$$

In particular, G_k is compact for $k = 0$ and $k = n - 1$.

For $n \geq 3$ and M a boundary of a domain in $\mathbb{C}^n$, resp. embedded and orientable, Theorem 1.4 is contained in [RS08] (resp. [S10]).

2. PROOFS

Proof of Theorem 1.3. We choose a local patch U where a local frame of vector fields is found for which (1.1) is fulfilled. The key point is to specify the convenient choices of q_o and φ in (1.1). Let $1 = \psi^{+2} + \psi^{-2} + \psi^{02}$ be a conic, smooth partition of the unity in the space $\mathbb{R}^{2n-1}$ dual to the space in which U is identified in local coordinates. Let $\Psi^{\pm}$ be the pseudodifferential operators with symbols $\psi^{\pm}$ and let $\text{id} = \Psi^+ \Psi^{+*} + \Psi^- \Psi^{-*} + \Psi^0 \Psi^{0*}$ be the corresponding microlocal decomposition of the identity. For a cut off function $\zeta^1 \in C_c^\infty(U)$ we decompose a form u as

$$(2.1) \quad u^{\pm} = \zeta^1 \Psi^{\pm} u \quad u \in \mathcal{B}_c^k(U), \quad \zeta^1|_{\text{supp } u} \equiv 1.$$

For u^+ we choose $q_o = 0$ and $\varphi = \varphi^\epsilon$. We also need to go back to Remark 1.2. Now, if a_ϵ has been chosen so that (1.3) is fulfilled, we remove T from our scalar products observing that, for large ξ , we have $\xi_{2n+1} > a_\epsilon$ over $\text{supp } \psi^+$. In the same

way as in Lemma 4.12 of [N06], we conclude that for $k \geq q$

$$\begin{aligned} & \sum'_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((c_{ij}T + \varphi_{ij}^\epsilon)u_{iK}^+, u_{jK}^+)_{\varphi^\epsilon} \\ & \geq \sum'_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((a_\epsilon c_{ij} + \varphi_{ij}^\epsilon)u_{iK}^+, u_{jK}^+)_{\varphi^\epsilon} \\ & \quad - C\|u^+\|_{\varphi^\epsilon}^2 - C_\epsilon\|u^+\|_{-1,\varphi^\epsilon}^2 - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^+\|_{\varphi^\epsilon}^2 \\ & \geq \epsilon^{-1}\|u^+\|_{\varphi^\epsilon}^2 - C_\epsilon\|u^+\|_{-1,\varphi^\epsilon}^2 - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^+\|_{\varphi^\epsilon}^2, \end{aligned}$$

where $\tilde{\Psi}^0 \succ \Psi^0$ and $\zeta^2 \succ \zeta^1$ in the sense that $\tilde{\psi}^0|_{\text{supp } \psi^0} \equiv 1$ and $\zeta^2|_{\text{supp } \zeta^1} \equiv 1$ respectively. (Here $\|u^+\|_{-1,\varphi^\epsilon} = \|\Lambda^{-1}u^+\|_{\varphi^\epsilon}$, where Λ^{-1} is the standard tangential pseudodifferential operator of order -1 in the local patch U .) Note that there is an additional term $-C_\epsilon\|u^+\|_{-1,\varphi^\epsilon}^2$ with respect to [N06]. The reason is that $(c_{ij}\xi_{2n-1} + \varphi_{ij}^\epsilon)$ can get negative values, even on $\text{supp } \psi^+$, when $\xi_{2n-1} < a_\epsilon$. Integration in this compact region produces the above error term. It follows that

$$(2.2) \quad \|u^+\|_{\varphi^\epsilon}^2 \leq \epsilon(\|\bar{\partial}_b u^+\|_{\varphi^\epsilon}^2 + \|\bar{\partial}_{b,\varphi^\epsilon}^* u^+\|_{\varphi^\epsilon}^2) + C_\epsilon\|u^+\|_{-1,\varphi^\epsilon}^2 + C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^+\|_{\varphi^\epsilon}^2, \quad k=1, \dots, n-1.$$

By taking the composition $\chi(\varphi^\epsilon)$ where $\chi = \chi(t)$ is a smooth function on $\mathbb{R}^+$ satisfying $\dot{\chi} > 0$ and $\ddot{\chi} > 0$, we get

$$(\chi(\varphi^\epsilon))_{ij} = \dot{\chi}\varphi_{ij}^\epsilon + \ddot{\chi}|\varphi_j^\epsilon|^2\kappa_{ij},$$

where κ_{ij} is the Kronecker symbol. We also notice that

$$|\bar{\partial}_{b,\chi(\varphi^\epsilon)}^* u|^2 \leq 2|\bar{\partial}_b^* u|^2 + 2\dot{\chi}^2 \sum'_{|K|=k-1} \left| \sum_{j=1}^{n-1} \varphi_j^\epsilon u_{jK} \right|^2.$$

Remember that $\{\varphi^\epsilon\}$ are uniformly bounded by 1. Thus, if we choose $\chi = \frac{1}{4}e^{(t-1)}$, then we have that $\ddot{\chi} \geq 2\dot{\chi}^2$ for $t = \varphi^\epsilon$. For this reason, with this modified weight, we can replace the weighted adjoint $\bar{\partial}_{b,\varphi^\epsilon}^*$ by the unweighted $\bar{\partial}_b^*$ in (2.2). By the uniform boundedness of the weights, we can also remove them from the norms and end up with the estimate

$$(2.3) \quad \|u^+\|^2 \leq \epsilon(\|\bar{\partial}_b u^+\|^2 + \|\bar{\partial}_b^* u^+\|^2) + C_\epsilon\|u^+\|_{-1}^2 + C_\epsilon\|\zeta^2\tilde{\Psi}^0 u\|^2, \quad k=q, \dots, n-1.$$

For u^- , we choose $q_0 = n-1$ and $\varphi = -\varphi^\epsilon$. Observe that for $|\xi|$ large we have $-\xi_{2n-1} \geq a_\epsilon$ over $\text{supp } \psi^-$ (cf. [N06], Lemma 4.13). Thus, we have in the current case, for $k \leq n-1-q$,

$$\begin{aligned} & \sum'_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((c_{ij}T - \varphi_{ij}^\epsilon)u_{iK}^-, u_{jK}^-)_{-\varphi^\epsilon} - \sum'_{|J|=k} \sum_{j=1}^{n-1} ((c_{jj}T - \varphi_{jj}^\epsilon)u_J^-, u_J^-)_{-\varphi^\epsilon} \\ & \geq - \sum'_{|K|=k-1} \sum_{i,j=1,\dots,n-1} ((a_\epsilon c_{ij} + \varphi_{ij}^\epsilon)u_{iK}^-, u_{jK}^-)_{-\varphi^\epsilon} \\ & \quad + \sum'_{|J|=k} \sum_{j=1}^{n-1} ((a_\epsilon c_{jj} + \varphi_{jj}^\epsilon)u_J^-, u_J^-)_{-\varphi^\epsilon} \\ & \quad - C\|u^-\|_{\varphi^\epsilon}^2 - C_\epsilon\|u^-\|_{-1,\varphi^\epsilon}^2 - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^-\|_{\varphi^\epsilon}^2 \\ & \geq \epsilon^{-1}\|u^-\|_{\varphi^\epsilon}^2 - C\|u^-\|_{\varphi^\epsilon}^2 - C_\epsilon\|u^-\|_{-1,\varphi^\epsilon}^2 - C_\epsilon\|\zeta^2\tilde{\Psi}^0 u^-\|_{\varphi^\epsilon}^2. \end{aligned}$$

Thus, we get the analogue of (2.2) for u^+ replaced by u^- and, again removing the weight from the adjoint $\bar{\partial}_{b,\varphi^\epsilon}^*$ and from the norms, we conclude that

$$(2.4) \quad \|u^-\|^2 \leq \epsilon(\|\bar{\partial}_b u^-\|^2 + \|\bar{\partial}_b^* u^-\|^2) + C_\epsilon \|u^-\|_{-1,\varphi^\epsilon}^2 + C_\epsilon \|\zeta^2 \tilde{\Psi}^0 u\|^2, \quad k = 0, \dots, n-1-q.$$

In addition to (2.3) and (2.4), we have elliptic estimates for u^0 :

$$(2.5) \quad \|u^0\|_1^2 \lesssim \|\bar{\partial} u^0\|^2 + \|\bar{\partial}_b^* u^0\|^2 + \|u\|_{-1}^2.$$

The same estimate also holds for u^0 replaced by $\zeta^2 \tilde{\Psi}^0 u$. We put together (2.3), (2.4) and (2.5) and notice that

$$(2.6) \quad \begin{aligned} \|\bar{\partial}_b(\zeta^1 \Psi^{\frac{\pm}{0}} u)\|^2 &\leq \|\zeta^1 \Psi^{\frac{\pm}{0}} \bar{\partial}_b u\|^2 + \|[\bar{\partial}_b, \zeta^1 \Psi^{\frac{\pm}{0}}]u\|^2 \\ &\leq \|\zeta^1 \Psi^{\frac{\pm}{0}} \bar{\partial}_b u\|^2 + \|\zeta^2 \tilde{\Psi}^{\frac{\pm}{0}} u\|^2 + \|\zeta^2 \tilde{\Psi}^0 u\|^2, \end{aligned}$$

for $\zeta^2 \succ \zeta^1$ and $\tilde{\Psi}^0 \succ \Psi^0$. A similar estimate holds for $\bar{\partial}_b$ replaced by $\bar{\partial}_b^*$. Since $\zeta^1|_{\text{supp } u} \equiv 1$, then

$$\begin{aligned} \|u\|^2 &\leq \sum_{+,-,0} \|\zeta^1 \Psi^{\frac{\pm}{0}} u\|^2 + Op^{-\infty}(u) \\ &\leq \epsilon \sum_{+,-,0} (\|(\bar{\partial}_b u)^{\frac{\pm}{0}}\|^2 + \|(\bar{\partial}_b^* u)^{\frac{\pm}{0}}\|^2) + C_\epsilon \|u\|_{-1}^2, \end{aligned}$$

and therefore

$$(2.7) \quad \|u\|^2 \leq \epsilon(\|\bar{\partial}_b u\|^2 + \|\bar{\partial}_b^* u\|^2) + C_\epsilon \|u\|_{-1}^2, \quad q \leq k \leq n-1-q.$$

We now consider u globally defined on the whole M instead of a local patch U . To pass from local to global compactness estimates is immediate (cf. e.g. [S10]). We cover M by $\{U_\nu\}$ so that in each patch there is a basis of forms in which the basic estimate holds. In the identification of U_ν to $\mathbb{R}^{2n-1}$, we suppose that the microlocal decomposition by the operators $\Psi^{\frac{\pm}{0}}$ which occur in (2.6) is well defined. We then get (2.7) and apply it to a decomposition $u = \sum_\nu \zeta_\nu u$ for a partition of the unity $\sum_\nu \zeta_\nu = 1$ on M . We point out that we first take summation over $+, -, 0$ on each patch U_ν and then summation over ν ; this is why orientability of M is needless.

We observe that $[\bar{\partial}_b, \zeta_\nu]$ and $[\bar{\partial}_b^*, \zeta_\nu]$ are 0-order operators and, since they come with a factor of ϵ , they are absorbed in the left side of (2.7); thus (2.7) holds for any $u \in \mathcal{B}^k$. Finally, we use the density of smooth forms $\mathcal{B}^k$ into Sobolev forms $(H^1)^k$ of $D_{\bar{\partial}_b^*}^k \cap D_{\bar{\partial}_b}^k$ for the graph norm and get (2.7). The proof is complete. $\square$

Proof of Theorem 1.4. We prove estimates in degree 0 (those in degree $n-1$ being similar). We first discuss the case $n > 2$. We make repeated use of (2.7) in degree 1. This first implies that $\bar{\partial}_b^*$ has closed range on 1-forms, that is,

$$\begin{aligned} \mathcal{H}^{0\perp} &= (\ker \bar{\partial}_b)^\perp \\ &= \text{range } \bar{\partial}_b^*. \end{aligned}$$

(Thus, if $u \in \mathcal{H}^{0\perp}$, then there exists a solution $v \in (L^2)^1$ to the equation $\bar{\partial}^* v = u$. Moreover, we can choose v belonging to $(\text{Ker}(\bar{\partial}_b^*))^\perp$.) This is a consequence of the following estimate:

$$(2.8) \quad \|v\|_0^2 \lesssim \|\bar{\partial}_b^* v\|_0^2 \quad \text{for any } v \in (\ker \bar{\partial}_b^*)^\perp.$$

This can be proved by contradiction. If (2.8) is violated, there exists a sequence $v_\nu \in (\ker \bar{\partial}_b^*)^\perp$ such that $\|v_\nu\|_0^2 \equiv 1$ and $\|\bar{\partial}_b^* v_\nu\|_0 \rightarrow 0$. Take a subsequential L^2 -weak limit v_0 of v_ν ; it satisfies $v_0 \in \ker(\bar{\partial}_b^*) \cap (\ker(\bar{\partial}_b^*))^\perp$ and in particular $\|v_\nu\|_{-1} \rightarrow 0$. This violates (2.7) and proves (2.8). We also have

$$(2.9) \quad \|v\|_{-1}^2 \leq \epsilon \|\bar{\partial}_b^* v\|_0^2 + c_\epsilon \|\bar{\partial}_b^* v\|_{-1}^2, \quad \text{for any } v \in (\ker \bar{\partial}_b^*)^\perp.$$

The argument is similar. If (2.9) is violated, then there is a sequence $v_\nu \in (\ker \bar{\partial}_b^*)^\perp$ such that $\|v_\nu\|_{-1} \equiv 1$, $\|\bar{\partial}_b^* v_\nu\|_{-1} \rightarrow 0$ and $\|\bar{\partial}_b^* v_\nu\|_0 \leq c$. By (2.7), $\|v_\nu\|_0 \leq C'$; hence there is a subsequential L^2 -weak limit $v_{\nu_k} \rightarrow v_0 \in (\ker(\bar{\partial}_b^*))^\perp \cap \ker(\bar{\partial}_b^*)$; thus $v_0 = 0$ and $\|v_{\nu_k}\|_{-1} \rightarrow 0$, a contradiction.

We now point out that $(\ker(\bar{\partial}_b^*))^\perp = \overline{\text{range}(\bar{\partial}_b)} \subset \ker(\bar{\partial})$; in particular, our solution v satisfies $\bar{\partial}_b v = 0$. We are ready to conclude the proof for $n > 2$. We use the notation lc and sc for a large and small constant respectively. We have for any function $u \in \mathcal{H}^\perp$

$$(2.10) \quad \begin{aligned} \|u\|^2 &= (u, \bar{\partial}_b^* v) \\ &= (\bar{\partial}_b u, v) \\ &\leq \|\bar{\partial}_b u\| \|v\| \\ &\stackrel{(2.7) \text{ for } v}{\leq} \|\bar{\partial}_b u\| (\epsilon \|\bar{\partial}_b^* v\| + c_\epsilon \|v\|_{-1}) \\ &\stackrel{(2.9)}{\lesssim} \|\bar{\partial}_b u\| (\epsilon \|u\| + c_\epsilon \|u\|_{-1}) \\ &\leq lc_1 \epsilon^2 \|\bar{\partial}_b u\|^2 + sc_1 \|u\|^2 + lc_2 c_\epsilon^2 \|u\|_{-1}^2 + sc_2 \|\bar{\partial}_b u\|^2 \\ &\leq \epsilon' \|\bar{\partial}_b u\|^2 + c_{\epsilon'} \|u\|_{-1}^2 + sc_1 \|u\|^2, \end{aligned}$$

for $\epsilon' = lc_1 \epsilon^2 + sc_2$ and $c_{\epsilon'} = lc_2 c_\epsilon^2$. By choosing sc_1 so that $sc_1 \|u\|^2$ is absorbed in the left, (2.10) yields (2.7) for u in degree 0. This concludes the proof of the case $n > 2$ for functions.

Let $n = 2$. We have only estimates for positively microlocalized 1-forms and for negatively microlocalized functions. We have to show how to get estimates for positively microlocalized functions (the argument for negative 1-forms being similar). We use our extra assumption of closed range for $\bar{\partial}_b$; thus for any $u \in (\ker \bar{\partial}_b)^\perp$ there is $v \in (\ker \bar{\partial}_b^*)^\perp$ such that $\bar{\partial}_b^* v = u$. On each U_ν we consider the positive microlocalization Ψ^+ , take a pair of cut-off functions $\zeta_\nu, \zeta_\nu^1 \in C_c^\infty(U_\nu)$ with $\zeta_\nu^1|_{\text{supp } \zeta_\nu} \equiv 1$, and define $\Psi_\nu^+ := \zeta_\nu^1 \Psi^+ \zeta_\nu$. Note that the commutators $[\bar{\partial}_b^*, \Psi_\nu^+]$ and $[\bar{\partial}_b, \Psi_\nu^+]$ are operators with symbols of types $\zeta_\nu^1 \psi^+ \zeta_\nu$, $\zeta_\nu^1 \psi^+ \zeta_\nu$ and $\zeta_\nu^1 \psi^+ \zeta_\nu$. All these symbols have support contained in the positive half-space $\xi_{2n-1} > 0$, and hence we have compactness estimates for 1-forms if their coefficients are subjected to the action of the corresponding pseudodifferential operators. We denote by a common symbol Φ_ν^+ all these operators coming from commutators. We have

$$(2.11) \quad \begin{aligned} \|\Psi_\nu^+ v\| &\leq \epsilon \|\bar{\partial}_b^* \Psi_\nu^+ v\| + c_\epsilon \|\Psi_\nu^+ v\|_{-1} + c_\epsilon \|\zeta_\nu^2 \tilde{\Psi}^0 \zeta_\nu v\| \\ &\leq \epsilon \|\Psi_\nu^+ \bar{\partial}_b^* v\| + \epsilon \|\Phi_\nu^+ v\| + c_\epsilon \|\Psi_\nu^+ v\|_{-1} + c_\epsilon \|\zeta_\nu^2 \tilde{\Psi}^0 \zeta_\nu v\| \\ &\stackrel{(2.8) \text{ and } (2.9) \text{ for } +}{\leq} \epsilon \|u\| + c_\epsilon \|u\|_{-1}. \end{aligned}$$

The same estimate also holds for $\|\Phi_\nu^+ v\|$. It follows that

$$\begin{aligned}
 \|\Psi_\nu^+ u\|^2 &= (\Psi_\nu^+ u, \Psi_\nu^+ \bar{\partial}_b^* v) \\
 &= (\Psi_\nu^+ \bar{\partial}_b u, \Psi_\nu^+ v) + (\Phi_\nu^+ u, \Psi_\nu^+ v) + (\Psi_\nu^+ u, \Phi_\nu^+ v) \\
 &\leq (\|\Psi_\nu^+ \bar{\partial}_b u\| + \|\Phi_\nu^+ u\| + \|\Psi_\nu^+ u\|)(\|\Phi_\nu^+ v\| + \|\Psi_\nu^+ v\|) \\
 &\stackrel{(2.11)}{\leq} (\|\Psi_\nu^+ \bar{\partial}_b u\| + \|u\|)(\epsilon\|u\| + c_\epsilon\|u\|_{-1}) \\
 &\lesssim \epsilon\|\Psi_\nu^+ \bar{\partial}_b u\|\|u\| + c_\epsilon\|\Psi_\nu^+ \bar{\partial}_b u\|\|u\|_{-1} + \epsilon\|u\|^2 + c_\epsilon\|u\|_{-1}\|u\| \\
 &\leq lc_1\epsilon^2\|\Psi_\nu^+ \bar{\partial}_b u\|^2 + sc_1\|u\|^2 + sc_2\|\Psi_\nu^+ \bar{\partial}_b u\|^2 + lc_2c_\epsilon^2\|u\|_{-1}^2 \\
 &\quad + \epsilon\|u\|^2 + sc_3\|u\|^2 + lc_3c_\epsilon^2\|u\|_{-1}^2 \\
 &\leq \epsilon'\|\Psi_\nu^+ \bar{\partial}_b u\|^2 + sc_4\|u\|^2 + c_{\epsilon'}\|u\|_{-1}^2,
 \end{aligned}
 \tag{2.12}$$

where $\epsilon' = lc_1\epsilon^2 + sc_2$, $c_{\epsilon'} = lc_2c_\epsilon^2 + lc_3c_\epsilon^2$ and $sc_4 = sc_1 + \epsilon + sc_3$. We have to recall now that the same estimate as (2.12) also holds for $\|\Psi_\nu^- u\|^2$ (the one for $\|\Psi_\nu^0 u\|^2$ being trivial by ellipticity). Taking summation over $+$, $-$ and 0 on each U_ν , we get

$$\|\zeta_\nu u\|^2 \leq \epsilon\|\zeta_\nu^1 \bar{\partial}_b u\|^2 + c_\epsilon\|u\|_{-1}^2 + sc\|u\|^2.$$

We now take summation over ν and choose sc so that the related term is absorbed by $\sum_\nu \|\zeta_\nu u\|^2 \sim \|u\|^2$ and end up with

$$\|u\|^2 \leq \epsilon\|\bar{\partial}_b u\|^2 + c_\epsilon\|u\|_{-1}^2 \quad \text{for any function } u. \quad \square$$

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TAN TAO UNIVERSITY, TAN TAO UNIVERSITY AVENUE, DUC HOA DISTRICT, LONG AN PROVINCE, VIETNAM

E-mail address: `khanh.tran@ttu.edu.vn`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY

E-mail address: `pinton@math.unipd.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY

E-mail address: `zampieri@math.unipd.it`