LINEAR ORTHOGONALITY PRESERVERS OF HILBERT C^* -MODULES OVER C^* -ALGEBRAS WITH REAL RANK ZERO

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ABSTRACT. Let A be a C^* -algebra with real rank zero. Let E and F be Hilbert A -modules with E being full. Suppose that $\theta : E \rightarrow F$ is a linear map preserving orthogonality, i.e.,

$$\langle \theta(x), \theta(y) \rangle = 0 \quad \text{whenever} \quad \langle x, y \rangle = 0.$$

We show in this article that if θ is an A -module map (not assumed to be bounded), then there exists a central positive multiplier $u \in M(A)$ such that

$$\langle \theta(x), \theta(y) \rangle = u \langle x, y \rangle \quad (x, y \in E).$$

In the case when A is a standard C^* -algebra, when A is a real rank zero properly infinite unital C^* -algebra, or when A is a W^* -algebra, we also get the same conclusion with the assumption of θ being an A -module map weakened to being a local map.

1. INTRODUCTION AND NOTATION

It is common knowledge that, together with linearity, the inner product and the norm structures of a Hilbert space H determine each other. It might be a bit less well known that the orthogonality structure also suffices to determine the inner product up to a scalar. This fact follows from the following easy observation: for any $x, y \in H$, $\|x\| = \|y\|$ if and only if $x + \lambda y$ is orthogonal to $x - \lambda y$ for some scalar λ with $|\lambda| = 1$ (see also [3, 6]).

It is natural and interesting to ask whether the linear structure and orthogonality structure of a (complex) Hilbert C^* -module determines its C^* -algebra-valued inner product. More precisely, let A be a (complex) C^* -algebra, and $\theta : E \rightarrow F$ be a $\mathbb{C}$ -linear map between Hilbert A -modules that preserves orthogonality (i.e. preserves zero A -valued inner products). We want to study to what extent θ will respect the A -valued inner products. When the underlying C^* -algebra is $\mathbb{C}$, it reduces to the case of Hilbert spaces.

We first note that without any further assumption on θ , the above question might have a negative answer.

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Example 1.1. Let H be an infinite dimensional (complex) Hilbert space and $A = \mathcal{K}(H)$ be the C^* -algebra of all compact operators on H . Suppose that $\bar{H}$ is a vector space that is conjugate-linear isomorphic to H . When equipped with the operations: $\langle \bar{\eta}_1, \bar{\eta}_2 \rangle := \eta_1 \otimes \eta_2$ and $\overline{\eta_1 T} := \overline{T^* \eta_1}$ ($\bar{\eta}_1, \bar{\eta}_2 \in \bar{H}; T \in A$), we see that $\bar{H}$ is a Hilbert A -module. Suppose that θ is any unbounded bijective $\mathbb{C}$ -linear map from $\bar{H}$ onto $\bar{H}$. Since $\langle x, y \rangle = 0$ if and only if $x = 0$ or $y = 0$, we see that both θ and θ^{-1} preserve orthogonality.

In fact, assuming norm continuity, Schweizer obtains his pioneering work in his thesis (see [20, 9.6]) on a characterization of *bounded* orthogonality-preserving $\mathbb{C}$ -linear maps. His result ensures that all *bounded* orthogonality-preserving $\mathbb{C}$ -linear maps from a full Hilbert C -module X into a full Hilbert D -module Y are exactly those maps T which induce a sort of “correspondence” between the C^* -algebras $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, where $\mathcal{K}(X)$ (respectively, $\mathcal{K}(Y)$) is generated by “rank one operators” $\theta_{\zeta, \eta}$ defined as $\theta_{\zeta, \eta}(\xi) := \zeta \langle \eta, \xi \rangle$ for $\zeta, \eta, \xi \in X$ (respectively, Y). More precisely, there is a “local morphism” $\pi : \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ such that

$$T(ax) = \pi(a)T(x) \quad (a \in \mathcal{K}(X)),$$

or equivalently,

$$\theta_{T\zeta, T\zeta} \leq \|T\|^2 \pi(\theta_{\zeta, \zeta}) \quad (\zeta \in X).$$

As a result, in the case $C = D$, we can see that even a *bounded* $\mathbb{C}$ -linear orthogonality preserver does not necessarily preserve the C^* -algebra inner products. Hence, we still need more assumptions to ensure our question to have a positive answer.

As we are dealing with Hilbert A -modules, a natural additional assumption is that θ is an A -module map, i.e., $\theta(xa) = \theta(x)a$ ($x \in E, a \in A$). In [9], Ilišević and Turnšek showed that if A is a standard C^* -algebra, then for every orthogonality-preserving A -module map $\theta : E \rightarrow F$, there is a scalar $\lambda \geq 0$ such that

$$\langle \theta(x), \theta(y) \rangle = \lambda \langle x, y \rangle \quad (x, y \in E).$$

In particular, all such θ are scalar multiples of isometries.

In [13], under a weaker assumption on θ , namely θ being “local”, we get a similar conclusion in the case when A is a commutative C^* -algebra (in fact, the main difficulties in [13] come from the fact that θ is not assumed to be an A -module map). Recall that a $\mathbb{C}$ -linear map $\theta : E \rightarrow F$ is *local* if

$$\theta(x)a = 0 \quad \text{whenever} \quad xa = 0 \quad (x \in E; a \in A).$$

Readers should find the idea of local linear maps familiar. For example, local linear maps on the space of smooth functions defined on a manifold modeled on $\mathbb{R}^n$ are exactly linear differential operators (see, e.g., [19, 16]). See [11, 2] for the vector-valued case, and [1] for the Banach $C^1[0, 1]$ -module setting. We also mention that there is a bimodule version of local maps as studied by Schweizer in [20, 21] (which is different from ours). Notice that every module map is local, but local linear maps, e.g., linear differential operators, might not be a module map. Nevertheless, it has been shown in [12, Proposition A.1] that every *bounded* local map between Hilbert C^* -modules is a module map.

The results in [9] and [13] lead to the following conjecture. We remark that the fullness assumption of E in this conjecture is a necessity. Without this, the conclusion does not hold even in the case when A is commutative (see [13, 3.6]). Here, a Hilbert A -module E is said to be *full* if the linear span of $\{\langle x, y \rangle : x, y \in E\}$ is dense in A .

Conjecture 1.2. *Let A be a C^* -algebra. Let E and F be Hilbert A -modules with E being full. Assume $\theta : E \rightarrow F$ is a ($\mathbb{C}$ -linear) local map preserving orthogonality; i.e. for any $x, y \in E$,*

$$\langle x, y \rangle = 0 \quad \text{implies} \quad \langle \theta(x), \theta(y) \rangle = 0.$$

Then there is a central positive element $u \in M(A)$ such that

$$\langle \theta(x), \theta(y) \rangle = u \langle x, y \rangle \quad (x, y \in E).$$

In this article, positive answers of this conjecture will be given in the following four cases:

- (1) A is a C^* -algebra of real rank zero and θ is an A -module map (Theorem 2.3).
- (2) A is a standard C^* -algebra (Corollary 3.2).
- (3) A is a properly infinite unital C^* -algebra of real rank zero (Corollary 3.3).
- (4) A is a W^* -algebra (Corollary 3.4).

We note that although the main result, Theorem 9.6, in [20] appears to be more general and more complete (as they deal with $\mathbb{C}$ -linear orthogonality preservers), they do not give too much hint to the answer of our conjecture. For example, in the situation of Example 1.1, what one can get from [20, 9.6] is the tautology that a bounded orthogonality-preserving $\mathbb{C}$ -linear map $\theta : \bar{H} \rightarrow \bar{H}$ is $\mathbb{C}$ -linear. This does not give any useful insight into the study of orthogonality-preserving $\mathcal{K}(H)$ -module maps in even such a simple case.

As a final remark for the introduction, we note that, unlike the situation in some of the other literature (e.g. [20, 7]), θ is *not* assumed to be bounded. It is because of the conceptual reason that A -linearity (or locality) and orthogonality structures should completely determine the A -inner products, as stated in the beginning of this section. However, we shall see that the boundedness of an A -linear orthogonality preserver will be an automatic consequence of our results.

2. ORTHOGONALITY-PRESERVING A -MODULE MAPS

First, we give some notation that will be used throughout this article. In the following, A is a C^* -algebra, E and F are Hilbert A -modules, and $\Psi, \theta : E \rightarrow F$ are orthogonality-preserving $\mathbb{C}$ -linear maps, which are not assumed to be bounded.

Let $a \in A_+$. We set $C^*(a)$ to be the C^* -subalgebra generated by a , and $\mathbf{c}(a)$ to be the *central cover* of a , i.e., the smallest central element in A_+^{**} dominating a (see, e.g., [18, 2.6.2]). If, in addition, $\alpha, \beta \in \mathbb{R}_+$, we put $e_a(\alpha, \beta]$ to be the spectral projections of a in A^{**} corresponding to the set $(\alpha, \beta] \cap \sigma(a)$.

We denote by $Z(A)$ the center and by $M(A)$ the space of all multipliers of A . On the other hand, $\text{Proj}_1(A)$ is the set of all norm-one (i.e., nonzero) projections in A . For any *open projection* $p \in \text{Proj}_1(A^{**})$ (i.e., there exists an increasing net $\{a_i\}$ in A_+ such that $a_i \uparrow p$ in the weak*-topology), we denote by $\text{her}(p) := pA^{**}p \cap A$ the hereditary C^* -subalgebra associated to p . See, e.g., [4] for more information about open projections.

We recall that A has *real rank zero* if every selfadjoint element in A can be approximated in norm by invertible selfadjoint elements. If A has real rank zero, then the real linear span of $\text{Proj}_1(A)$ is norm dense in A_{sa} (see, e.g., [5]).

Let us start with the following easy lemma, part (a) of which might be well known, but we give an argument here for completeness.

Lemma 2.1. (a) *If $p \in \text{Proj}_1(A^{**})$ and $b \in Z(pA^{**}p)_+$, then $\|\mathbf{c}(b)\| = \|b\|$, $\mathbf{c}(b)p = b$ and $\mathbf{c}(b)\mathbf{c}(p) = \mathbf{c}(b)$.*

(b) *Suppose that A has real rank zero and E is full. If $q \in A^{**} \setminus \{0\}$ is an open projection, there are $r \in \text{Proj}_1(A)$ and $y \in Er$ such that $r = \langle y, y \rangle \leq q$.*

Proof. (a) Since $b \leq \|b\|1$, we see that $0 \leq b \leq \mathbf{c}(b) \leq \|b\|1$ and $\|\mathbf{c}(b)\| = \|b\|$. Clearly, $\mathbf{c}(b)p = p\mathbf{c}(b)p \geq pbp = b$. Conversely, as $Z(pA^{**}p) = Z(A^{**})p$ (see e.g. [10, 5.5.6]), there is $a \in Z(A^{**})_+$ with $b = ap$ (note that $b^{1/2} \in Z(A^{**})p$). Thus, we have $b = a^{1/2}pa^{1/2} \leq a^{1/2}\mathbf{c}(p)a^{1/2} = a\mathbf{c}(p)$. As $a\mathbf{c}(p)$ is central, $\mathbf{c}(b) \leq a\mathbf{c}(p)$ and $\mathbf{c}(b)p = p\mathbf{c}(b)p \leq ap = b$. The last equality follows from [18, 2.6.4] and the fact that $b\mathbf{c}(p) = b$.

(b) Note that $\text{her}(q) \neq (0)$ and also has real rank zero (see e.g. [5]). Moreover, $E_0 := E \cdot \text{her}(q)$ is a full (and, hence nonzero) Hilbert $\text{her}(q)$ -module. Pick any $x \in E_0$ such that $a := \langle x, x \rangle$ is a norm-one element in $\text{her}(q)$. Let $t \in (0, 1/3)$ and $b \in \text{her}(q)_+$ such that $\|a - b\| < t$ and $\sigma(b) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \leq \dots \leq \lambda_n = \|b\|$ (see e.g. [5]). Since $\|b\| > 2/3$, we can choose $s \in [t, \|b\|] \setminus \sigma(b)$. If we set $r := e_b(s, 2] \in \text{Proj}_1(A)$, then $\|r - rar\| \leq \|r - rbr\| + \|b - a\| < 1$. If $c := r + \sum_{n=1}^{\infty} (r - rar)^n \in A_+$, then $(rar)c = c(rar) = r$ and so, $\langle xc^{1/2}, xc^{1/2} \rangle = c^{1/2}rar c^{1/2} = r$. Finally, $xc^{1/2} \in Er$ as $c^{1/2}$ is in the C^* -subalgebra $rAr + \mathbb{C}r$. $\square$

Proposition 2.2. *Let A be a unital C^* -algebra of real rank zero. Suppose that $\theta : E \rightarrow F$ is an A -module map preserving orthogonality, and there is an element $x_0 \in E$ such that $\langle x_0, x_0 \rangle = 1$. Then one can find $u \in Z(A)_+$ satisfying*

$$\langle \theta(x), \theta(y) \rangle = u\langle x, y \rangle \quad (x, y \in E).$$

Proof. Let $u := \langle \theta(x_0), \theta(x_0) \rangle \in A_+$. For any symmetry $w \in A$, as $x_0 + x_0w$ and $x_0 - x_0w$ are orthogonal to each other, so are $\theta(x_0) + \theta(x_0)w$ and $\theta(x_0) - \theta(x_0)w$. Consequently, $u + wu - uw - wuw = 0$ and $u + uw - wu - wuw = 0$ (by taking the adjoint). This tells us that $u = wuw$, and so $u \in Z(A)_+$ (as A is generated by projections, and thus also by symmetries). Pick any $z \in E$ with $\langle x_0, z \rangle = 0$. Then $z + x_0\langle z, z \rangle^{1/2}$ is also orthogonal to $z - x_0\langle z, z \rangle^{1/2}$. It follows from the orthogonality-preserving property that

$$\langle \theta(z), \theta(z) \rangle = \langle z, z \rangle^{1/2} \langle \theta(x_0), \theta(x_0) \rangle \langle z, z \rangle^{1/2} = u\langle z, z \rangle.$$

For any $y \in E$, the element $z = y - x_0\langle x_0, y \rangle$ is orthogonal to x_0 . Hence,

$$\langle \theta(y), \theta(y) \rangle = \langle y, x_0 \rangle \langle \theta(x_0), \theta(x_0) \rangle \langle x_0, y \rangle + \langle \theta(z), \theta(z) \rangle = u\langle y, y \rangle.$$

A polarization type argument implies that $\langle \theta(x), \theta(y) \rangle = u\langle x, y \rangle$ ($x, y \in E$). $\square$

Theorem 2.3. *Let A be a C^* -algebra of real rank zero. Suppose that E is full, and $\theta : E \rightarrow F$ is an orthogonality-preserving A -module map (not assumed to be bounded). There is $u \in Z(M(A))_+$ such that*

$$\langle \theta(x), \theta(y) \rangle = u\langle x, y \rangle \quad (x, y \in E).$$

In particular, θ is automatically bounded.

Proof. Set

$$P := \{(x, p) \in E \times \text{Proj}_1(A) : \langle x, x \rangle = p\}.$$

Lemma 2.1(b) tells us that $P \neq \emptyset$. Suppose that $(x, p) \in P$. Then $xp = x$ since we have $\langle xp - x, xp - x \rangle = p\langle x, x \rangle p - \langle x, x \rangle p - p\langle x, x \rangle + \langle x, x \rangle = 0$. Also, Ep is a full Hilbert pAp -module and the restriction of θ on Ep is an orthogonality-preserving pAp -module map. Since p is the identity of the C^* -algebra pAp and $\theta(Ep) \subseteq Fp$, one can apply Proposition 2.2 to obtain $b_p \in Z(pAp)_+$ which satisfies

$$\langle \theta(x)p, \theta(y)p \rangle = b_p \langle xp, yp \rangle \quad (x, y \in E).$$

By Lemma 2.1(a), we have

$$(2.1) \quad p(\langle \theta(x), \theta(y) \rangle - \mathbf{c}(b_p)\langle x, y \rangle)p = 0 \quad (x, y \in E).$$

As the weak- $*$ -closed linear span, I , of $\{\langle \theta(x), \theta(y) \rangle - \mathbf{c}(b_p)\langle x, y \rangle : x, y \in E\}$ is an ideal of A^{**} , there is a central projection $q_I \in A^{**}$ with $I = q_I A^{**}$. Since $pq_I = p q_I p = 0$, we have $\mathbf{c}(p) \leq 1 - q_I$. Consequently,

$$(2.2) \quad \mathbf{c}(p)\langle \theta(x), \theta(y) \rangle = \mathbf{c}(p)\mathbf{c}(b_p)\langle x, y \rangle \quad (x, y \in E).$$

Now, let $\mathcal{D} := \{D \subseteq P : \mathbf{c}(p)\mathbf{c}(q) = 0 \text{ whenever } (x, p), (y, q) \in D\}$. If we equip $\mathcal{D}$ with the usual inclusion, then Zorn's Lemma gives a maximal element $D_0 = \{(x_\gamma, p_\gamma)\}_{\gamma \in \Gamma} \in \mathcal{D}$. Set $q_0 := \bigvee_{\gamma \in \Gamma} \mathbf{c}(p_\gamma)$ in $\text{Proj}_1(A^{**})$, which is a central element. Observe that $1 - q_0$ will not dominate a nontrivial open projection. Indeed, if $0 \neq q \leq 1 - q_0$ is an open projection, then Lemma 2.1(b) produces an element $(y, r) \in P$ such that $r \leq q$. Therefore, $D_0 \cup \{(y, r)\} \in \mathcal{D}$, which contradicts the maximality of D_0 . We now claim that the $*$ -homomorphism $\Phi : A \rightarrow q_0 A \subseteq A^{**}$ defined by $\Phi(a) = q_0 a$ is injective. Suppose on the contrary that there exists $a \in A_+$ with $\|a\| = 1$ and $\Phi(a) = 0$. Take any $\epsilon \in (0, 1)$, and put q_ϵ to be the nonzero open projection $e_a(\epsilon, 1]$. As $a - \epsilon q_\epsilon \geq 0$, we have $q_\epsilon q_0 = q_0 q_\epsilon q_0 \leq q_0 a q_0 / \epsilon = 0$. So, $q_\epsilon \leq 1 - q_0$, which implies the contradiction that $q_\epsilon = 0$.

As $x_\gamma \mathbf{c}(p_\gamma) = x_\gamma p_\gamma \mathbf{c}(p_\gamma) = x_\gamma$ ($\gamma \in \Gamma$), we see that x_γ and $x_{\gamma'}$ are orthogonal if $\gamma \neq \gamma'$. We now claim that the $\mathbf{c}(b_\gamma)$'s are uniformly bounded (where $b_\gamma \in Z(p_\gamma A p_\gamma)_+$ is the element associated with $(x_\gamma, p_\gamma) \in P$ that satisfies (2.2)). Suppose on the contrary that there are $\mathbf{c}(b_{\gamma_n})$ with $\|\mathbf{c}(b_{\gamma_n})\| = \|b_{\gamma_n}\| \geq n^3$ ($n \in \mathbb{N}$). Note that the orthogonal sum $x := \sum_n \frac{x_{\gamma_n}}{n}$ is convergent in norm in E . By the orthogonality-preserving property of θ , Lemma 2.1(a) as well as equality (2.1), for any $m \in \mathbb{N}$,

$$\begin{aligned} \langle \theta(x), \theta(x) \rangle &= \langle \theta(x_{\gamma_m}/m), \theta(x_{\gamma_m}/m) \rangle + \langle \theta(x - x_{\gamma_m}/m), \theta(x - x_{\gamma_m}/m) \rangle \\ &\geq \langle \theta(x_{\gamma_m}/m), \theta(x_{\gamma_m}/m) \rangle = \frac{\mathbf{c}(b_{\gamma_m})\langle x_{\gamma_m}, x_{\gamma_m} \rangle}{m^2} = \frac{b_{\gamma_m}}{m^2}. \end{aligned}$$

As the norm of the last term goes to infinity as $n \rightarrow \infty$, we reach a contradiction.

Finally, let d be the weak- $*$ -limit in A^{**} of finite sums of the uniformly bounded mutually orthogonal elements $\mathbf{c}(b_\gamma)$ (see Lemma 2.1(a)). By relation (2.2) and the fact that q_0 is the weak- $*$ -limit of finite sums of $\mathbf{c}(p_\gamma)$'s, we have

$$dq_0 \langle x, y \rangle = q_0 \langle \theta(x), \theta(y) \rangle \in q_0 A \quad (x, y \in E).$$

Since E is full, we see that d induces an element $m \in Z(M(q_0A))_+$ such that $m q_0 \langle x, y \rangle = q_0 \langle \theta(x), \theta(y) \rangle$ ($x, y \in E$). Since $\Phi : A \rightarrow q_0A$ extends to a $*$ -isomorphism $\tilde{\Phi} : M(A) \rightarrow M(q_0A)$, there is $u \in Z(M(A))_+$ such that $\tilde{\Phi}(u) = m$. This means that

$$\Phi(u \langle x, y \rangle - \langle \theta(x), \theta(y) \rangle) = 0 \quad (x, y \in E),$$

which gives the required conclusion. $\square$

Remark 2.4. Let A be a general C^* -algebra. Suppose that there exist Hilbert A^{**} -modules $\tilde{E}$ and $\tilde{F}$ containing E and F respectively such that the Hilbert A^{**} -module structures extend the corresponding Hilbert A -module structures and θ extends to an orthogonality-preserving A^{**} -module map $\tilde{\theta} : \tilde{E} \rightarrow \tilde{F}$. Then one can use Theorem 2.3 to show that θ satisfies the conclusion of Conjecture 1.2 (since A^{**} has real rank zero). In the situation when θ is a bounded orthogonality-preserving A -module map, we have tried $\tilde{E} = E^{**}$ and $\tilde{F} = F^{**}$ but encountered some difficulties in showing that θ^{**} is orthogonality preserving. It was claimed in [7] that, when θ is bounded, such $\tilde{E}$, $\tilde{F}$ and $\tilde{\theta}$ could be found. However, instead of manipulating the difficulties in the arguments in [7], we are working on a proof for the case of general C^* -algebras, *without* the boundedness assumption on θ (but θ is assumed to be an A -module map), using completely different ideas from those in this article and in [7], [9], and [13].

3. ORTHOGONALITY-PRESERVING $\mathbb{C}$ -LINEAR LOCAL MAPS

In this section, we consider ($\mathbb{C}$ -linear) local maps (see the Introduction) that preserve orthogonality. Let us first give the following useful observation.

Lemma 3.1. *Let A_0 be the $*$ -algebra generated by all the idempotents in A . If $\Psi : E \rightarrow F$ is a local map, then Ψ is an A_0 -module map.*

Proof. Let $p \in A$ be an idempotent and $x \in E$. As Ψ is local, one has $\Psi(x - xp)p = 0$. If $\{u_i\}$ is an approximate unit for A , then $(1 - p)u_i \in A$ will strictly converge to $(1 - p)$. Since $\Psi(xp)(1 - p)u_i = 0$, we have $\langle y, \Psi(xp) \rangle (1 - p) = \lim \langle y, \Psi(xp) \rangle (1 - p)u_i = 0$ ($y \in F$). This implies that $\Psi(xp) - \Psi(xp)p = \Psi(xp)(1 - p) = 0$. Thus, $\Psi(x)p = \Psi(xp)$, and so $\Psi(xv) = \Psi(x)v$ for any $v \in A_0$. $\square$

Note that if A has real rank zero, then A_0 is dense in A . We remark however that A_0 can be $\{0\}$ (e.g. if $A = C_0(0, 1)$).

Recall that A is a *standard C^* -algebra* on a Hilbert space H if $\mathcal{K}(H) \subseteq A \subseteq \mathcal{L}(H)$. In this case, A_0 contains a “big enough” ideal $\mathcal{F}(H)$ of A , in the sense that $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$ is an essential ideal. As a consequence, we can use Lemma 3.1 and Theorem 2.3 to give a self-contained proof of the following slight extension of [9, 2.3] (note that the A -linearity is replaced by the local property).

Corollary 3.2 (cf. [9, 2.3]). *Suppose that A is a standard C^* -algebra on a Hilbert space H . If $\Psi : E \rightarrow F$ is local and orthogonality preserving, then there is $\lambda \in \mathbb{R}_+$ such that*

$$\langle \Psi(x), \Psi(y) \rangle = \lambda \langle x, y \rangle \quad (x, y \in E).$$

Proof. Consider $E_0 := E \cdot \mathcal{K}(H)$ and $F_0 := F \cdot \mathcal{K}(H)$ (both of them being Hilbert $\mathcal{K}(H)$ -modules). Let $\{v_\gamma\}_{\gamma \in \Gamma}$ be an approximate unit in $\mathcal{K}(H)$ consisting of finite rank positive operators. By Lemma 3.1, $\Psi(xv) = \Psi(x)v$ for every finite rank operator v and every $x \in E$. On the other hand, for any $y \in E_0$, there exist $a \in \mathcal{K}(H)$ and $x \in E$ with $y = xa$ (by the Cohen factorization theorem), and so

$$\Psi(y)v_\gamma = \Psi(xav_\gamma) = \Psi(x)av_\gamma \quad (\gamma \in \Gamma),$$

which shows that $\|\Psi(y)v_\gamma - \Psi(x)a\| \rightarrow 0$ (along γ). Define $\Phi : E_0 \rightarrow F_0$ by setting $\Phi(y)$ to be the norm limit of $\Psi(y)v_\gamma$. As $\Phi(y) = \Psi(x)a$ as well, we see that $\Phi(y)$ does not depend on the choice of $\{v_\gamma\}_{\gamma \in \Gamma}$, nor on the decomposition $y = xa$. If $b \in \mathcal{K}(H)$, then

$$\Phi(yb) = \Phi(xab) = \Psi(x)ab = \Phi(y)b.$$

Moreover, if $x, y \in E_0$ with $\langle x, y \rangle = 0$, then $\langle \Psi(x), \Psi(y) \rangle = 0$, which implies that $\langle \Psi(x)v_\gamma, \Psi(y)v_{\gamma'} \rangle = 0$ ($\gamma, \gamma' \in \Gamma$), and so, $\langle \Phi(x), \Phi(y) \rangle = 0$. On the other hand, since $\mathcal{K}(H)$ is simple, we see that either E_0 is a full $\mathcal{K}(H)$ -module or $E_0 = \{0\}$. By Theorem 2.3, there exists $\lambda \in Z(M(\mathcal{K}(H)))_+ = \mathbb{R}_+$ such that for every $x, y \in E_0$, one has $\langle \Phi(x), \Phi(y) \rangle = \lambda \langle x, y \rangle$ (note that one can take any λ if $E_0 = \{0\}$). Thus, for any $x, y \in E$ and $\gamma, \gamma' \in \Gamma$,

$$v_\gamma \langle \Psi(x), \Psi(y) \rangle v_{\gamma'} = \langle \Phi(xv_\gamma), \Phi(yv_{\gamma'}) \rangle = \lambda \langle xv_\gamma, yv_{\gamma'} \rangle = \lambda v_\gamma \langle x, y \rangle v_{\gamma'}.$$

Consequently, if $b_{x,y} := \langle \Psi(x), \Psi(y) \rangle - \lambda \langle x, y \rangle \in A$, then $v_\gamma b_{x,y} v_{\gamma'} = 0$ ($\gamma, \gamma' \in \Gamma$), which shows that $b_{x,y} = 0$ (as $v_\gamma \rightarrow 1$ in the strong operator topology). $\square$

We recall that a unital C^* -algebra A is said to be *properly infinite* if there exists $p \in \text{Proj}_1(A)$ such that $p \sim 1 \sim 1 - p$.

Corollary 3.3. *Let A be a properly infinite real rank zero unital C^* -algebra. If E is full and $\Psi : E \rightarrow F$ is an orthogonality-preserving local map, then there is $u \in Z(A)_+$ such that*

$$\langle \Psi(x), \Psi(y) \rangle = u \langle x, y \rangle \quad (x, y \in E).$$

Proof. Let A_1 be the linear span of projections in A . By [15, Corollary 2.2], we see that $A = A_1$. Therefore, Lemma 3.1 tells us that Ψ is an A -module map. Now, we can apply Theorem 2.3 to obtain the conclusion. $\square$

Corollary 3.4. *Let A be a W^* -algebra. If E is full and $\Psi : E \rightarrow F$ is an orthogonality-preserving local map, then there is $u \in Z(A)_+$ such that*

$$\langle \Psi(x), \Psi(y) \rangle = u \langle x, y \rangle \quad (x, y \in E).$$

Proof. By Theorem 2.3, it suffices to show that Ψ is an A -module map. Recall that there are mutually orthogonal central projections q_{11} , q_{21} and q_∞ in A summing up to 1 such that $q_{11}A$ is a finite W^* -algebra of type I, $q_{21}A$ is a finite W^* -algebra of type II, and $q_\infty A$ is a properly infinite W^* -algebra (see, e.g., [14, 6.1.9]). Thus, $E = Eq_{11} \oplus Eq_{21} \oplus Eq_\infty$. The restriction $\Psi|_{Eq_\infty}$ is a $(q_\infty A)$ -module map because of Lemma 3.1 and the fact that every element in $q_\infty A$ is a sum of at most five idempotents (see [17, Theorem 4]). Similarly, the restriction $\Psi|_{Eq_{21}}$ is a $(q_{21}A)$ -module map since every element in $q_{21}A$ is a complex linear combination of at most twenty-four projections [8, Theorem 2]. Thus, it remains to verify the case when A is a finite W^* -algebra of type I.

In this case, for each $n \in \mathbb{N}$, there exist a hyperstonean space Ω_n (could be empty) and a projection $q_n \in Z(A)$ such that $\{q_n\}$ are orthogonal to one another, $\sum_n q_n$ weak- $*$ -converges to 1 and $q_n A \cong C(\Omega_n) \otimes M_n$ (see e.g. [14, 6.7.7]). Here we use the convention that $C(\Omega_n) = \{0\}$ if $\Omega_n = \emptyset$. Let $n \in \mathbb{N}$ such that $\Omega_n \neq \emptyset$ and $e \in C(\Omega_n)$ be the identity. Pick any rank one projection $p \in M_n$. If $r := e \otimes p \in \text{Proj}_1(q_n A)$, then rAr is isomorphic to $C(\Omega_n)$. By Lemma 3.1, the induced map $\Psi_r : Er \rightarrow Fr$ is an orthogonality-preserving local map between Hilbert rAr -modules. Using [13, 3.5], we see that Ψ_r is an rAr -module map. In particular, for any $a \in C(\Omega_n)$ and $x \in E$, one has

$$\Psi(x(a \otimes p)) = \Psi_r(xr(a \otimes I)r) = \Psi_r(xr)r(a \otimes I)r = \Psi(x)(a \otimes p),$$

where $I \in M_n$ is the identity. Now, let $\{e_{kl}\}_{k,l=1}^n$ be the matrix unit of M_n . As $\frac{e_{kl} + e_{kl}^* + e_{kk} + e_{ll}}{2}$ and $\frac{i(e_{kl} - e_{kl}^*) + e_{kk} + e_{ll}}{2}$ are rank one projections, we see that e_{kl} is a linear combination of rank one projections ($k, l \in \{1, \dots, n\}$). Since any $a \in q_n A$ is of the form $a = \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$ ($a_{ij} \in C(\Omega_n)$), we see that $\Psi(xa) = \Psi(x)a$ ($x \in E; a \in q_n A$). It follows that for any $x \in E$ and $a \in A$,

$$\Psi(xa) \sum_{k=1}^n q_k = \Psi \left(xa \sum_{k=1}^n q_k \right) = \Psi(x) \left(a \sum_{k=1}^n q_k \right) = (\Psi(x)a) \sum_{k=1}^n q_k \quad (n \in \mathbb{N}).$$

Consequently, for any $y \in F$, we have $\langle y, \Psi(xa) - \Psi(x)a \rangle \sum_{k=1}^n q_k = 0$, which implies that $\langle y, \Psi(xa) - \Psi(x)a \rangle = 0$, and so $\Psi(xa) = \Psi(x)a$ as required. $\square$

We end this article with a result that could be a first step towards a positive answer for Conjecture 1.2 in the case when A has real rank zero (with θ not being assumed to be an A -module map nor bounded). This result is also interesting by itself and gives us a rough idea what kind of difficulties we will come across without the A -linearity.

Proposition 3.5. *Let A be a unital C^* -algebra of real rank zero, and A_0 be the $*$ -algebra generated by the idempotents in A . Suppose that there is an element $x_0 \in E$ such that $\langle x_0, x_0 \rangle = 1$. If $\Psi : E \rightarrow F$ is a local map preserving orthogonality, then one can find $u \in Z(A)_+$ as well as an A_0 -submodule $E_0 \subseteq E$ containing x_0 with $E_0^\perp = \{0\}$ such that*

$$\langle \Psi(x), \Psi(y) \rangle = u \langle x, y \rangle \quad (x, y \in E_0).$$

Proof. Define $u := \langle \Psi(x_0), \Psi(x_0) \rangle \in A_+$. Note that by Lemma 3.1, Ψ is an A_0 -module map. Thus, $\Psi(xw) = \Psi(x)w$ for any symmetry $w \in A$ (as $w \in A_0$). Now, the argument of Proposition 2.2 tells us that $u \in Z(A)_+$. Let $z \in E$ such that $\langle x_0, z \rangle = 0$ and $\langle z, z \rangle \in A_0$. Then $z + x_0$ is also orthogonal to $z - x_0 \langle z, z \rangle$. It follows from the orthogonality-preserving property and the A_0 -linearity of Ψ that

$$\langle \Psi(z), \Psi(z) \rangle = \langle \Psi(x_0), \Psi(x_0) \rangle \langle z, z \rangle = u \langle z, z \rangle.$$

Let $\mathcal{D} := \{D \subseteq E : x_0 \in D; \langle x, x \rangle \in A_0 \text{ and } \langle x, y \rangle = 0 \text{ for any } x \neq y \in D\}$. Take any maximal element M in $\mathcal{D}$, and define E_0 to be the linear spans of $x \cdot a$ ($x \in M; a \in A_0$). For any $y \in E_0$, we know that $\langle y, y \rangle, \langle x_0, y \rangle \in A_0$. Thus, $z = y - x_0 \langle x_0, y \rangle$ is orthogonal to x_0 and $\langle z, z \rangle = \langle y, y \rangle - \langle y, x_0 \rangle \langle x_0, y \rangle \in A_0$. Hence, the above implies that

$$\langle \Psi(y), \Psi(y) \rangle = \langle y, x_0 \rangle \langle \Psi(x_0), \Psi(x_0) \rangle \langle x_0, y \rangle + \langle \Psi(z), \Psi(z) \rangle = u \langle y, y \rangle.$$

A polarization type argument tells us that $\langle \Psi(x), \Psi(y) \rangle = u\langle x, y \rangle$ ($x, y \in E_0$). Suppose on the contrary that there exists $z \in E$ with $\|z\| = 1$ and $\langle z, x \rangle = 0$ for any $x \in E_0$. Let $a := \langle z, z \rangle$ and $q_n := e_a(\frac{1}{2^n}, 1]$ ($n \in \mathbb{N}$). There exist $d, b \in C^*(a)_+$ such that $q_5 \leq ab \leq 1$, $d \leq a$, $dq_4 = aq_4$ and $dq_5 = d$. As $b^{1/2}d^{1/2} \leq 1$, $b^{1/2}d^{1/2}q_4 = b^{1/2}a^{1/2}q_4 = q_4$ and $b^{1/2}d^{1/2}q_5 = b^{1/2}d^{1/2}$, we see that

$$\|z - zb^{1/2}d^{1/2}\|^2 = \|a - 2ab^{1/2}d^{1/2} + abd\| \leq 2\|a(1 - b^{1/2}d^{1/2})\| < 1/8.$$

Since $d^{1/2} \in \text{her}(q_5)$, there exists $c \in A_0 \cap \text{her}(q_5)_+$ such that $\|d^{1/2} - c\| < 1/8$ (because $\text{her}(q_5)$ also has real rank zero; see e.g. [5]). If $z' := zb^{1/2}c$, then $\langle z', z' \rangle = cq_5abq_5c = c^2 \in A_0$ (as $abq_5 = q_5$) and $\|z - z'\| \leq \|z - zb^{1/2}d^{1/2}\| + \|zb^{1/2}d^{1/2} - zb^{1/2}c\| \leq 1/2$, which implies that $z' \neq 0$. Moreover, since $\langle x, z' \rangle = \langle x, z \rangle b^{1/2}c = 0$ for any $x \in M$, we see that $M \cup \{z'\} \in \mathcal{D}$, which is a contradiction. $\square$

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